Level Crossing Rate of Phase Noise IM-DD Optical Systems in Presence of Interference

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Abstract: In this paper the joint probability density function of the envelope and the first time derivative of envelope, and phase and first time derivative of phase for signals in IM-DD optical systems are derived. The level crossing rate and the average fade duration for envelope and phase of sine wave plus narrow band Gaussian noise are given. The envelope of signal and its derivative are independent. The first time derivative of the signals has Gaussian distribution.

Keywords: Gaussian noise, diversity system, level crossing rate, IM-DD system, probability density function, average fade duration.

1 Introduction

Joint probability density function of an envelope and its first time derivative and phase and its first time derivative of a sum of sine wave and narrow band Gaussian noise are derived. A pioneering work in this matter was given by Rice in his classical paper [1]. In this paper the level crossing rate and average average fade duration of the envelope, phase and frequency of signal plus narrow band Gaussian random process is obtained. Statistical characterization of the received signal envelope in terms of the average level crossing rate and average time fade duration in IM-DD optical systems is useful in design of these systems and the analysis of its performance. For instance, the level crossing rate and average fade duration provide information about the statistic of burst error which the design and selection of error correction technique [2–4].

The level crossing rates and the average fade durations are two quantities which statistically characterize a communication channel. The level crossing rate is defined as the number of times per unit duration that the envelope of a fading channel
crosses a given value in the negative direction [5]. The average fade duration corresponds to the average length of time the envelope remains under this value once it crosses it in the negative direction. These quantities reflect the correlation properties and thus the second order statistics of a fading channel. They provide a dynamic representation of the channel. They complement the probability density function and cumulative distribution function, which are first order statistics and can only be used to obtain static metrics associated the channel, such as the bit error rate. The level crossing rate and average fade duration have found a variety of applications in the modelling and design of communication systems, such as the finite-state Markow modelling of channels [6], the analysis of handoff algorithms [7] and the estimation of packet error rates [8]. In digital systems, for signals below a given threshold burst error occur. Level crossing rate and average fade duration are important second order statistical quantities which have been extensively explored in the literature. The aim of this paper is to evaluate, in an exact manner, the LCR and AFD for envelope, phase and the first time derivative of the phase of sum of sine wave and narrow band Gaussian noise.

2 The Narrow Band Gaussian noise

A random process is said to be a narrow band random process of the width of the significant region of its spectral density is small compared to the center frequency \( f_c \) of that region. A sample function of such a random process can be expressed in the form

\[
n(t) = x(t) \cos \omega t + y(t) \sin \omega t,
\]

where \( x(t) \) and \( y(t) \) are Gaussian random variables. Their means are zero and variances are \( \sigma^2 \). The random process \( x(t) \) and \( y(t) \) are slowly varying functions of time. The frequency components of \( x(t) \) and \( y(t) \) are therefore confined to a narrow band about zero frequency. The derivations of Gaussian random variables are Gaussian random variables.

The covariance matrix of the random variables \( x(t) \) and \( y(t) \) and it’s the first derivative \( \dot{x}(t) \) and \( \dot{y}(t) \), is

\[
\begin{bmatrix}
1 & 0 & 0 & R_s'(0) \\
0 & 1 & -R_s'(0) & 0 \\
0 & -R_c'(0) & R_c''(0) & 0 \\
R_s'(0) & 0 & 0 & -R_c''(0)
\end{bmatrix}
\]

where is
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\[ -R_s'(0) = \frac{\int (\omega - \omega_0)|H(\omega)|^2 d\omega}{\int |H(\omega)|^2 d\omega} = a \]
\[ -R_c'(0) = \frac{\int (\omega - \omega_0)|H(\omega)|^2 d\omega}{\int |H(\omega)|^2 d\omega} = b \]

The joint probability density function of \( x, y, \dot{x} \) and \( \dot{y} \) is

\[ P_{xy\dot{x}\dot{y}}(x\dot{x}y\dot{y}) = \frac{1}{(2\pi)^2(b-a^2)} e^{-\frac{1}{8\sigma^2(b-a^2)} [b(x^2+y^2)+(x^2+y^2)-2a(xy-\dot{y}\dot{x})]} \]

where

\[ D = (b-a^2)^2 \]

If \( a = 0 \), then \( x, y, \dot{x} \) and \( \dot{y} \) are uncorrelated and also statistically independent.

3 The Amplitude of Signal is Constant

We shall derive expression for the joint probability density functions of the envelope, phase and its time derivatives of the sum of a sine wave and a narrow band Gaussian random process

\[ z = A \cos(\omega t) + x(t) \cos(\omega t) + y(t) \sin(\omega t) = r(t) \cos[(\omega t) + \varphi(t)] \]

where \( A \) is a constant. The random variables \( x(t) \) and \( y(t) \) are independent Gaussian random variables with zero means and variances \( \sigma^2 \). Random variables \( r(t) \) and \( \varphi(t) \) are envelope and phase of the random process \( z \), respectively. It then follows from (5)

\[ x + A = r \cos \phi \]
\[ y = r \sin \phi \]

By differentiating \( x \) and \( y \), with respect to \( r \) and \( \varphi \), we can obtain

\[ \dot{x} = \dot{r} \cos \varphi - r \dot{\varphi} \sin \varphi \]
\[ \dot{y} = \dot{r} \sin \varphi + r \dot{\varphi} \cos \varphi \]
that of $x$, $y$, $\dot{x}$ and $\dot{y}$. From (6) and (7) the Jacobian of the transformation from $x$, $\dot{x}$, $y$ and $\dot{y}$ to $r$, $\dot{r}$, $\varphi$ and $\dot{\varphi}$ is

$$J = \frac{\partial(x, \dot{x}, y, \dot{y})}{\partial(r, \dot{r}, \varphi, \dot{\varphi})} = r^2$$  

(8)

It therefore follows from (6), (7) and (8) that

$$p_{rr\varphi\dot{r}\dot{\varphi}}(r\dot{r}\varphi\dot{\varphi}) = \frac{r^2}{4\pi^2\sigma^4(b - a^2)}e^{-\frac{b(r^2 - 2a\cos\varphi + A^2) + r^2 + 2\sigma^2\varphi^2 - 2a^2 + 2a(r\cos\varphi + r\varphi\cos\varphi)}{2\sigma^2(b - a^2)}}$$  

(9)

We can now determine the joint probability density function of $r$ and its first time derivative $\dot{r}$ by integrating this result over $\varphi$ and $\dot{\varphi}$. Thus

$$P_{rr}(r\dot{r}) = \frac{r^2}{4\pi^2\sigma^4b}e^{-\frac{1}{2\sigma^2}(r^2 + A^2 + \frac{\dot{r}^2}{b})} \int_{-\infty}^{\pi} \int_{-\infty}^{\pi} e^{-\frac{2\varphi^2}{2\sigma^2b}} \frac{A\cos\varphi}{\sigma^2} d\varphi d\dot{\varphi}$$

(10)

By integrating the expression (9), over $r$ and $\dot{r}$, the joint probability density function of the phase and its first time derivative of the sum of a sine wave and narrow band Gaussian random noise can be obtained in the form

$$p_{\varphi\dot{\varphi}}(\varphi\dot{\varphi}) = \frac{1}{\sqrt{2\pi\sigma^2}^3b}e^{-\frac{\dot{\varphi}^2}{2\sigma^2}} \int_0^{\infty} r^2 e^{-\frac{r^2}{2\sigma^2}(1 + \frac{\dot{\varphi}^2}{b}) + \frac{A\sigma}{\sigma^2}\cos\varphi} dr$$  

(11)

By integrating the expression (11) over $\varphi$, the probability density function of the frequency of the sum of a sine wave and narrow band Gaussian random noise can be obtained in the form

$$p_{\phi}(\phi) = \frac{1}{\sqrt{2\pi\sigma^2}^3b} \int_0^{\infty} r^2 I_0\left(\frac{A\sigma}{\sigma^2}\right) e^{-\frac{r^2}{2\sigma^2}(1 + \frac{\dot{\varphi}^2}{b})} dr$$

(12)

$$= \frac{1}{\sqrt{b\varphi^2}} F_1\left(\frac{3}{2}, 1, \frac{A^2}{2\sigma^2V}\right) e^{-\frac{\phi^2}{2\sigma^2}}$$

where

$$V = 1 + \frac{\dot{\varphi}^2}{b}$$
The envelope of the sum of sine wave and narrow band Gaussian noise is

\[ r = \sqrt{(x - A)^2 + y^2} \]  

(13)

From (13), we can obtain

\[ y = \frac{\sqrt{r^2 - (x - A)^2}}{\sqrt{(x - A)^2 + y^2}} \]  

(14)

By differentiating \( r \), with respect to \( x \) and \( y \), can be obtained

\[ \dot{r} = \frac{(x - A)\dot{x} + y\dot{y}}{\sqrt{(x - A)^2 + y^2}} \]  

(15)

By differentiating \( y \), with respect to \( r \) and \( x \), can be obtained

\[ \dot{y} = \frac{r\dot{r} - (x - A)\dot{x}}{\sqrt{r^2 - (x - A)^2}} \]  

(16)

If \( x, \dot{x}, y, \) and \( \dot{y} \) are independent Gaussian random variables, their joint probability density function then is

\[ P_{xy\dot{y}}(x\dot{x}y) = \frac{1}{2\pi b\sigma^4} e^{-\frac{b[(x^2 + y^2) + (\dot{x}^2 + \dot{y}^2)]}{2\sigma^4b}} \]  

(17)

The conditional joint probability density for \( r \) and \( \dot{r} \) is

\[ p_{r\dot{r}}(r\dot{r}/x\dot{x}) = Jp_{y\dot{y}}\left(\sqrt{r^2 - (x - A)^2}, \frac{r\dot{r} - (x - A)\dot{x}}{\sqrt{r^2 - (x - A)^2}}\right) \]  

(18)

where \( J \) is the Jacobian transformation from \( y \) and \( \dot{y} \) to \( r \) and \( \dot{r} \).

By averaging the expression (18) with respect to \( x \) and \( \dot{x} \), the joint probability density function for envelope \( r \) and its the first time derivative can be obtained as.

\[ p_{rr}(r\dot{r}) = \int dx \int d\dot{x} Jp_{x\dot{xy}}(x, \dot{x}, \sqrt{r^2 - (x - A)^2}, \frac{r\dot{r} - (x - A)\dot{x}}{\sqrt{r^2 - (x - A)^2}}) \]  

(19)

The phase of the sum of a sine wave and narrow band Gaussian random process is

\[ \varphi = \arctan \frac{y}{x - A} \]  

(20)

Random variable \( y \) can be obtained from (20) in the form

\[ y = (x - A)\tan \varphi \]  

(21)
By differentiating \( \varphi \), with respect to \( x \) and \( y \), can be given

\[
\dot{\varphi} = \frac{\dot{y}(x-A) - \dot{x}y}{(x-A)^2 + y^2}
\]  

(22)

By differentiating \( y \), with respect to \( \varphi \) and \( x \), can be given

\[
\dot{y} = \dot{x} \tan \varphi + (x-A)\varphi
\]  

(23)

The conditional joint probability density function of \( \varphi \) and \( \dot{\varphi} \), with respect \( x \) and \( \dot{x} \), is

\[
p_{\varphi\dot{\varphi}}(\varphi\dot{\varphi}/x\dot{x}) = J p_{y\dot{y}} \left( (x-A)\tan \varphi, \frac{\dot{y}(x-A) - \dot{x}y}{(x-A)^2 + y^2} \right)
\]  

(24)

By averaging the expression (24), with respect to \( x \) and \( \dot{x} \), the joint probability density function for phase \( \varphi \) and it’s the first time derivative of the sum of a sine wave and narrow band Gaussian noise can be obtained as

\[
p_{\varphi\dot{\varphi}}(\varphi\dot{\varphi}) = \int dx \int d\dot{x} J p_{x\dot{x}y}(x, \dot{x}, (x-A)\tan \varphi, \frac{\dot{y}(x-A) - \dot{x}y}{(x-A)^2 + y^2})
\]  

(25)

The average level crossing rate of envelope level, \( r \), is defined as the rate at which a signal envelope crosses level \( R \) in a positive or negative going direction. The average level crossing of the envelope \( r \), is

\[
N_r = \int_0^\infty r p_r(r\dot{r}) \, dr
\]

\[
= \int_0^\infty \frac{r}{\sigma^3 b \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(r^2+\dot{A}^2+\dot{b}^2)} I_0 \left( \frac{Ar}{\sigma^2} \right) \, dr
\]

(26)

\[
= \frac{r}{\sigma^3 b \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(r^2+\dot{A}^2)} I_0 \left( \frac{Ar}{\sigma^2} \right) \int_0^\infty \dot{r} e^{-\frac{\dot{r}^2}{2\sigma^2} b} \, d\dot{r}
\]

The phase level crossing rate of phase level, \( \varphi \), is defined as the rate at which a signal phase crosses level \( \varphi \) in a positive going direction. The average level crossing of the phase \( \varphi \), is

\[
N_\varphi = \int_0^\infty \dot{\varphi} p_{\varphi\dot{\varphi}}(\varphi\dot{\varphi}) \, d\varphi
\]

\[
= \int_0^\infty \frac{1}{\sqrt{(2\pi\sigma^2)^3} b} e^{-\frac{\dot{\varphi}^2}{2\sigma^2}} \int_0^\infty r^2 e^{-\frac{r^2}{2\sigma^2} (1+\dot{b}^2)+\frac{\dot{A}^2}{\sigma^2}} \cos \varphi \, dV
\]  

(27)
The average fade duration for envelope is defined as the envelope time that the envelope remains below a specified level offer crossing that level in a downward direction and is given by

$$AFD_r = \frac{P_0}{\int_0^\infty V p_{VV}(V) dV}$$

where

$$P_0 = \int_0^{V_r} p_V(V) dV$$

The phase fade duration for phase is defined as the average time that the phase remains below a specified level after crossing that level in a downward direction and is given by

$$AFD_\phi = \frac{P_0}{\int_0^\infty \phi p_{\phi\phi}(\phi) d\phi}$$

The joint probability density function of envelope $r$ and its the first time derivative and its the second time derivative, when the amplitude of signal is constant, is

$$p_{VV\dot{V}}(V \dot{V}) = \frac{V}{\sigma^3 b \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(V^2 + \dot{V}^2)} I_0(\frac{AV}{\sigma^2}) d\dot{V} \frac{1}{\sqrt{2\pi \beta_2}} e^{-\frac{\dot{V}^2}{2\beta_2}}$$

The joint probability density function of phase $\phi$ and its the first time derivative and its the second time derivative of the sum a sine wave out narrow band Gaussian noise, when the amplitude of signal is known, is

$$p_{\phi\phi\phi}(\phi \dot{\phi} \ddot{\phi}) = \frac{e^{-\frac{A^2}{2\sigma^2}}}{\sqrt{(2\pi \sigma^2)^3 b}} \int_0^\infty V^2 e^{-\frac{V^2}{2\sigma^2} (1 + \frac{\dot{\phi}^2}{b^2}) + \frac{AV}{\sigma^2} \cos \phi} dV \frac{1}{\sqrt{2\pi \beta_2}} e^{-\frac{\dot{\phi}^2}{2\beta_2}}$$

By integrating the expression (31), the joint probability density function of frequency and its the time derivation can be obtained. Level crossing rate of frequency is

$$N_\phi = \int \phi p_{\phi\phi\phi}(\phi \dot{\phi} \ddot{\phi}) d\phi$$
4 The Amplitude of Signals is Random Variable

The envelope of the sum signals and narrow band Gaussian noise can be obtained in the form

\[ r = \sqrt{(x-A)^2 + y^2} \]  \hspace{1cm} (33)

From (33), random variables \( A \) and \( y \) can be given in the form

\[ A = x - \sqrt{r^2 - y^2} \]
\[ y = \sqrt{r^2 - (x-A)^2} \]

Differentiating \( r \) with respect to \( A \), \( x \) and \( y \), the \( \dot{r} \) can be given by

\[ \dot{r} = \frac{(x-A)\dot{x} - \dot{A} + yy'}{\sqrt{(x-A)^2 + y^2}} \]  \hspace{1cm} (34)

Differentiating \( A \) with respect to \( r \), \( x \) and \( y \), the \( \dot{A} \) can be given by

\[ \dot{A} = \dot{x} - \frac{r\dot{r} - yy'}{\sqrt{r^2 - y^2}} \]

Differentiating \( y \) with respect to \( A \), \( x \) and \( r \), the \( \dot{y} \) can be given by

\[ \dot{y} = \frac{r\dot{r} - (x-A)(\dot{x} - \dot{A})}{\sqrt{(x-A)^2 + y^2}} \]

The conditional joint probability density function of the envelope and its the first time derivative is

\[ p_{rr}(r\dot{r} / A\dot{A}) = \int dx \int d\dot{x} J_{xxyy} \left( x\ddot{x}, \sqrt{r^2 - (x-A)^2}, \frac{r\dot{r} - (x-A)(\dot{x} - \dot{A})}{\sqrt{(x-A)^2 + y^2}} \right) \]  \hspace{1cm} (35)

By averaging the expression (35) with respect of \( A \) and, the joint probability density function of envelope and its the time derivative of the sum sine waves and narrow band Gaussian noise can be expressed in the form:

\[ p_{rr}(r\dot{r}) = \int dA \int dA' p(r\dot{r} / A\dot{A}) p_{AA}(A\dot{A}) \]  \hspace{1cm} (36)

The phase of the sum of sine waves and narrow band noise can be obtained from (5) in the form:
\[ \varphi = \arctan \frac{y}{x - A} \]  
(37)

By differentiating, \( \varphi \), with respect to \( x \), \( A \) and \( y \) we can write

\[ \varphi = \frac{y(x - A) - y(\dot{x} - \dot{A})}{(x - A)^2 + y^2} \]  
(38)

From (37), we can write

\[ y = (x - A) \tan \varphi \]  
(39)

By differentiating, \( y \), with respect to \( r \), \( A \) and \( x \) we can write:

\[ \dot{y} = (\dot{x} - \dot{A}) \tan \varphi + \frac{(x - A) \dot{\varphi}}{\cos^2 \varphi} \]  
(40)

In this case, the conditional joint probability density function of phase of sum sine wave and narrow band Gaussian noise is:

\[ p_{\varphi \varphi}(\varphi \varphi / A\dot{A}) = \int dx \int d\dot{x} p_{x\dot{x}y}(x, \dot{x}, (x - A) \tan \varphi, (\dot{x} - \dot{A}) \tan \varphi + \frac{(x - A) \dot{\varphi}}{\cos^2 \varphi}) \]  
(41)

The joint probability density function of amplitude \( A \), its the first time derivative and its the second time derivative is:

\[ p_{A\dot{A}\ddot{A}}(A\dot{A}\ddot{A}) = \frac{A}{\sigma_1} e^{-\frac{A^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi} \beta_A} e^{-\frac{\dot{A}^2}{2\beta_A^2}} \frac{1}{\sqrt{2\pi} \gamma_A} e^{-\frac{\ddot{A}^2}{2\gamma_A^2}} \]  
(42)

The joint probability density function of Gaussian variable \( x \), its first time derivative and its the second derivative is:

\[ p_{xx\ddot{x}}(x\dot{x}\ddot{x}) = \frac{x^2}{\sigma} e^{-\frac{x^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi} \beta_x} e^{-\frac{\dot{x}^2}{2\beta_x^2}} \frac{1}{\sqrt{2\pi} \gamma_x} e^{-\frac{\ddot{x}^2}{2\gamma_x^2}} \]  
(43)

The joint probability density function of Gaussian variable \( y \), its first time derivative and its the second derivative is:

\[ p_{yy\ddot{y}}(y\dot{y}\ddot{y}) = \frac{y^2}{\sigma} e^{-\frac{y^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi} \beta_y} e^{-\frac{\dot{y}^2}{2\beta_y^2}} \frac{1}{\sqrt{2\pi} \gamma_y} e^{-\frac{\ddot{y}^2}{2\gamma_y^2}} \]  
(44)

The envelope of sum of sine wave and narrow band noise is

\[ r = \sqrt{(x - A)^2 + y^2} \]  
(45)
The first time derivative of envelope is

$$
\dot{r} = \frac{(x-A)(\dot{x}-\dot{A}) + y\dot{y}}{\sqrt{(x-A)^2 + y^2}}
$$

(46)

The second time derivative of envelope is

$$
\ddot{r} = \frac{[(\dot{x}-\dot{A})^2 + (x-A)(\dot{x}-\dot{A}) + y^2 + y\dot{y}][(x-A)^2 + y^2] - [(x-A)(\dot{x}-\dot{A}) + y\dot{y}]^2}{[(x-A)^2 + y^2]^{\frac{3}{2}}}
$$

(47)

From the expression (33) we can obtain

$$
y = \sqrt{r^2 - (x-A)^2}
$$

(48)

The first and second time derivative of $y$ are

$$
\dot{y} = \frac{r\ddot{r} - (x-A)(\dot{x}-\dot{A})}{\sqrt{r^2 - (x-A)^2}}
$$

$$
\ddot{y} = \frac{[r^2 + r\ddot{r} - (x-A)^2 - (x-A)(\dot{x}-\dot{A})][r^2 - (x-A)^2] - [r\ddot{r} - (x-A)(\dot{x}-\dot{A})]^2}{[r^2 - (x-A)^2]^{\frac{3}{2}}}
$$

(49)

The amplitude $A$, the first time derivative of $A$ and the second time derivative of $A$, respectively, are:

$$
A = x - \sqrt{r^2 - y^2}
$$

$$
\dot{A} = \dot{x} - \frac{r\ddot{r} - y\dot{y}}{\sqrt{r^2 - y^2}}
$$

$$
\ddot{A} = \ddot{x} - \frac{(r^2 - r\ddot{r} - y^2 - y\dot{y})(r^2 - y^2) - (r\ddot{r} - y\dot{y})}{(r^2 - y^2)^{\frac{3}{2}}}
$$

(50)

The conditional joint probability density function of envelope $r$ and its the first and second time derivative is:

$$
p_{rr\dot{r}}(r\dot{r} \dot{A} \dot{A}) = J p_{xx\dot{y}y}(x, \dot{x}, \dot{y}, y, \ddot{y})
$$

(51)

Joint probability density function of envelope $r$, the first time derivative of envelope $r$ and second time derivative of envelope $r$ is:

$$
p_{rr\dot{r}}(r\dot{r}) = \int dx \int d\dot{x} \int d\ddot{x} |J| \int dA \int d\dot{A} \int d\ddot{A} p_{xx\dot{y}y}(x, \dot{x}, \dot{y}, y, \ddot{y}) p_{\dot{A}\ddot{A}}(A, \dot{A})
$$

(52)
The phase of sum sine wave and narrow band Gaussian noise, its the first and second derivative are:

\[
\varphi = \arctan \frac{y}{x - A}
\]

\[
\dot{\varphi} = \frac{\dot{y}(x - A) - y(\dot{x} - \dot{A})}{(x - A)^2 + y^2}
\]

\[
\ddot{\varphi} = \frac{[\dot{y}(x - A) - y(\dot{x} - \dot{A})][(x - A)^2 + y^2]^2}{[(x - A)^2 + y^2]^2}
\]

\[
- \frac{2(x - A)(\dot{x} - \dot{A}) + 2yy][\dot{y}(x - A) - y(\dot{x} - \dot{A})]}{[(x - A)^2 + y^2]^2}
\]

Gaussian variable \( y \), its the first and second derivative are:

\[
y = (x - A)\tan \varphi
\]

\[
\dot{y} = (\dot{x} - \dot{A})\tan \varphi + \frac{(x - A)\dot{\varphi}}{\cos^2 \varphi}
\]

\[
\ddot{y} = (\dot{x} - \dot{A})\tan \varphi + \frac{(\dot{x} - \dot{A})\dot{\varphi}}{\cos^2 \varphi}
\]

\[
+ \frac{[(\dot{x} - \dot{A})\dot{\varphi} + (x - A)\ddot{\varphi}]\cos^2 \varphi + 2\dot{\varphi}(x - A)\cos \varphi \sin \varphi}{\cos^4 \varphi}
\]

Conditional joint probability density of phase \( \varphi \), its the first time derivative and its the second time derivative is:

\[
p_{\varphi \dot{\varphi} \ddot{\varphi}}(\varphi \dot{\varphi} \ddot{\varphi} / A\dot{A}\ddot{A}) = \int dx \int d\dot{x} \int d\ddot{x} J p_{\dddot{x}x\dddot{y}y}(xx\dddot{y}y\dddot{y})
\]

The joint probability density function of phase of sum sine wave and narrow band Gaussian noise and its the first time derivative and its the second time derivative is:

\[
p_{\varphi \dot{\varphi} \ddot{\varphi}}(\varphi \dot{\varphi} \ddot{\varphi}) = \int dA \int d\dot{A} \int d\ddot{A} \int dx \int d\dot{x} \int d\ddot{x} J p_{\dddot{x}x\dddot{y}y}(xx\dddot{y}y\dddot{y})p_{\dot{A}\ddot{A}}(A\dot{A}\ddot{A})
\]

Fig.1. illustrates the joint probability density function of envelope of sum of sine wave and narrow band Gaussian noise and its the first time derivative. By this density we can determine the level crossing rate and average fade duration of envelope.

In fig. 2 the level crossing rate is shown. By this the level crossing rate we can determine the average fade duration and we can calculate the spectral density power of signals.
Fig. 1. The joint probability density function of envelope of sum of sine wave and narrow band Gaussian noise and its the first time derivative.

Fig. 2. The level crossing rate of anvelope.
5 Conclusion

In this paper, the joint probability density functions of envelope and its the first time derivative and phase and its the first derivation of the sum of a sine wave and narrow band Gaussian random process in IM-DD optical systems is determined. The average level crossing rate defined as the rate of which a signal envelope crosses level in a positive going direction is determined for envelope, phase and time derivative of phase. The average fade duration defined as the average time that the envelope, phase and frequency remains below in a downward direction is obtained. By the expressions obtained in this paper, the bit error probability of IM-DD optical system in the presence of Gaussian noise can be determined.

References