CONTINUOUS REPRESENTATION OF INTERVAL ORDERS
BY MEANS OF DECREASING SCALES

Gianni Bosi

Abstract. We characterize the representability of an interval order on a topological space through a pair of continuous real-valued functions which in addition represent two total preorders associated to the given interval order. Such a continuous representation is obtained by using the notion of a decreasing scale.

1. Introduction

Interval orders are reflexive and total binary relations which are not transitive in general. Such a model may be viewed as the simplest one fulfilling these requirements, in the sense that interval orders may be fully represented by a pair of real-valued functions. The real representability of interval orders was first deeply studied by Fishburn (see e.g. Fishburn [18,19], and then considered by other authors (see e.g. Bridges [9–11], Bridges and Mehta [13], and Oloriz et al. [22]).

Some authors were concerned with the existence of a (semi)continuous representation of an interval order on a topological space (see e.g. Bridges [12], Candeal et al. [15], Chateauneuf [16], Bosi [3], Bosi and Isler [4], and Bosi et al. [5]). In particular, Chateauneuf [16] provided a characterization of the existence of a pair of continuous real-valued functions representing an interval order on a connected topological space. A characterization of the existence of a continuous representation of an interval order on a topological space has been recently obtained by Bosi et al. [6] by using a suitable notion of order separability, called i.o. separability.

In this paper we provide a characterization of the existence of a pair \((U, V)\) of continuous real-valued functions representing an interval order \(\preceq\) on a topological space \((X, \tau)\) (in the sense that, for all \(x, y \in X\), \(x \preceq y\) if and only if \(U(x) \leq V(y)\)). The functions \(U\) and \(V\) may be chosen so that they represent two total preorders associated to the interval order \(\preceq\). In order to obtain such a characterization, we use the notion of a decreasing scale which was first introduced by Burgess and Fitzpatrick [14], and then considered by other authors (see e.g. Herden [20], Alcantud et al. [1], Bosi and Mehta [7] and Bosi and Zuanon [8]).

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2. Notation and preliminaries

An interval order \( \preceq \) on an arbitrary nonempty set \( X \) is a binary relation on \( X \) which is reflexive and in addition verifies the following condition for all \( x, y, z, w \in X \):

\[
(x \preceq z \quad \text{and} \quad (y \preceq w) \Rightarrow (x \preceq w) \quad \text{or} \quad (y \preceq z)).
\]

The strict part of a given interval order \( \preceq \) will be denoted by \( \prec \) (i.e., for all \( x, y \in X \), \( x \prec y \) if and only if \( \neg (y \preceq x) \)). An interval order \( \preceq \) on a set \( X \) is necessarily total, in the sense that, for any two elements \( x, y \in X \), either \( x \preceq y \) or \( y \preceq x \) (see Oloriz et al. [22]).

If \( \preceq \) is an interval order on a set \( X \), then we may consider the binary relations \( \preceq^* \) and \( \preceq^{**} \) on \( X \) defined as follows:

\[
x \preceq^* y \iff (z \preceq x \Rightarrow z \preceq y \text{ for every } z \in X) \quad (x, y \in X)
\]

\[
x \preceq^{**} y \iff (y \preceq z \Rightarrow x \preceq z \text{ for every } z \in X) \quad (x, y \in X)
\]

Fishburn [19] proved that the binary relations \( \preceq^* \) and \( \preceq^{**} \) associated to any interval order \( \preceq \) on a set \( X \) are total preorders on \( X \) (i.e., they are reflexive, transitive and total). It is clear that, for any two elements \( x, y \in X \), if either \( x \preceq^* y \) or \( x \preceq^{**} y \), then we have that \( x \preceq y \).

Obviously every total preorder \( \preceq \) on a set \( X \) is an interval order on \( X \). In this case, we have that \( \preceq = \preceq^* = \preceq^{**} \). The importance of interval orders in economics lies on the fact that they are not transitive in general.

A total preorder \( \preceq \) on a set \( X \) is representable by means of a real-valued function \( U \) on \( X \) if, for all \( x, y \in X \):

\[
x \preceq y \iff U(x) \leq U(y).
\]

We also say that \( U \) is a utility function for the total preorder \( \preceq \) on the set \( X \).

An interval order \( \preceq \) on a set \( X \) is said to be representable through a pair \( (U, V) \) of real-valued functions on \( X \) if, for all \( x, y \in X \):

\[
x \preceq y \iff U(x) \leq V(y).
\]

If \( \preceq \) is an interval order on a set \( X \), then a subset \( G \) of \( X \) is said to be \( \preceq \)-decreasing if, for all \( x, y \in X \), \( x \preceq y \) and \( y \in G \) imply \( x \in G \).

An interval order \( \preceq \) on a topological space \( (X, \tau) \) is said to be upper (lower) semicontinuous if \( L_{\preceq}(x) = \{ y \in X : y \prec x \} \) (\( U_{\preceq}(x) = \{ y \in X : x \prec y \} \)) is a \( \tau \)-open subset of \( X \) for every \( x \in X \). If \( \preceq \) is both upper and lower semicontinuous, then it is said to be continuous.

3. Continuous representability

We present a characterization of the existence of a pair of continuous real-valued functions representing an interval order on a topological space, where the
two functions are utilities for two total preorders naturally associated with the given interval order.

**Theorem 3.1.** The following conditions are equivalent for an interval order $\preceq$ on a topological space $(X, \tau)$:

i) The interval order $\preceq$ is representable through a pair of continuous real-valued functions $(U, V)$ with values in $[0, 1]$, where $U$ is a representation for the total preorder $\preceq^*$ and $V$ is a representation for the total preorder $\preceq^*$;

ii) There exist two families $\{G_r^1\}_{r \in Q \cap [0, 1]}$ and $\{G_r^2\}_{r \in Q \cap [0, 1]}$ of open subsets of $(X, \tau)$ with $G_1 = G_1^* = X$ satisfying the following conditions:

a) $x \preceq y$ and $y \in G_r^*$ imply $x \in G_r^*$ for every $x, y \in X$ and $r \in Q \cap [0, 1]$;

b) $G_r^*$ is $\preceq^*$-decreasing and $G_r^*$ is $\preceq^*$-decreasing for every $r \in Q \cap [0, 1]$;

c) $G_r^* \subseteq G_r^{*1}$ and $G_r^{*2} \subseteq G_r^{*1}$ for every $r_1, r_2 \in Q \cap [0, 1]$ such that $r_1 < r_2$;

d) For every $x, y \in X$ such that $x \prec y$ there exist $r_1, r_2 \in Q \cap [0, 1]$ such that $r_1 < r_2$, $x \in G_r^*$, $y \notin G_r^{*2}$.

**Proof.** i) $\Rightarrow$ ii). If $(U, V)$ is a representation of the interval order $\preceq$ with the indicated properties, then just define $G_r^* = V^{-1}([0, r[), G_r^{*2} = U^{-1}([0, r[)$ for every $r \in Q \cap [0, 1]$, and $G_1^* = G_1^* = X$ in order to immediately verify that $\{G_r^*\}_{r \in Q \cap [0, 1]}$ and $\{G_r^{*2}\}_{r \in Q \cap [0, 1]}$ are two families of open subsets of $(X, \tau)$ satisfying conditions (a) through (d).

ii) $\Rightarrow$ i). Assume that the above condition ii) holds. Define two functions $U, V : X \to [0, 1]$ as follows:

$U(x) = \inf\{r \in Q \cap [0, 1] : x \in G_r^*\}$ \quad ($x \in X$),

$V(x) = \inf\{r \in Q \cap [0, 1] : x \in G_r^*\}$ \quad ($x \in X$).

We claim that $(U, V)$ is a pair of continuous functions on $(X, \tau)$ with values in $[0, 1]$ representing the interval order $\preceq$ where $U$ is a representation for the total preorder $\preceq^*$ and $V$ is a representation for the total preorder $\preceq^*$.

From the definition of the functions $U$ and $V$, it is clear that they both take values in $[0, 1]$. Let us first show that the pair $(U, V)$ represents the interval order $\preceq$. Consider any two elements $x, y \in X$ such that $x \preceq y$, and observe that, for every $r \in Q \cap [0, 1]$, if $y \in G_r^*$ then it must be that $x \in G_r^*$ by the above condition (a). Hence it must be that $U(x) \leq V(y)$ from the definition of $U$ and $V$. Now consider any two elements $x, y \in X$ such that $y \prec x$. Then by condition (d), there exist $r_1, r_2 \in Q \cap [0, 1]$ such that $r_1 < r_2$, $y \in G_r^*, x \notin G_r^*$. Hence we have that $V(y) \leq r_1 < r_2 \leq U(x)$, which obviously implies that $V(y) < U(x)$.

Let us now prove that $V$ is a representation for the total preorder $\preceq^*$. From the first part of condition (b) we have that $G_r^*$ is a $\preceq^*$ decreasing subset of $X$ for every $r \in Q \cap [0, 1]$. Hence if $x, y$ are any two elements of $X$ such that $x \preceq^* y$, then it must be that $V(x) \leq V(y)$ from the definition of $V$. Now consider any two elements $x, y \in X$ such that $y \prec^* x$. Hence there exists another element $z \in X$ such that $y \prec z \preceq x$. So, by condition (d), there exist $r_1, r_2 \in Q \cap [0, 1]$ such that $r_1 < r_2,$
\[ y \in G_{r_1}^*, \ z \notin G_{r_2}^{**}. \] By condition (a), we have that \( x \notin G_{r_2}^* \) since \( z \preceq x \). Finally, we may guarantee the existence of \( r_1, r_2 \in \mathbb{Q} \cap [0, 1] \) such that \( r_1 < r_2, \ y \in G_{r_1}^*, \ x \notin G_{r_2}^* \). Hence from the definition of \( V \), we have that \( V(y) \leq r_1 < r_2 \leq V(x) \) which obviously implies that \( V(y) < V(x) \).

Analogously it may be shown that \( U \) is a representation for the total preorder \( \preceq^{**} \).

To conclude the proof, let us show that \( U \) and \( V \) are both continuous functions by condition (c). We only prove that \( U \) is continuous. Then analogous arguments will show that also \( V \) is continuous. Let us first prove that \( U \) is upper semicontinuous. Consider any \( x \in X \), and \( \alpha \in \mathbb{R} \cap [0, 1] \) such that \( U(x) < \alpha \). Then from the definition of \( U \), there exists \( r \in \mathbb{Q} \cap [0, 1] \) such that \( U(x) \leq r < \alpha \), \( x \in G_r^{**} \). Observe that \( U(z) \geq \alpha (z \in X) \) implies that \( U(z) > r \) which in turn implies that \( z \notin G_r^{**} \). Hence \( G_r^{**} \) is an open subset of \( X \) containing \( x \) such that \( U(z) < \alpha \) for every \( z \in G_r^{**} \). In order to show that \( U \) is lower semicontinuous, let us first prove that

\[ U(x) = \inf \{ r \in \mathbb{Q} \cap [0, 1] : x \in G_r^{**} \} \text{ for every } x \in X. \]

Since \( G_r^{**} \subseteq G_{r'}^{**} \) for every \( r \in \mathbb{Q} \cap [0, 1] \), it is clear that, for every \( x \in X \),

\[ \inf \{ r \in \mathbb{Q} \cap [0, 1] : x \in G_r^{**} \} \leq \inf \{ r \in \mathbb{Q} \cap [0, 1] : x \in G_r \}. \]

Now assume that there exists \( x \in X \) with

\[ \inf \{ r \in \mathbb{Q} \cap [0, 1] : x \in G_r^{**} \} < \inf \{ r \in \mathbb{Q} \cap [0, 1] : x \in G_r^{**} \}. \]

Consider \( r_1, r_2 \in \mathbb{Q} \cap [0, 1] \) such that

\[ \inf \{ r \in \mathbb{Q} \cap [0, 1] : x \in G_r^{**} \} < r_1 < r_2 < \inf \{ r \in \mathbb{Q} \cap [0, 1] : x \in G_r^{**} \}. \]

Then we have that \( x \in G_{r_1}^*, \ x \notin G_{r_2}^{**} \), and this is contradictory, since \( G_{r_1}^{**} \subseteq G_{r_2}^{**} \). So it must be that, for every \( x \in X \),

\[ \inf \{ r \in \mathbb{Q} \cap [0, 1] : x \in G_r^{**} \} = \inf \{ r \in \mathbb{Q} \cap [0, 1] : x \in G_r^{**} \}. \]

Now consider any \( x \in X \), and any \( \alpha \in \mathbb{R} \cap [0, 1] \) such that \( \alpha < U(x) \). Further let \( r_1, r_2 \in \mathbb{Q} \cap [0, 1] \) be such that \( \alpha < r_1 < r_2 < U(x) \). Then we have that \( x \notin G_{r_1}^{**} \) because otherwise \( x \in G_{r_1}^{**} \) implies that \( x \in G_{r_2}^{**} \) and this contradicts the fact that \( U(x) > r_2 \). Observe that \( U(z) \leq \alpha (z \in X) \) implies that \( U(z) < r_1 \) which in turn implies that \( z \in G_{r_1}^{**} \) since \( U(x) = \inf \{ r \in \mathbb{Q} \cap [0, 1] : x \in G_r^{**} \} \) for every \( x \in X \). Hence \( X \setminus G_{r_1}^{**} \) is an open subset of \( X \) containing \( x \) such that \( \alpha < U(z) \) for every \( z \in X \setminus G_{r_1}^{**} \). This consideration completes the proof.

**Remark 3.2.** The family \( \{ G_r^* \}_{r \in \mathbb{Q} \cap [0, 1]} \) is a \( \preceq^* \)-decreasing scale according to the definition introduced by Burgess and Fitzpatrick [14].

As an application of the previous characterization, in the following proposition we present a generalization of the Theorem in Chateauneuf [16]. Chateauneuf showed that a strongly separable interval order \( \preceq \) on a connected topological space
(X, τ) is representable through a pair of continuous real-valued functions (U, V), where U is a representation for the total preorder ⪯∗∗ and V is a representation for the total preorder ⪯*, provided that the total preorders ⪯* and ⪯∗∗ are both continuous.

We recall that an interval order ⪯ on a set X is said to be strongly separable if there exists a countable set D ⊆ X such that, for every x, y ∈ X with x ⪯ y, there exists d₁, d₂ ∈ D with x ⪯ d₁ ⪯ d₂ ⪯ y. D is said to be an order dense subset of X (see Chateauneuf [16]).

Observe that, in contrast to the Chateauneuf Representation Theorem, ours does not need any connectedness assumption on the topological space. The following proposition was already proved by Bosi [3] by using the proof of the existence of a continuous utility function provided by Jaffray [21].

**Proposition 3.3.** Let ⪯ be a strongly separable interval order on a topological space (X, τ), and assume that the total preorders ⪯* and ⪯∗∗ are both continuous. Then the interval order ⪯ is representable through a pair of continuous real-valued functions (U, V) with values in [0, 1], where U is a representation for the total preorder ⪯∗∗ and V is a representation for the total preorder ⪯*.

**Proof.** Let ⪯ be a strongly separable interval order on a topological space (X, τ), and assume that the associated total preorders ⪯* and ⪯∗∗ are both continuous. Then from the Proposition in Chateauneuf [16], we have that ⪯ is continuous. Further strong separability of ⪯ implies order separability of ⪯∗ and ⪯∗∗. In particular, if D is an order dense subset of X, then for all x, y ∈ X with x ⪯∗ y there exists d ∈ D with x ⪯∗ d ⪯∗ y. Without loss of generality, we may assume that (D, ⪯∗∗) is actually a totally ordered set (or a chain) without extreme points. Therefore by using considerations in Birkhoff [2] and following a construction analogous to Construction 3.3 in Alcantud et al. [1], we may conclude that there exists an order-preserving function f : (D, ⪯∗∗) → ([Q]∩[0, 1], ≤). Further we may assume that the mapping f is onto.

For reader’s convenience we recall that any countable chain (C, ⪯) is order isomorphic with a subchain of ([Q], ≤) (see Theorem 22 on page 200 in Birkhoff [2]) and in particular order isomorphic with ([Q], ≤) if (C, ⪯) is dense in itself and has neither a minimal nor a maximal element (see Theorem 23 on page 200 in Birkhoff [2]). Therefore we may conclude that any countable chain (C, ⪯) with the above properties is also order isomorphic with ([Q]∩[0, 1], ≤).

Let us now go back to our case and consider an order-preserving function f : (D, ⪯∗∗) → ([Q]∩[0, 1], ≤) which is also onto. If f⁻¹(r) = d (r ∈ [Q]∩[0, 1]), then define

\[ G_r^* = L_{<}(d), \quad G_r^{∗∗} = L_{<∗∗}(d) \quad (r ∈ [Q]∩[0, 1]), \]

and set \( G_1^* = G_1^{∗∗} = X \). We claim that \{G_r^*\}_{r ∈ [Q]∩[0, 1]} and \{G_r^{∗∗}\}_{r ∈ [Q]∩[0, 1]} are two families of subsets of X satisfying condition ii) of Theorem 3.1. It is clear that \( G_r^* \) is open and ⪯*-decreasing, and \( G_r^{∗∗} \) is open and ⪯∗*-decreasing for every \( r ∈ [Q]∩[0, 1] \), so that condition (b) holds. In order to show that condition (a) is verified, just
observe that, for every $d \in D$, if $x \preceq y$ and $y \prec d$, then $x \prec^* d$. In order to prove that condition (c) holds, observe that for all $d_1, d_2 \in D$ such that $d_1 \prec^* d_2$ there exists $z \in X$ such that $d_1 \preceq z \prec d_2$, and therefore we have that

$$L_{<}(d_1) \subseteq L_{<}(d_2),$$

$$L_{<^*}(d_1) \subseteq L_{<^*}(d_2),$$

where $L_{<^*}(z) = \{ w \in X : w \preceq^* z \}$ and $L_{<^*}(d_1) = \{ w \in X : w \preceq^* d_1 \}$ are closed subsets of $X$. Finally, condition (d) of Theorem 3.1 holds, since strong separability of the interval order $\preceq$ implies that for all $x, y \in X$ such that $x \prec y$ there exist $d_1, d_2 \in D$ such that $x \prec d_1 \prec^* d_2 \prec^* y$, and therefore $x \in L_{<}(d_1)$ and $y \notin L_{<^*}(d_2)$. This consideration completes the proof.

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Dipartimento di Matematica Applicata “Bruno de Finetti”, Università di Trieste, Piazzale Europa 1, 34127 Trieste, Italy

E-mail: giannibo@econ.units.it