ON CERTAIN MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY USING A DIFFERENTIAL OPERATOR

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Abstract. In this paper, we introduce the subclass \( S_j(n, p, q, \alpha) \) of analytic and \( p \)-valent functions with negative coefficients defined by new operator \( D_n^p \). In this paper we give some properties of functions in the class \( S_j(n, p, q, \alpha) \) and obtain numerous sharp results including (for example) coefficient estimates, distortion theorem, radii of close-to-convexity, starlikeness and convexity and modified Hadamard products of functions belonging to the class \( S_j(n, p, q, \alpha) \). Finally, several applications involving an integral operator and certain fractional calculus operators are also considered.

1. Introduction

Let \( T(j, p) \) denote the class of functions of the form

\[
f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k \quad (a_k \geq 0; \ p, j \in \mathbb{N} = \{1, 2, \ldots \}),
\]

which are analytic and \( p \)-valent in the open unit disc \( U = \{z : |z| < 1\} \). A function \( f(z) \in T(j, p) \) is said to be \( p \)-valently starlike of order \( \alpha \) if it satisfies the inequality

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U; \ 0 \leq \alpha < p; \ p \in \mathbb{N}).
\]

We denote by \( T^*_j(p, \alpha) \) the class of all \( p \)-valently starlike functions of order \( \alpha \). Also a function \( f(z) \in T(j, p) \) is said to be \( p \)-valently convex of order \( \alpha \) if it satisfies the inequality

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U; \ 0 \leq \alpha < p; \ p \in \mathbb{N}).
\]

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We denote by \( C_j(p, \alpha) \) the class of all \( p \)-valently convex functions of order \( \alpha \). We note that (see for example Duren [5] and Goodman [6])

\[
f(z) \in C_j(p, \alpha) \iff \frac{zf'(z)}{p} \in T^*_j(p, \alpha) \quad (0 \leq \alpha < p; \ p \in N).
\] (1.4)

The classes \( T^*_j(p, \alpha) \) and \( C_j(p, \alpha) \) were studied by Owa [12].

For each \( f(z) \in T(j, p) \), we have (see [3])

\[
f^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=0}^\infty \frac{k^n}{k!(k+q)!} a_k z^{k-q} \quad (q \in N_0 = N \cup \{0\}; \ p > q).
\] (1.5)

For a function \( f(z) \) in \( T(j, p) \), we define

\[
D^0_p f^{(q)}(z) = f^{(q)}(z),
\]

\[
D^1_p f^{(q)}(z) = Df^{(q)}(z) = \frac{z}{(p-q)} (f^{(q)}(z))^\prime = \frac{z}{(p-q)} f^{(1+q)}(z)
\]

\[
= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=0}^\infty \frac{k^n}{k!(k+q)!} \left( \frac{k-q}{p-q} \right) a_k z^{k-q},
\]

\[
D^2_p f^{(q)}(z) = D(D^1_p f^{(q)}(z))
\]

\[
= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=0}^\infty \frac{k^n}{k!(k+q)!} \left( \frac{k-q}{p-q} \right)^2 a_k z^{k-q},
\]

and

\[
D^n_p f^{(q)}(z) = D(D^{n-1}_p f^{(q)}(z)) \quad (n \in N)
\]

\[
= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=0}^\infty \frac{k^n}{k!(k+q)!} \left( \frac{k-q}{p-q} \right)^n a_k z^{k-q}
\]

\[
(p, j \in N; \ q \in N_0; \ p > q).
\] (1.8)

We note that, by taking \( q = 0 \) and \( p = 1 \), the differential operator \( D^0_1 = D^1 \) was introduced by Salagean [13].

With the help of the differential operator \( D^n_p \), we say that a function \( f(z) \) belonging to \( T(j, p) \) is in the class \( S_j(n, p, q, \alpha) \) if and only if

\[
\text{Re} \left\{ \frac{z(D^p_{p} f^{(q)}(z))^\prime}{D^p_{p} f^{(q)}(z)} \right\} > \alpha \quad (p \in N; \ q, n \in N_0)
\] (1.9)

for some \( \alpha (0 \leq \alpha < p-q, p > q) \) and for all \( z \in U \).

We note that, by specializing the parameters \( j, p, n, q \) and \( \alpha \), we obtain the following subclasses studied by various authors:

(i) \( S_j(0, p, q, \alpha) = S_j(p, q, \alpha) \) and \( S_j(1, p, q, \alpha) = C_j(p, q, \alpha) \) (Chen et al. [3]);

(ii) \( S_j(n, 1, 0, \alpha) = P(j, \alpha, n) \) \( (j \in N; \ n \in N_0; \ 0 \leq \alpha < 1) \) (Aouf and Srivastava [1]).
On certain multivalent functions defined by a differential operator

(iii) $S_1(n, 1, 0, \alpha) = T(n, \alpha) \ (n \in N_0; \ 0 \leq \alpha < 1)$ (Hur and Oh [7]);

(iv) $S_j(0, p, 0, \alpha) = \left\{ \begin{array}{ll}
T_j(p, \alpha) & (p, j \in N; \ 0 \leq \alpha < p), \\
T_{\alpha}(p, j) & (Yamakawa [20])
\end{array} \right.$

(v) $S_j(1, p, 0, \alpha) = \left\{ \begin{array}{ll}
C_j(p, \alpha) & (Owa [12]), \\
CT_{\alpha}(p, j) & (Yamakawa [20])
\end{array} \right.$ (p, j \in N; \ 0 \leq \alpha < p).

(vi) $S_1(0, p, 0, \alpha) = T^*(p, \alpha)$ and $S_1(1, p, 0, \alpha) = C(p, \alpha)$ (Owa [11] and Salagean et al. [14]);

(vii) $S_j(0, 1, 0, \alpha) = T_{\alpha}(j)$ and $S_j(1, 1, 0, \alpha) = C_{\alpha}(j)$ (Srivastava et al. [19])

(viii) $S_j(n, p, 0, \alpha) = S_j(n, p, \alpha)$ (p, j \in N; \ n \in N_0; \ 0 \leq \alpha < p), where $S_j(n, p, \alpha)$ represents the class of functions $f(z) \in T(j, p)$ satisfying the inequality

$$\text{Re} \left\{ \frac{z(D_p^nf(z))'}{D_p^nf(z)} \right\} > \alpha \quad (z \in U). \quad (1.10)$$

In our present paper, we shall make use of the familiar integral operator $J_{c,p}$ defined by (cf. [2], [8] and [9]; see also [18])

$$(J_{c,p}f)(z) = \frac{c + p}{z^c} \int_0^z t^{c-1} f(t) \, dt \quad (1.11)$$

($f \in T(j, p); \ c > -p; \ p \in N$) as well as the fractional calculus operator $D_{z}^{\mu}$ for which it is well known that (see for details [10] and [16]; see also Section 5 below)

$$D_{z}^{\mu}\{z^{\rho}\} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-\mu)} z^{\rho-\mu} \quad (\rho > -1; \ \mu \in R) \quad (1.12)$$

in terms of Gamma functions.

2. Coefficient estimates

THEOREM 1. Let the function $f(z)$ be defined by (1.1). Then $f(z) \in S_j(n, p, q, \alpha)$ if and only if

$$\sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right)^n (k-q-\alpha) \delta(k, q) a_k \leq (p-q-\alpha) \delta(p, q) \quad (2.1)$$

(0 \leq \alpha < p-q; \ p, j \in N; \ q, n \in N_0; \ p > q) where

$$\delta(p, q) = \frac{p!}{(p-q)!} = \left\{ \begin{array}{ll}
p(p-1) \cdots (p-q+1), & q \neq 0, \\
1, & q = 0.
\end{array} \right. \quad (2.2)$$
Proof. Assume that inequality (2.1) holds true. Then we find that

$$ \left| z \left( D^n_p f^{(q)}(z) \right)' \right| - (p - q) \leq \frac{\sum_{k=j+p}^{\infty} (k - p) \left( \frac{k - q}{p - q} \right)^n \delta(k, q) a_k |z|^{k-p} \delta(p, q) - \sum_{k=j+p}^{\infty} \left( \frac{k - q}{p - q} \right)^n \delta(k, q) a_k |z|^{k-p} \delta(p, q) - \sum_{k=j+p}^{\infty} \left( \frac{k - q}{p - q} \right)^n \delta(k, q) a_k}{\sum_{k=j+p}^{\infty} \left( \frac{k - q}{p - q} \right)^n \delta(k, q) a_k} \leq p - q - \alpha. $$

This shows that the values of the function

$$ \Phi(z) = \frac{z \left( D^n_p f^{(q)}(z) \right)'}{D^n_p f^{(q)}(z)} \quad (2.3) $$

lie in a circle which is centered at \( w = (p - q) \) and whose radius is \( (p - q - \alpha) \). Hence \( f(z) \) satisfies the condition (1.9).

Conversely, assume that the function \( f(z) \) is in the class \( S_j(n, p, q, \alpha) \). Then we have

$$ \text{Re} \left\{ \frac{z \left( D^n_p f^{(q)}(z) \right)'}{D^n_p f^{(q)}(z)} \right\} = \text{Re} \left\{ \frac{(p - q)\delta(p, q) - \sum_{k=j+p}^{\infty} \left( \frac{k - q}{p - q} \right)^n \delta(k, q) a_k z^{k-p}}{\delta(p, q) - \sum_{k=j+p}^{\infty} \left( \frac{k - q}{p - q} \right)^n \delta(k, q) a_k z^{k-p}} \right\} > \alpha, \quad (2.4) $$

for some \( \alpha \) (0 \( \leq \alpha < p - q) \), \( p, j \in N \), \( q, n \in N_0 \), \( p > q \) and \( z \in U \). Choose values of \( z \) on the real axis so that \( \Phi(z) \) given by (2.3) is real. Upon clearing the denominator in (2.4) and letting \( z \to 1^- \) through real values, we can see that

$$ (p - q)\delta(p, q) - \sum_{k=j+p}^{\infty} \left( \frac{k - q}{p - q} \right)^n \delta(k, q) a_k \geq \alpha \left\{ \delta(p, q) - \sum_{k=j+p}^{\infty} \left( \frac{k - q}{p - q} \right)^n \delta(k, q) a_k \right\}. \quad (2.5) $$

Thus we have the inequality (2.1). \( \blacksquare \)

Corollary 1. Let the function \( f(z) \) defined by (1.1) be in the class \( S_j(n, p, q, \alpha) \). Then

$$ a_k \leq \frac{(p - q - \alpha)\delta(p, q)}{\left( \frac{k - q}{p - q} \right)^n \delta(k, q)} \quad (k \geq j + p; \ p, j \in N; \ q, n \in N_0; \ p > q). \quad (2.6) $$
The result is sharp for the function \( f(z) \) given by
\[
f(z) = z^p - \frac{(p - q - \alpha)\delta(p, q)}{(k - q)^n(k - q - \alpha)\delta(k, q)} z^k
\]  
(\( k \geq j + p; \ p, j \in N; \ q, n \in N_0; \ p > q \)).

**Remark 1.** (i) Putting \( n = 0 \) in Theorem 1, we obtain the result obtained by Chen et al. [3, Theorem 1].

(ii) Putting \( n = 1 \) in Theorem 1, we obtain the result obtained by Chen et al. [3, Theorem 2].

### 3. Distortion theorem

**Theorem 3.** If the function \( f(z) \) defined by (1.1) is in the class \( S_j(n, p, q, \alpha) \), then
\[
\left\{ \frac{p!}{(p - m)!} - \frac{(p - q - \alpha)\delta(p, q)(j + p - q)!}{(j + p - q - \alpha)\delta(j + p, q)!} \right\} |z|^m \leq |f^{(m)}(z)| \leq \left\{ \frac{p!}{(p - m)!} + \frac{(p - q - \alpha)\delta(p, q)(j + p - q)!}{(j + p - q - \alpha)\delta(j + p, q)!} \right\} |z|^m
\]  
(\( z \in U; \ 0 \leq \alpha < p - q; \ p, j \in N; \ q, n, m \in N_0; \ p > \max\{q, m\} \)). The result is sharp for the function \( f(z) \) given by
\[
f(z) = z^p - \frac{(p - q - \alpha)\delta(p, q)}{(j + p - q - \alpha)\delta(j + p, q)} z^{j + p}
\]  
(\( p, j \in N; \ q, n \in N_0; \ p > q \)).

**Proof.** In view of Theorem 1, we have
\[
\frac{(j + p - q - \alpha)\delta(j + p, q)}{(p - q - \alpha)\delta(p, q)} \sum_{k=j+p}^{\infty} k!a_k \leq \sum_{k=j+p}^{\infty} \frac{(k - q - \alpha)\delta(k, q)}{(p - q - \alpha)\delta(p, q)} a_k \leq 1
\]  
which readily yields
\[
\sum_{k=j+p}^{\infty} k!a_k \leq \frac{(p - q - \alpha)\delta(p, q)(j + p - q)!}{(j + p - q - \alpha)\delta(j + p, q)}.
\]  
(3.3)

Now, by differentiating both sides of (1.1) \( m \) times, we have
\[
f^{(m)}(z) = \frac{p!}{(p - m)!} z^{p - m} - \sum_{k=j+p}^{\infty} \frac{k!}{(k - m)!} a_k z^{k - m}
\]  
(3.4)

(\( k \geq j + p; \ p, j \in N; \ q, m \in N_0; \ p > \max\{q, m\} \)) and Theorem 2 follows from (3.3) and (3.4).
**Remark 2.** (i) Putting \( n = 0 \) in Theorem 2, we obtain the result obtained by Chen et al. [3, Theorem 7].

(ii) Putting \( n = 1 \) in Theorem 2, we obtain the result obtained by Chen et al. [3, Theorem 8].

4. Radii of close-to-convexity, starlikeness and convexity

**Theorem 3.** Let the function \( f(z) \) defined by (1.1) be in the class \( S_j(n, p, q, \alpha) \). Then

(i) \( f(z) \) is \( p \)-valently close-to-convex of order \( \varphi \) \( (0 \leq \varphi < p) \) in \( |z| < r_1 \), where

\[
    r_1 = \inf_k \left\{ \left( \frac{k-q}{p-q} \right)^n(k-q-\alpha)\delta(k, q) \left( \frac{p-\varphi}{k} \right)^{\frac{1}{p}} \right\}^{\frac{1}{k-p}} \quad (4.1)
\]

\( (k \geq j + p; \, p, j \in \mathbb{N}; \, q, n \in \mathbb{N}_0; \, p > q) \).

(ii) \( f(z) \) is \( p \)-valently starlike of order \( \varphi \) \( (0 \leq \varphi < p) \) in \( |z| < r_2 \), where

\[
    r_2 = \inf_k \left\{ \left( \frac{k-q}{p-q} \right)^n(k-q-\alpha)\delta(k, q) \left( \frac{p-\varphi}{k-\varphi} \right)^{\frac{1}{p}} \right\}^{\frac{1}{k-p}} \quad (4.2)
\]

\( (k \geq j + p; \, p, j \in \mathbb{N}; \, q, n \in \mathbb{N}_0; \, p > q) \).

(iii) \( f(z) \) is \( p \)-valently convex of order \( \varphi \) \( (0 \leq \varphi < p) \) in \( |z| < r_3 \), where

\[
    r_3 = \inf_k \left\{ \left( \frac{k-q}{p-q} \right)^n(k-q-\alpha)\delta(k, q) \left( \frac{p-\varphi}{k(k-\varphi)} \right)^{\frac{1}{p}} \right\}^{\frac{1}{k-p}} \quad (4.3)
\]

\( (k \geq j + p; \, p, j \in \mathbb{N}; \, q, n \in \mathbb{N}_0; \, p > q) \). Each of these results is sharp for the function \( f(z) \) given by (2.7).

**Proof.** It is sufficient to show that

\[
    \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \varphi \quad (|z| < r_1; \, 0 \leq \varphi < p; \, p \in \mathbb{N}) \quad (4.4)
\]

\[
    \left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \varphi \quad (|z| < r_2; \, 0 \leq \varphi < p; \, p \in \mathbb{N}) \quad (4.5)
\]

and that

\[
    \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p - \varphi \quad (|z| < r_3; \, 0 \leq \varphi < p; \, p \in \mathbb{N}) \quad (4.6)
\]

for a function \( f(z) \in S_j(n, p, q, \alpha) \), where \( r_1, r_2 \) and \( r_3 \) are defined by (4.1), (4.2) and (4.3), respectively. The details involved are fairly straightforward and may be omitted. ■
5. Modified Hadamard products

For the functions

\[ f_\nu(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,\nu} z^k \quad (a_{k,\nu} \geq 0; \ \nu = 1, 2) \]  

we denote by \((f_1 \ast f_2)(z)\) the modified Hadamard product (or convolution) of the functions \(f_1(z)\) and \(f_2(z)\), where

\[ (f_1 \ast f_2)(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,1} \cdot a_{k,2} z^k. \]  

**Theorem 4.** Let the functions \(f_\nu(z)\) \((\nu = 1, 2)\) defined by (5.1) be in the class \(S_j(n, p, q, \alpha)\). Then \((f_1 \ast f_2)(z) \in S_j(n, p, q, \gamma)\), where

\[ \gamma = (p - q) - \frac{j(p - q - \alpha)^2 \delta(p, q)}{(\frac{j^2 + p - q}{p - q})^n (j + p - q - \alpha)^2 \delta(j + p, q) - (p - q - \alpha)^2 \delta(p, q)}. \]  

The result is sharp for the functions \(f_\nu(z)\) \((\nu = 1, 2)\) given by

\[ f_\nu(z) = z^p - \frac{(p - q - \alpha)^2 \delta(p, q)}{(\frac{j^2 + p - q}{p - q})^n (j + p - q - \alpha)^2 \delta(j + p, q)} z^{j+p} \]  

\((p, j \in N; \ q, n \in N_0; \ p > q; \ \nu = 1, 2)\).

**Proof.** Employing the technique used earlier by Schild and Silverman [15], we need to find the largest \(\gamma\) such that

\[ \sum_{k=j+p}^{\infty} \frac{(k-q-\gamma)^n (p-q-\gamma)^{\delta(k, q)}}{(p-q-\gamma)^{\delta(p, q)}} a_{k,1} \cdot a_{k,2} \leq 1 \]  

\((f_\nu(z) \in S_j(n, p, q, \alpha), \ \nu = 1, 2)\). Since \(f_\nu(z) \in S_j(n, p, q, \alpha)\) \((\nu = 1, 2)\), we readily see that

\[ \sum_{k=j+p}^{\infty} \frac{(k-q-\gamma)^n (p-q-\gamma)^{\delta(k, q)}}{(p-q-\gamma)^{\delta(p, q)}} a_{k,\nu} \leq 1 \quad (\nu = 1, 2). \]  

Therefore, by the Cauchy-Schwarz inequality, we obtain

\[ \sum_{k=j+p}^{\infty} \frac{(k-q-\gamma)^n (k-q-\gamma)^{\delta(k, q)}}{(p-q-\gamma)^{\delta(p, q)}} \sqrt{a_{k,1} \cdot a_{k,2}} \leq 1. \]  

Thus we only need to show that

\[ \frac{(k-q-\gamma)}{(p-q-\gamma)} a_{k,1} \cdot a_{k,2} \leq \frac{(k-q-\alpha)}{(p-q-\alpha)} \sqrt{a_{k,1} \cdot a_{k,2}} \]  

\((k \geq j+p; \ p, j \in N)\), or, equivalently, that

\[ \sqrt{a_{k,1} \cdot a_{k,2}} \leq \frac{(p-q-\gamma)(k-q-\alpha)}{(p-q-\alpha)(k-q-\gamma)} \]  

(5.9)
\((k \geq j + p; \ p, j \in \mathbb{N})\). Hence, in the light of inequality (5.7), it is sufficient to prove that
\[
\frac{(p - q - \alpha)\delta(p, q)}{(\frac{k - q}{p - q})^n(k - q - \alpha)\delta(k, q)} \leq \frac{(p - q - \gamma)(k - q - \alpha)}{(p - q - \alpha)(k - q - \gamma)}
\]  
(5.10)

\((k \geq j + p; \ p, j \in \mathbb{N})\). It follows from (5.10) that
\[
\gamma \leq (p - q) - \frac{(k - p)(p - q - \alpha)^2\delta(p, q)}{(\frac{k - q}{p - q})^n(k - q - \alpha)^2\delta(k, q) - (p - q - \alpha)^2\delta(p, q)}
\]  
(5.11)

\((k \geq j + p; \ p, j \in \mathbb{N})\).

Now, defining the function \(G(k)\) by
\[
G(k) = (p - q) - \frac{(k - p)(p - q - \alpha)^2\delta(p, q)}{(\frac{k - q}{p - q})^n(k - q - \alpha)^2\delta(k, q) - (p - q - \alpha)^2\delta(p, q)}
\]  
(5.12)

\((k \geq j + p; \ p, j \in \mathbb{N})\), we see that \(G(k)\) is an increasing function of \(k\). Therefore, we conclude that
\[
\gamma \leq G(j + p) = (p - q) - \frac{j(p - q - \alpha)^2\delta(p, q)}{(\frac{j + p - q}{p - q})^n(j + p - q - \alpha)^2\delta(j + p, q) - (p - q - \alpha)^2\delta(p, q)}
\]  
(5.13)

which evidently completes the proof of Theorem 4. \(\blacksquare\)

Putting \(n = 0\) and \(n = 1\) in Theorem 4, we obtain

**Corollary 2.** Let the functions \(f_\nu(z)\) \((\nu = 1, 2)\) defined by (5.1) be in the class \(S_j(p, q, \alpha)\). Then \((f_1 \oplus f_2)(z) \in S_j(p, q, \gamma)\), where
\[
\gamma = (p - q) - \frac{j(p - q - \alpha)^2\delta(p, q)}{(j + p - q - \alpha)^2\delta(j + p, q) - (p - q - \alpha)^2\delta(p, q)}.
\]  
(5.14)

The result is sharp.

**Remark 3.** We note that the result obtained by Chen et al. [3, Theorem 5] is not correct. The correct result is given by (5.14).

**Corollary 3.** Let the functions \(f_\nu(z)(\nu = 1, 2)\) defined by (5.1) be in the class \(C_j(p, q, \alpha)\). Then \((f_1 \oplus f_2)(z) \in C_j(p, q, \gamma)\), where
\[
\gamma = (p - q) - \frac{j(p - q - \alpha)^2\delta(p, q + 1)}{(j + p - q - \alpha)^2\delta(j + p, q + 1) - (p - q - \alpha)^2\delta(p, q + 1)}.
\]  
(5.15)

The result is sharp.

**Remark 4.** We note that the result obtained by Chen et al. [3, Theorem 6] is not correct. The correct result is given by (5.15).

Using arguments similar to those in the proof of Theorem 4, we obtain the following results.
Theorem 5. Let the functions $f_1(z)$, resp. $f_2(z)$ defined by (5.1) be in the class $S_j(n,p,q,\alpha)$, resp. $S_j(n,p,q,\tau)$. Then $(f_1 \circ f_2)(z) \in S_j(n,p,q,\zeta)$, where
\[
\zeta = (p - q) - \frac{j(p - q - \alpha)(p - q - \tau) \delta(p, q)}{(j + p - q - \omega)(j + p - q - \omega) \delta(p, q) - \Omega \delta(p, q)},
\]
where \[
\Omega = (p - q - \alpha)(p - q - \tau).
\]
The result is the best possible for the functions
\[
f_1(z) = z^p - \frac{(p - q - \alpha) \delta(p, q)}{(j + p - q - \omega) \delta(p, q)} z^{j + p} \quad (p, j \in N; q, n \in N_0; p > q)
\]
(5.18)
\[
f_2(z) = z^p - \frac{(p - q - \tau) \delta(p, q)}{(j + p - q - \omega) \delta(p, q)} z^{j + p} \quad (p, j \in N; q, n \in N_0; p > q).
\]
(5.19)

Theorem 6. Let the functions $f_\nu(z)$ ($\nu = 1, 2$) defined by (5.1) be in the class $S_j(n,p,q,\alpha)$. Then the function
\[
h(z) = z^p - \sum_{k=j+p}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k
\]
(5.20)
belongs to the class $S_j(n,p,q,\xi)$, where
\[
\xi = (p - q) - \frac{2j(p - q - \alpha)^2 \delta(p, q)}{(j + p - q - \omega)^2 \delta(p, q) - 2(p - q - \omega)^2 \delta(p, q)}.
\]
(5.21)
The result is the sharp for the functions $f_\nu(z)$ ($\nu = 1, 2$) defined by (5.4).

6. Applications of fractional calculus

Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in the literature rather extensively (cf., e.g., [4], [10], [17] and [18]; see also the various references cited therein). For our present investigation, we recall the following definitions.

Definition 1. The fractional integral of order $\mu$ is defined, for a function $f(z)$, by
\[
D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{1-\mu}} d\zeta \quad (\mu > 0),
\]
(6.1)
where the function $f(z)$ is analytic in a simply-connected domain of the complex $z$-plane containing the origin and the multiplicity of $(z - \zeta)^{-1}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

Definition 2. The fractional derivative of order $\mu$ is defined, for a function $f(z)$, by
\[
D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{\mu}} d\zeta \quad (0 \leq \mu < 1),
\]
(6.2)
where the function $f(z)$ is constrained, and the multiplicity of $(z - \zeta)^{-\mu}$ is removed, as in Definition 1.
**Definition 3.** Under the hypotheses of Definition 2, the fractional derivative of order \( n + \mu \) is defined, for a function \( f(z) \), by

\[
D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} \{ D_z^\mu f(z) \} \quad (0 \leq \mu < 1; \ n \in \mathbb{N}_0).
\]  

(6.3)

In this section, we shall investigate the growth and distortion properties of functions in the class \( S_j(n,p,q,\alpha) \), involving the operators \( J_{c,p} \) and \( D_z^\mu \). In order to derive our results, we need the following lemma given by Chen et al. [4].

**Lemma 1.** [4] Let the function \( f(z) \) be defined by (1.1). Then

\[
D_z^\mu \{(J_{c,p}f)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} z^{p-\mu} - \sum_{k=j+p}^{\infty} \frac{(c+p)\Gamma(p+1)}{(c+k-\mu)\Gamma(p+1-\mu)} a_k z^{k-\mu}
\]  

(6.4)

(\( \mu \in \mathbb{R}; \ c > -p; \ p, j \in \mathbb{N} \)) and

\[
J_{c,p}(D_z^\mu \{f(z)\}) = \frac{(c+p)\Gamma(p+1)}{(c+p-\mu)\Gamma(p+1-\mu)} z^{p-\mu} - \sum_{k=j+p}^{\infty} \frac{(c+p)\Gamma(k+1)}{(c+k-\mu)\Gamma(k+1-\mu)} a_k z^{k-\mu}
\]  

(6.5)

(\( \mu \in \mathbb{R}; \ c > -p; \ p, j \in \mathbb{N} \)), provided that no zeros appear in the denominators in (6.4) and (6.5).

**Theorem 7.** Let the function \( f(z) \) defined by (1.1) be in the class \( S_j(n,p,q,\alpha) \). Then

\[
|D_z^{-\mu} \{(J_{c,p}f)(z)\}| \geq \left\{ \frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} - \frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)(\frac{p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)} |z|^j \right\} |z|^{p+\mu} \]  

(6.6)

(\( z \in U; \ 0 \leq \alpha < p-q; \ \mu > 0; \ c > -p; \ p, j \in \mathbb{N}; \ q, n \in \mathbb{N}_0; \ p > q \)) and

\[
|D_z^{-\mu} \{(J_{c,p}f)(z)\}| \leq \left\{ \frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} + \frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)(\frac{p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)} |z|^j \right\} |z|^{p+\mu}
\]  

(6.7)

(\( z \in U; \ 0 \leq \alpha < p-q; \ \mu > 0; \ c > -p; \ p, j \in \mathbb{N}; \ q, n \in \mathbb{N}_0; \ p > q \)). Each of the assertions (6.6) and (6.7) is sharp.

**Proof.** In view of Theorem 1, we have

\[
\frac{(\frac{p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)}{(p-q-\alpha)\delta(p,q)} \sum_{k=j+p}^{\infty} a_k \leq \sum_{k=j+p}^{\infty} \frac{(\frac{k-q}{p-q})^n(k-q-\alpha)\delta(k,q)}{(p-q-\alpha)\delta(p,q)} a_k \leq 1,
\]  

(6.8)
which readily yields

$$\sum_{k=j+p}^{\infty} a_k \leq \frac{(p-q-\alpha)\delta(p,q)}{(j+p-q-\alpha)\delta(j+p,q)}.$$  \hfill (6.9)

Consider the function $F(z)$ defined in $U$ by

$$F(z) = \frac{\Gamma(p+1+\mu)}{\Gamma(p+1)} z^{-\mu} D_{-z}^{-\mu} \{ (J_{c,p} f)(z) \}$$

$$= z^p - \sum_{k=j+p}^{\infty} \frac{(c+p)\Gamma(k+1+\mu)\Gamma(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)\Gamma(p+1)} a_k z^k$$

$$= z^p - \sum_{k=j+p}^{\infty} \Phi(k) a_k z^k \quad (z \in U)$$

where

$$\Phi(k) = \frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)\Gamma(p+1)} \quad (k \geq j+p; \; p, j \in \mathbb{N}; \; \mu > 0).$$  \hfill (6.10)

Since $\Phi(k)$ is a decreasing function of $k$ when $\mu > 0$, we get

$$0 < \Phi(k) \leq \Phi(j+p) = \frac{(c+p)\Gamma(j+p+1)\Gamma(p+1+\mu)}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)}$$  \hfill (6.11)

$(c > -p; \; p, j \in \mathbb{N}; \; \mu > 0)$. Thus, by using (6.9) and (6.11), we deduce that

$$|F(z)| \geq |z|^p - \Phi(j+p) |z|^{j+p} \sum_{k=j+p}^{\infty} a_k \geq |z|^p -$$

$$\frac{(c+p)\Gamma(j+p+1)\Gamma(p+1+\mu)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)} |z|^{j+p}$$

$(z \in U)$ and

$$|F(z)| \leq |z|^p + \Phi(j+p) |z|^{j+p} \sum_{k=j+p}^{\infty} a_k \leq |z|^p +$$

$$\frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)} |z|^{j+p}$$

$(z \in U)$, which yield the inequalities (6.6) and (6.7) of Theorem 7. The equalities in (6.6) and (6.7) are attained for the function $f(z)$ given by

$$D_{-z}^{-\mu} \{ (J_{c,p} f)(z) \} = \left\{ \begin{array}{l}
\frac{\Gamma(p+1+\mu)}{\Gamma(p+1)} - \\
\frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)} z^j
\end{array} \right\} z^{p+\mu}$$  \hfill (6.12)

or, equivalently, by

$$(J_{c,p} f)(z) = z^p - \frac{(c+p)(p-q-\alpha)\delta(p,q)}{(c+j+p)(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)} z^{j+p}.$$  \hfill (6.13)

Thus we complete the proof of Theorem 7.
Using arguments similar to those in the proof of Theorem 7, we obtain the following result.

**Theorem 8.** Let the function \( f(z) \) defined by (1.1) be in the class \( S_j(n, p, q, \alpha) \). Then

\[
|D^p_z \{(J_{c,p}f)(z)\}| \geq \left\{ \frac{\Gamma(p + 1)}{\Gamma(p + 1 - \mu)} - \frac{(c + p)\Gamma(j + p + 1)(p - q - \alpha)\delta(p, q)}{(c + j + p)\Gamma(j + p + 1 - \mu)(\frac{zp - q}{p - q})^n(j + p - q - \alpha)\delta(j + p, q)} |z|^j \right\} |z|^{p-\mu}
\]

(1.14)

\[
|D^p_z \{(J_{c,p}f)(z)\}| \leq \left\{ \frac{\Gamma(p + 1)}{\Gamma(p + 1 - \mu)} + \frac{(c + p)\Gamma(j + p + 1)(p - q - \alpha)\delta(p, q)}{(c + j + p)\Gamma(j + p + 1 - \mu)(\frac{zp - q}{p - q})^n(j + p - q - \alpha)\delta(j + p, q)} |z|^j \right\} |z|^{p-\mu}
\]

(1.15)

\((z \in U; \quad 0 \leq \alpha < p - q; \quad 0 \leq \mu < 1; \quad c > -p; \quad p, j \in N; \quad q, n \in N_0; \quad p > q)\) and

Each of the assertions (1.14), and (1.15) is sharp.

**Remark 5.** Putting \( n = 0 \) and \( n = 1 \) in Theorem 7 and Theorem 8, we obtain the corresponding results for the classes \( S_j(p, q, \alpha) \) and \( C_j(p, q, \alpha) \), respectively.

**References**


On certain multivalent functions defined by a differential operator


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