EQUITORSION CONFORM MAPPINGS OF GENERALIZED RIEEMANNIAN SPACES

Mića S. Stanković, Ljubica S. Velimirović, Svetislav M. Minčić and Milan Lj. Zlatanović

Abstract. We define an equitorsion conform mapping of two generalized Riemannian spaces and obtain some invariant geometric objects of this mapping, generalizing the tensor of conform curvature.

0. Introduction

A generalized Riemannian space $GR_N$ in the sense of Eisenhart’s definition [5] is a differentiable $N$-dimensional manifold, equipped with nonsymmetric basic tensor $g_{ij}$. The use of non-symmetric basic tensor and non-symmetric connection became especially actual after appearance of the works of A. Einstein [1]–[4] related to creation of the Unified Field Theory (UFT). Remark that at UFT the symmetric part $g_{ij}$ of the basic tensor $g_{ij}$ is related to the gravitation, and antisymmetric one $g_{ij}$ to the electromagnetism. M Prvanović [14] and S. Minčić [8] gave geometric interpretations of the torsion and curvature tensors of non-symmetric affine connection.

Consider two $N$-dimensional generalized Riemannian spaces $GR_N$ and $G\overline{R}_N$. Generalized Cristoffel’s symbols of the first kind of the space $GR_N$ and $G\overline{R}_N$ are given by

\[
\Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}) \quad \text{and} \quad \Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}), \quad (0.1)
\]

where, for example, $g_{ij,k} = \partial g_{ij}/\partial x^k$. Connection coefficients of these spaces are generalized Cristoffel’s symbols of the second kind $\Gamma^i_{jk} = g^{ip}\Gamma_{p.jk}$ and $\overline{\Gamma}^i_{jk} = g^{ip}\overline{\Gamma}_{p.jk}$.

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119
\[ \bar{g}_{ij}^\varpi \Gamma_{p,jk} \] respectively, where \((g^{\frac{1}{2}}) = (g_{ij})^{-1} \) and \( ij \) denote symmetrisation with division by indices \( i \) and \( j \). Generally it is \( \bar{g}_{ik}^j \neq \Gamma_{k,i}^j \). We suppose that \( g = \det(g_{ij}) \neq 0, \bar{g} = \det(\bar{g}_{ij}) \neq 0 \).

One says that a reciprocal one-valued mapping \( f : GR_N \to G\bar{R}_N \) is conform if for the basic tensors \( g_{ij} \) and \( \bar{g}_{ij} \) of these spaces the condition

\[ \bar{g}_{ij} = e^{2\psi} g_{ij} \]  

is satisfied, where \( \psi \) is an arbitrary function of \( x \)'s, and the spaces are considered in the common by this mapping system of local coordinates \( x^i \). In this case for the Cristoffel’s symbols of the first kind of the spaces \( GR_N \) and \( G\bar{R}_N \) the relation

\[ \Gamma_{i,jk} = e^{2\psi} (\Gamma_{i,jk} + g_{ij} \psi_{,k} - g_{jk} \psi_{,i} + g_{ik} \psi_{,j}) \]  

holds true, and for the Cristoffel’s symbols of the second kind

\[ \Gamma_{i}^{jk} = \Gamma_{j}^{ik} + g^{\varpi}_{lp}(g_{jp} \psi_{,k} - g_{jk} \psi_{,p} + g_{pk} \psi_{,j}) \]  

holds. Let us denote \( \psi_k = \psi_{,k} = \partial \psi/\partial x^k \) and \( \psi^i = g^{\varpi}_{lp} \psi_{,p} \). Now from (0.4) we have

\[ \Gamma_{i}^{jk} = \Gamma_{j}^{ik} + g^{\varpi}_{lp}(g_{jp} \psi_{,k} - g_{jk} \psi_{,p} + g_{pk} \psi_{,j}) + g^{\varpi}_{lp}(g_{jp} \psi_{,k} - g_{jk} \psi_{,p} + g_{pk} \psi_{,j}), \]

i.e.

\[ \Gamma_{i}^{jk} = \delta_{j}^{i} \psi_{,k} + \delta_{k}^{i} \psi_{,j} - \psi_{,j}^{i} g_{jk} + \xi_{jk}^{i}, \]  

where

\[ \xi_{jk}^{i} = g^{\varpi}_{lp}(g_{jp} \psi_{,k} - g_{jk} \psi_{,p} + g_{pk} \psi_{,j}) = -\xi_{kj}^{i} \]  

and \( ij \) denotes an antisymmetrisation with division. In the corresponding points \( M(x) \) and \( M(x) \) of conform mapping we can put

\[ \Gamma_{i}^{jk} = \Gamma_{j}^{ik} + P_{jk}^{i} \quad (i,j,k = 1, \ldots, N), \]  

where \( P_{jk}^{i} \) is the deformation tensor of the connection \( \Gamma \) of \( GR_N \) according to the conform mapping \( f : GR_N \to G\bar{R}_N \).

Notice that in \( GR_N \) we have

\[ \Gamma_{lp}^{i} = 0, \]  

(eq. (2.10) in [13]).

In a generalized Riemannian space one can define four kinds of covariant derivatives [10, 11]. For example, for a tensor \( a_{ij}^{l} \) in \( GR_N \) we have

\[ a_{lj}^{i} = a_{lj}^{i} + \Gamma_{lp}^{i} a_{jl}^{p} - \Gamma_{jp}^{i} a_{lj}^{p}, \]

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Denote by \( \theta \) a covariant derivative of the kind \( \theta \) in \( GR_N \) and \( G\bar{R}_N \) respectively. We have [7]

\[
\frac{\partial g_{ij|ma}}{\partial \theta} = \frac{\partial g_{ij|ma}}{\partial \theta} = 0.
\]

In the case of the space \( GR_N \) we have five independent curvature tensors [9] (in [9] \( R_N \) is denoted by \( \tilde{R} \)):

\[
\begin{align*}
R^1_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm} \Gamma^i_{pn} - \Gamma^p_{jn} \Gamma^i_{pm}, \\
R^2_{jmn} &= \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj} \Gamma^i_{np} - \Gamma^p_{nj} \Gamma^i_{mp}, \\
R^3_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm} \Gamma^i_{np} - \Gamma^p_{jn} \Gamma^i_{mp} + \Gamma^p_{nm} (\Gamma^i_{pj} - \Gamma^i_{jp}), \\
R^4_{jmn} &= \Gamma^i_{jmn} - \Gamma^i_{nj,m} + \Gamma^p_{jm} \Gamma^i_{np} - \Gamma^p_{nj} \Gamma^i_{mp} + \Gamma^p_{nm} (\Gamma^i_{pj} - \Gamma^i_{jp}), \\
R^5_{jmn} &= \frac{1}{2} (\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{jn,m} + \Gamma^p_{jm} \Gamma^i_{pn} + \Gamma^p_{mj} \Gamma^i_{np} \\
&\quad - \Gamma^p_{jn} \Gamma^i_{mp} - \Gamma^p_{nj} \Gamma^i_{pm}).
\end{align*}
\]

We use the conform mapping \( f : GR_N \to G\bar{R}_N \) to obtain tensors \( \overline{R}^i_{jmn} \) (\( \theta = 1, \ldots, 5 \)), where for example

\[
\overline{R}^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm} \Gamma^i_{pn} - \Gamma^p_{jn} \Gamma^i_{pm}.
\]

In the case of conform mapping \( f : R_N \to \overline{R}_N \) of Riemannian spaces \( R_N \) and \( \overline{R}_N \) [6, 15] we have an invariant geometric object

\[
C^i_{jmn} = R^i_{jmn} + \delta^i_m P_{jm} - \delta^i_n P_{jm} + P^i_m g_{jn} - P^i_n g_{jm}
\]

where

\[
P_{jm} = \frac{1}{N-2} (R_{jm} - \frac{1}{2(N-1)} R g_{jm}),
\]

and \( R^i_{jmn} \) is Riemann-Cristoffel’s curvature tensor of the space \( R_N \), \( R_{jm} \) Ricci’s tensor and \( R \) a scalar curvature.

The object \( C^i_{jmn} \) is called a conform curvature tensor [6, 15]. Having a conform mapping of two generalized Riemannian spaces, we cannot find a generalization of the tensor of conform curvature as an invariant of conform mapping in general case. For that reason we define a special conform mapping.

A mapping \( f : GR_N \to G\bar{R}_N \) is an equitorsion conform mapping if the torsion tensors of the spaces \( GR_N \) and \( G\bar{R}_N \) are equal. Then from (0.5) and (0.6) we have

\[
\xi^i_{jk} = 0.
\]

In [12] we have investigated equitorsion geodesic mappings of generalized Riemannian spaces.
1. Equitorsion conform curvature tensor of the first kind

Using (0.6), we get a relation between the first kind curvature tensors of the spaces $GR_N$ and $G\tilde{R}_N$ [12, 16]

$$\bar{R}'_{jmn} = R^i_{jmn} + P^i_{jm|n} - P^i_{jn|m} + P^p_{jm}P^i_{pn} - P^p_{jn}P^i_{pm} + 2\Gamma^p_{mn}P^i_{jp}. $$

Substituting $P$ with respect to (0.5, 6, 10), and using (0.7'), we obtain

$$\bar{R}'_{jmn} = R^i_{jmn} + \delta^i_j (\psi^i_m - \psi^i_n) - \psi^i_j (\psi^i_m - \psi^i_n)g_{jm} + (\psi^i_j - \psi^i_m)g_{jn} - \psi^i_n \psi^i_p g_{jm} + \delta^i_p \psi^i_q g_{jm} + 2\delta^i_j \Gamma^i_{mn} \psi^i_p + 2\Gamma^i_{mn} \psi^i_j - 2\Gamma^i_{j,mn} \psi^i_j. $$

Denoting

$$\psi^i_{ij} = \psi^i_{1|j} - \psi^i_j, \quad \psi^i_j = g^{pq} \psi^i_{pj}$$

and using the relation

$$\psi^i_{mn} - \psi^i_{nm} = -2\Gamma^p_{mn} \psi^i_p$$

in (1.1), we get

$$\bar{R}'_{jmn} = R^i_{jmn} + \delta^i_j (\psi^i_m - \psi^i_n) - \psi^i_j (\psi^i_m - \psi^i_n)g_{jm} + (\psi^i_j - \psi^i_m)g_{jn} - \psi^i_n \psi^i_p g_{jm} + \delta^i_p \psi^i_q g_{jm} + 2\delta^i_j \Gamma^i_{mn} \psi^i_p + 2\Gamma^i_{mn} \psi^i_j - 2\Gamma^i_{j,mn} \psi^i_j. $$

Further, let us denote

$$\Delta^2 \psi = g^{pq} \psi^i_{pi}q$$

Then we have

$$\psi^i_p = \psi^i_{pq}g^{pq} = (\psi^i_{pq} - \psi^i_p \psi^i_q)g^{pq} = \Delta^2 \psi - \Delta \psi.$$ 

Contracting by indices $i$ and $n$ in (1.4) we get

$$\bar{R}'_{jmn} = R^i_{jmn} - (N - 2)\psi^i_{jm} - \Delta^2 \psi + (N - 2) \Delta \psi | g_{jm} - 2\Gamma^i_{j,mn} \psi^i. $$

From (0.2) we get

$$\bar{g}^{2\psi} = e^{-2\psi} g^{ij}. $$

In (1.6) multiplying by $g^{jm}$ and contracting by $j$ and then by $m$ we get

$$e^{2\psi} \bar{R}' = R - 2(N - 1)\Delta^2 \psi - (N - 1)(N - 2) \Delta \psi,$$
where $\mathcal{R}_1 = g^{pq} R_{1 pq}$, and $R = g^{pq} R_{pq}$ are scalar curvature of the first kind of the spaces $G\mathcal{R}_N$ and $GR_N$ respectively. From (1.8) we have

$$\Delta_2\psi = \frac{1}{2(N - 1)} (R - e^{2\psi} \overline{R}) - \frac{N - 2}{2} \Delta_1\psi. \quad (1.9)$$

Substituting (1.9) in (1.6) we get

$$\frac{1}{(N - 2)} \psi_{jm} = R_{jm} - \overline{R}_{jm} - \frac{1}{2(N - 1)} (R - e^{2\psi} \overline{R}) g_{jm}$$

$$- \frac{N - 2}{2} \Delta_1\psi g_{jm} - 2\Gamma_{j,m,p} \psi^p. \quad (1.10)$$

Let us denote in the space $GR_N$

$$P_{1 jm} = \frac{1}{N - 2} (R_{jm} - \overline{R}_{jm}) \quad (1.10')$$

and analogously $\overline{P}_{jm}$ in the space $G\mathcal{R}_N$. In this case for $\psi_{jm}$ we obtain

$$\psi_{jm} = P_{1 jm} - \overline{P}_{1 jm} - \frac{1}{2} \Delta_1\psi g_{jm} - \frac{2}{N - 2} \Gamma_{j,m,p} \psi^p. \quad (1.11)$$

Substituting (1.11) in (1.4), we get

$$\mathcal{R}^i_{1 jmn} = R^i_{1 jmn} + \delta^i_n (P_{1 jm} - \overline{P}_{1 jm}) - \delta^i_m (P_{1 jm} - \overline{P}_{1 jm})$$

$$+ P_{1 nm} g_{jm} - P_{1 jm} g_{jm} + \Gamma_{1 jm}^{\psi}$$

$$+ \frac{2}{N - 2} (\delta^i_n \Gamma_{j,m,p} - \delta^i_m \Gamma_{j,n,p} + \Gamma_{j,m,\psi} g_{jm} - \Gamma_{j,n,\psi} g_{jm}) \psi^p$$

$$+ 2\Gamma_{1 nm} \psi_j - 2\Gamma_{j,m,n} \psi^i. \quad (1.12)$$

We can see that it follows from (0.2)

$$\psi^i = \frac{1}{2N} (\frac{\partial}{\partial x^i} \ln g - \frac{\partial}{\partial x^i} \ln \overline{g}) \quad (1.13)$$

where $g = \det (g_{ij})$, $\overline{g} = \det (\overline{g}_{ij})$. From (0.10) and (1.13) we obtain

$$\Gamma_{j,n,m} \psi^i = \frac{1}{2N} \Gamma_{j,n,m} g^{\psi q} \frac{\partial}{\partial x^q} \ln g - \frac{1}{2N} \Gamma_{j,n,m} g^{\psi q} \frac{\partial}{\partial x^q} \ln g \quad (1.14)$$

and

$$\Gamma_{q n} g_{m j} \psi^q = \frac{1}{2N} \Gamma_{q n} g_{m j} g^{\psi q} \frac{\partial}{\partial x^q} \ln g - \frac{1}{2N} \Gamma_{q n} g_{m j} g^{\psi q} \frac{\partial}{\partial x^q} \ln g. \quad (1.15)$$

Taking into account (1.13, 14,15), we can write the relation (1.12) in the form

$$\overline{C}^i_{1 jmn} = C^i_{1 jmn}. \quad (1.16)$$
where

\[
C_{1jmn}^i = R_{1jmn}^i + \delta_m^i P_{2j}^{jn} - \delta_n^i P_{2j}^{jm} + P_{1m}^i g_{jm} - P_{1n}^i g_{jm} \\
+ \frac{1}{N(N-2)} (\delta_m^i \Gamma_{j, np}^{vp} - \delta_n^i \Gamma_{j, mp}^{vp} + \Gamma_{mp}^i g_{jn} - \Gamma_{np}^i g_{jm}) g^{pq} \frac{\partial}{\partial x^q} \ln g \\
+ \frac{1}{N} (\Gamma_{jmn}^i \delta_j^p - \delta_j^m \Gamma_{jnp}^i - \delta_j^n \Gamma_{jmp}^i + \Gamma_{jmp}^i) g^{pq} \frac{\partial}{\partial x^p} \ln g
\]

(1.17)

and analogously for \( C_{1jmn}^i \). From (1.16) we see that the tensor \( C_{1jmn}^i \) is an invariant of equitorsion conform mapping, and one can call it the equitorsion conform curvature tensor of the first kind. So, we have

**Theorem 1.** Let generalized Riemannian spaces \( GR_N \) and \( G\bar{R}_N \) be defined by virtue of their nonsymmetric basic tensors \( g_{ij} \) and \( \bar{g}_{ij} \) respectively. The equitorsion conform curvature tensor of the first kind \( C_{1jmn}^i \) is an invariant of the equitorsion conform mapping \( f : GR_N \to G\bar{R}_N \), defined by (0.2), (0.5), (0.10), i.e. (1.16) is in force, where the tensor \( P_{1i} \) is given by (1.10′).

2. Equitorsion conform curvature tensor of the second kind

For the second kind curvature tensors of the spaces \( GR_N \) and \( G\bar{R}_N \) we get the relation [12, 16]

\[
\bar{T}_{1jmn}^i = R_{1jmn}^i + \delta_m^i \psi_{2jm} - \delta_n^i \psi_{2jm} + \psi_{2m}^i g_{mj} - \psi_{2n}^i g_{mj} \\
+ \frac{1}{N(N-2)} (\delta_m^i \Gamma_{j, np}^{vp} - \delta_n^i \Gamma_{j, mp}^{vp} + \Gamma_{mp}^i g_{jn} - \Gamma_{np}^i g_{jm}) g^{pq} \frac{\partial}{\partial x^q} \ln g \\
+ \frac{1}{N} (\Gamma_{jnm}^i \psi_{2p} - \psi_{2p} \Gamma_{jnp}^i - \psi_{2p} \Gamma_{jmp}^i + \Gamma_{jmp}^i) g^{pq} \frac{\partial}{\partial x^p} \ln g
\]

(2.1)

i.e., using (0.5,6,10) one obtains

\[
\bar{T}_{1jmn}^i = R_{1jmn}^i + \delta_m^i \psi_{2jm} - \delta_n^i \psi_{2jm} + \psi_{2m}^i g_{mj} - \psi_{2n}^i g_{mj} \\
+ \frac{1}{N(N-2)} (\delta_m^i \Gamma_{j, np}^{vp} - \delta_n^i \Gamma_{j, mp}^{vp} + \Gamma_{mp}^i g_{jn} - \Gamma_{np}^i g_{jm}) g^{pq} \frac{\partial}{\partial x^q} \ln g \\
+ \frac{1}{N} (\Gamma_{jnm}^i \psi_{2p} - \psi_{2p} \Gamma_{jnp}^i - \psi_{2p} \Gamma_{jmp}^i + \Gamma_{jmp}^i) g^{pq} \frac{\partial}{\partial x^p} \ln g
\]

(2.2)

Now, analogously to previous case, we get the invariant object of the equitorsion conform mapping \( f : GR_N \to G\bar{R}_N \)

\[
C_{2jmn}^i = R_{2jmn}^i + \delta_m^i P_{2j}^{jn} - \delta_n^i P_{2j}^{jm} + P_{2m}^i g_{jm} - P_{2n}^i g_{jm} \\
+ \frac{1}{N(N-2)} (\delta_m^i \Gamma_{j, np}^{vp} - \delta_n^i \Gamma_{j, mp}^{vp} + \Gamma_{mp}^i g_{jn} - \Gamma_{np}^i g_{jm}) g^{pq} \frac{\partial}{\partial x^q} \ln g \\
+ \frac{1}{N} (\Gamma_{jnm}^i \psi_{2p} - \psi_{2p} \Gamma_{jnp}^i - \psi_{2p} \Gamma_{jmp}^i + \Gamma_{jmp}^i) g^{pq} \frac{\partial}{\partial x^p} \ln g
\]

(2.3)
where
\[ P_{jm}^2 \equiv \frac{1}{N-2} \left( \frac{R_{jm}}{2} - \frac{1}{2(N-1)^2} R_{ij} \right), \]  
(2.4)

\( R_{jm} \) is Ricci’s curvature tensor of the second kind and \( \bar{R} \) is a scalar curvature tensor of the second kind. The object \( C_{jm}^i \) is a tensor and we call it equitorsion conform curvature tensor of the second kind. Accordingly, we have

**Theorem 2.** Starting from the curvature tensor \( R_{jm}^2 \), under conditions as in Theorem 1, one obtains an invariant tensor \( C_{jm}^i \) (2.3) of the equitorsion conform mapping of generalized Riemannian spaces, where \( P \) is given according to (2.4).

### 3. Equitorsion conform curvature tensor of the third kind

In the case of the third kind curvature tensors of the spaces \( GR_N \) and \( G\bar{R}_N \) we get the relation [12, 16]

\[ T_{jm}^i = R_{jm}^i + P_{jm}^i, \]

\[ P_{jm}^i = P_{jm}^i - P_{nm}^p P^m_{ip} - P_{nj}^i P^m_{jn}, \]

(3.1)
i.e., because of (0.5, 6, 10), (1.2a,b) and (2.2),

\[ T_{jm}^i = R_{jm}^i + \delta^i_{m} \psi_j - \delta^i_{n} \psi_j + \psi^i_{m} g_{jm} \]

\[ + (\delta^i_{m} g_{nj} - \delta^i_{n} g_{jm}) \Delta_1 \psi + 2 \psi^i_{m} \Gamma^i_{nj} + 2 \psi^i_{n} \Gamma^i_{mj} - 2 \psi^i_{j} g_{nm} \Gamma^i_{pj}, \]

Also, the following is satisfied

\[ \psi_{mn}^{ij} = \psi_{mn} + 2 \Gamma^p_{mn} \psi^i_p, \quad \psi_{n}^{ij} = \psi_{n} + 2 g^{ip} \Gamma^p_{mn} \psi^i_q, \]  
(3.2)

From (3.1), (3.2) and (10.1) we get

\[ T_{jm}^i = R_{jm}^i + \delta^i_{m} \psi_j - \delta^i_{n} \psi_j + \psi^i_{m} g_{jm} \]

\[ + (\delta^i_{m} g_{nj} - \delta^i_{n} g_{jm}) \Delta_1 \psi + 2 \psi^i_{m} \Gamma^i_{nj} + 2 \psi^i_{n} \Gamma^i_{mj} - 2 \psi^i_{j} g_{nm} \Gamma^i_{pj}, \]

\[ + 2 \delta^i_{m} \Gamma^p_{jm} \psi^i_p - 2 g^{ip} \Gamma^p_{jm} \psi^i_q g_{jm}. \]

(3.3)

Contracting (3.3) with respect to \( i \) and \( n \), and using (1.5), we get

\[ \bar{R}_{jm} = R_{jm} - (N-2) \psi_j - \Delta_2 \psi - (N-2) \Delta_1 \psi g_{jm} - \psi^p \Gamma_{m,p,j}. \]

(3.4)

Multiplying (3.4) by \( g_{jm} \) and contracting we get

\[ \Delta_2 \psi = \frac{1}{2(N-1)} \left( \frac{R}{3} - e^{-2 \psi} \bar{R} \right) - \frac{N-2}{2} \Delta_1 \psi. \]

(3.5)
Substituting (3.5) in (3.4) and denoting
\[ P_{3jm} = \frac{1}{N-2} \left( R_{3jm} - \frac{1}{2(N-1)} R g_{jm} \right) \] (3.6)
in $GR_N$ and analogously in $GR'_N$, in this case for $\psi_{jm}$ we obtain
\[ \psi_{jm} = P_{3jm} - \frac{1}{2} \Delta \psi g_{jm} - \frac{2}{N-2} \Gamma_{m,pj} \psi^p. \] (3.7)
Substituting (3.7) in (3.3) and using (1.14,15) we get
\[ C_{3}^{i}jmn = C_{3}^{i}jmn \] (3.8)
where
\[ C_{3}^{i}jmn = R_{3}^{i}jmn + \delta_{i}^{m} P_{3jm} - \delta_{m}^{n} P_{3jm} + P_{3}^{i}g_{nj} - P_{3}^{i}g_{jm} \]
\[ + \frac{1}{N(N-2)} \left( \delta_{m}^{i} \Gamma_{n,pj} - \delta_{m}^{n} \Gamma_{m,pj} \right) g_{pq} \frac{\partial}{\partial x^q} \ln g \]
\[ + \frac{1}{N} \left( g_{np} \Gamma_{mn} g_{jm} - \delta_{m}^{n} \Gamma_{i}^{n} g_{jm} - \delta_{m}^{i} \Gamma_{n}^{n} g_{jm} \right) \]
\[ + \Gamma_{pj} g_{nm} g_{pq} \frac{\partial}{\partial x^q} \ln g \] (3.9)
And analogously for $C_{3}^{i}jmn$ of the space $GR'_N$. From (3.8) we can see that the tensor $C_{3}^{i}jmn$ is an invariant of equitorsion conform mapping, and one can call it the equitorsion conform curvature tensor of the third kind. Now we have

**Theorem 3.** From the curvature tensor $R_{3}^{i}jmn$, under the conditions as in Theorem 1, we obtain an invariant tensor $C_{3}^{i}jmn$ (3.9) of the equitorsion conform mapping $f : GR_N \rightarrow GR'_N$, where $P_{3}$ is given according to (3.6).

4. Equitorsion conform curvature tensor of the fourth kind

For curvature tensors of the fourth kind we get [12, 16]
\[ \overline{R}_{4}^{i}jmn = R_{4}^{i}jmn + \delta_{i}^{n} P_{4jm} - \delta_{i}^{n} P_{4jm} + P_{4}^{i}g_{nj} - P_{4}^{i}g_{jm} \]
\[ + 2 P_{mn}^{p} \Gamma_{p}^{i} + 2 P_{mn}^{p} \Gamma_{pj}^{i} \]
i.e.
\[ \overline{R}_{4}^{i}jmn = R_{4}^{i}jmn + \delta_{i}^{n} \psi_{jn} - \delta_{i}^{n} \psi_{jm} + \psi_{i}^{n}g_{nj} - \psi_{i}^{n}g_{jm} \]
\[ + \left( \delta_{m}^{n} g_{nj} - \delta_{m}^{n} g_{jm} \right) \Delta \psi + 2 \psi_{n} \Gamma_{mj}^{i} + 2 \psi_{m} \Gamma_{nj}^{i} - 2 \psi^{p} g_{mn} \Gamma_{pj}^{i} \]
\[ + 2 \delta_{m}^{i} \Gamma_{pj}^{p} - 2 g_{mn}^{p} \Gamma_{p}^{i} \psi_{q} g_{jm}. \]
In this case, analogously to previous case, we get an invariant object of the equitorsion conform mapping in the form

\[
C^i_{4}jmn = R^i_{4jmn} + \delta^i_m P_{jm} - \delta^i_n P_{jm} + P^i_{4mj}g_{nj} - P^i_{4nj}g_{jm} \\
+ \frac{1}{N(N-2)}(\delta^i_m \Gamma_{n,pj} - \delta^i_n \Gamma_{m,pj})g^{pq} \frac{\partial}{\partial x^q} \ln g \\
+ \frac{1}{N}(g^{ip} \Gamma_{mn,p}^q g_{jm} - \delta^i_m \Gamma^q_{n,j} - \delta^i_n \Gamma^q_{m,j}) \\
+ \Gamma_{pj}^m g_{mn} g^{pq} - \delta^i_m \Gamma^q_{n,j} \frac{\partial}{\partial x^q} \ln g, \\
P_{4jm} = \frac{1}{N-2}(R^i_{4jmn} - \frac{1}{2(N-1)} R^i_{ijmn}), \tag{4.2}
\]

where \( R^i_{4jm} \) is Ricci's curvature tensor of the fourth kind and \( R \) a scalar curvature of the fourth kind. The object \( C^i_{4jmn} \) is a tensor and we call it equitorsion conform curvature tensor of the fourth kind of the equitorsion conform mapping. So, the next theorem is valid.

**Theorem 4.** From the curvature tensor \( R^i_{4jmn} \), under the conditions as in Theorem 1, one obtains an invariant tensor \( C^i_{4jmn} \) of the equitorsion conform mapping of generalized Riemannian spaces, where \( P \) is given with respect to (4.2).

### 5. Equitorsion conform curvature tensor of the fifth kind

For the curvature tensors of the fifth kind of the spaces \( GR_N \) and \( G\overline{R}_N \) we find the relation [12, 16]

\[
\overline{R}^i_{5jmn} = R^i_{5jmn} + \frac{1}{2}(P^i_{jmn} - P^i_{jnm} + P^i_{mjn} - P^i_{njm}) \\
+ P^p_{jm} P^i_{pn} - P^p_{jn} P^i_{mp} + P^p_{mj} P^i_{np} - P^p_{nj} P^i_{pm}) \\
i.e.
\overline{R}^i_{5jmn} = R^i_{5jmn} + \frac{1}{2}[\delta^i_m (\psi_{jn} + \psi_{ijn} - 2\psi_{j} \psi_{m}) - \delta^i_n (\psi_{jm} + \psi_{jmn} - 2\psi_{j} \psi_{m}) \\
+ (\psi^i_m + \psi^i_n - 2\psi_m \psi^n)g_{jm} - (\psi^i_j + \psi^i_n - 2\psi_n \psi^i)g_{mj} \\
+ 2(\delta^i_m g_{jn} - \delta^i_n g_{jm}) \psi_p \psi_p]. \tag{5.1}
\]

Let us denote

\[
\psi_{jn} = \frac{1}{2}(\psi_{jn} + \psi_{ijn} - 2\psi_{j} \psi_{n}), \quad \psi^i_{jn} = g^{ip} \psi_{pj}, \quad \Delta_1 \psi = g^{pq} \psi_p \psi_q. \tag{5.2}
\]

Then

\[
\overline{R}^i_{5jmn} = R^i_{5jmn} + \delta^i_m \psi_{jn} - \delta^i_n \psi_{jm} + \psi^i_{jn} g_{jm} - \psi^i_{jm} g_{jn} \\
+ (\delta^i_m g_{jn} - \delta^i_n g_{jm}) \Delta_1 \psi. \tag{5.3}
\]
Contracting by indices \(i, n\) and denoting
\[
\overline{R}^{ip}_{5jmp} = R_{5jm}, \quad R^{ip}_{5jmp} = R_{5jm}, \quad \Delta_{34}\psi = \frac{1}{2}g^{pq}_{3}((\psi_{p|q}^{i} + \psi_{q|p}^{i})_{3}),
\]  
(5.4)
we obtain
\[
\overline{R}_{5jm} = R_{5jm} - (N - 2)\psi_{34m}^{j} - [\Delta_{34}\psi + (N - 2)\Delta_{1}\psi]g_{jm},
\]  
(5.5)
wherefrom, multiplying by \(g_{jm} = e^{-2\psi}g_{jm}\) and contracting by \(j\) and then by \(m\) one obtains
\[
\Delta_{34}\psi = \frac{1}{2}(N - 1)(R_{5jm} - e^{2\psi}\overline{R}_{5jm}) - \frac{N - 2}{2}\Delta_{1}\psi.
\]  
(5.6)
From (5.5) and (5.6) we get
\[
\psi_{34 jm} = P_{5 jm} - P_{5 jm} - \frac{1}{2}\Delta_{1}\psi g_{jm}
\]  
(5.7)
where we denoted
\[
P_{5 jm} = \frac{1}{N - 2}(R_{5 jm} - \frac{1}{2(N - 1)}Rg_{jm})
\]  
(5.8)
in \(GR_{N}\) and analogously \(\overline{P}_{5 jm}\) in \(G\overline{R}_{N}\).

Analogously to previous cases eliminating \(\psi_{34 jm}\) from (5.3) we can write
\[
\overline{C}_{5 jmnn}^{i} = C_{5 jmnn}^{i},
\]  
(5.9)
where we denoted
\[
C_{5 jmnn}^{i} = P_{5 jm} + \delta_{m}^{i}P_{5 jm} - \delta_{n}^{i}P_{5 jm} + P_{5 mn}g_{nj} - P_{5 nm}g_{jm}.
\]  
(5.10)
The object \(C_{5 jmnn}^{i}\) is an invariant of the equitorsion conform mapping. We call it equitorsion conform curvature tensor of the fifth kind. So, we have

**Theorem 5.** Starting from the curvature tensor \(R_{5 jmnn}^{i}\), under the conditions as in the Theorem 1, we obtain an invariant tensor \(C_{5 jmnn}^{i}\) (5.10) of the equitorsion mapping \(f: GR_{N} \rightarrow G\overline{R}_{N}\), where \(P_{5}\) is given according to (5.8).

If \(GR_{N}(G\overline{R}_{N})\) reduces to \(R_{N}(R_{N})\), then the objects \(C_{5 jmnn}^{i}\) \((\theta = 1, \ldots, 5)\) reduce to the conform curvature tensor (0.9).

**References**


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Prirodno-matematički fakultet, Višegradska 33, 18000 Niš, Serbia

E-mail: stmica@ptt.rs