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Coupled Common Fixed Point Theorems in Partially Ordered G-metric Spaces for Nonlinear Contractions

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Abstract. The aim of this paper is to prove coupled coincidence and coupled common fixed point theorems for a mixed g-monotone mapping satisfying nonlinear contractive conditions in the setting of partially ordered G-metric spaces. Present theorems are true generalizations of the recent results of Choudhury and Maity [Math Comput. Modelling **54** (2011), 73–79], and Luong and Thuan [Math. Comput. Modelling **55** (2012), 1601–1609].

1. Introduction and Preliminaries

In [12], Mustafa together with Sims introduced the generalized structure of metric spaces, called G-metric spaces. Afterwards, numerous fixed point theorems in this generalized structure were proved by different authors. Works noted in [1,6,9,11,13,14,17,21] are some examples in this direction. Bhaskar and Lakshmikantham [3] introduced the notion of coupled fixed point and proved some coupled fixed point theorems for a mapping satisfying mixed monotone property in partially ordered metric spaces. As an application, they discussed the existence and uniqueness of solution for a periodic boundary value problem. Lakshmikantham and Ćirić [8] extended the notion of mixed monotone property to mixed g-monotone property and generalized the results of Bhaskar and Lakshmikantham [3] by establishing the existence of coupled coincidence point results using a pair of commutative mappings. These results have been extended and generalized by several authors. References [7, 15, 16] are some examples of these works. Now-a-days authors have keen interest in proving fixed point theorems in partially ordered metric spaces subjected to nonlinear contractive conditions, see [2, 4, 10, 18–20]. Our paper deals with the establishment of some coupled coincidence and

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coupled common fixed point results for a mixed g-monotone mapping satisfying nonlinear contractive conditions in partially ordered G-metric spaces. Our results generalize the recent results of Choudhury and Maity [5] and Luong and Thuan [9]. We give also an example to illustrate our results. We now recall some definitions and properties in G-metric spaces (see [12]).

Definition 1.1. Let X be a nonempty set. Suppose that $G : X \times X \times X \to [0, +\infty)$ is a function satisfying the following conditions:

(G1) $G(x, y, z) = 0$ if and only if $x = y = z$;
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a G-metric on X and (X, G) is called a G-metric space.

Definition 1.2. Let (X, G) be a G-metric space. We say that $\{x_n\}$ is:

(i) a G-Cauchy sequence if, for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ (the set of all positive integers) such that for all $n, m, l \geq N$, $G(x_n, x_m, x_l) < \varepsilon$;
(ii) a G-convergent sequence to $x \in X$ if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, $G(x, x_n, x_m) < \varepsilon$.

A G-metric space (X, G) is said to be complete if every G-Cauchy sequence in X is G-convergent in X.

Proposition 1.1. *Let (X, G) be a G-metric space. The following are equivalent:*

(i) *$\{x_n\}$ is G-convergent to x;*
(ii) *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$;*
(iii) *$G(x_n, x, x) \to 0$ as $n \to +\infty$;*
(iv) *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Proposition 1.2. *Let (X, G) be a G-metric space. Then the following are equivalent:*

(i) *the sequence $\{x_n\}$ is G-Cauchy;*
(ii) *$G(x_n, x_m, x_m) \to 0$ as $n, m \to +\infty$.*

Proposition 1.3. *Let (X, G) be a G-metric space. A mapping $g : X \to X$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{g(x_n)\}$ is G-convergent to $g(x)$.*

Proposition 1.4. *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 1.3 ([5])**.** Let (X,G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively, $\{F(x_n, y_n)\}$ is G-convergent to $F(x,y)$.

An interesting observation is that any G-metric space (X,G) induces a metric d_G on X given by

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \text{ for all } x,y \in X.$$

Moreover, (X,G) is G-complete if and only if (X,d_G) is complete.

Now, we recall some definitions introduced in [3, 8]. Let $(X,\preceq)$ be a partially ordered set and $g : X \to X$ be a mapping. The mapping g is said to be non-decreasing if for all $x,y \in X$, $x \preceq y$ implies $g(x) \preceq g(y)$. Similarly, g is said to be non-increasing, if for all $x,y \in X$, $x \preceq y$ implies $g(x) \succeq g(y)$.

Bhaskar and Lakshmikantham [3] introduced the following notions of mixed monotone mapping and coupled fixed point.

Definition 1.4. Let $(X,\preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$, for any $y \in X$ and for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2)$, for any $x \in X$.

The concept of the mixed monotone property was generalized by Lakshmikantham and Ćirić [8] as follows.

Definition 1.5 ([8])**.** Let $(X,\preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to have the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for all $x_1, x_2 \in X$, $g(x_1) \preceq g(x_2)$ implies $F(x_1, y) \preceq F(x_2, y)$, for any $y \in X$ and for all $y_1, y_2 \in X$, $g(y_1) \preceq g(y_2)$ implies $F(x, y_1) \succeq F(x, y_2)$, for any $x \in X$.

Clearly, if g is the identity mapping, then Definition 1.5 reduces to Definition 1.4.

Definition 1.6. An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

Definition 1.7. An element $(x,y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = g(x)$ and $F(y,x) = g(y)$.

Definition 1.8. Let $F : X \times X \to X$ and $g : X \to X$. Then, F and g are said to be commutative if $g(F(x,y)) = F(g(x), g(y))$ for all $x,y \in X$.

Definition 1.9 ([10])**.** Let Φ denote all functions $\phi : [0,+\infty) \to [0,+\infty)$ such that

(i) ϕ is continuous and increasing;
(ii) $\phi(t) = 0$ iff $t = 0$;
(iii) $\phi(t+s) \leq \phi(t) + \phi(s)$, for all $t, s \in [0, +\infty)$.

Definition 1.10 ([9]). Let Θ denote all functions $\theta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ such that $\lim_{(t_1,t_2)\to(r_1,r_2)} \theta(t_1, t_2) > 0$ for all $(r_1, r_2) \in [0, +\infty) \times [0, +\infty)$ with $r_1 + r_2 > 0$.

Let $(X, \preceq)$ be a partially ordered set, $F : X \times X \to X$ be a mapping having the mixed monotone property and suppose there exists a G-metric such that (X, G) is a G-metric space. Choudhury and Maity [5] established some fixed point results for the mapping F under the following contractive condition

$$G(F(x, y), F(u, v), F(w, z)) \leq \frac{k}{2}(G(x, u, w) + G(y, v, z))$$

for $w \preceq u \preceq x$ and $y \preceq v \preceq z$, where $k \in [0, 1)$.
Recently, Luong and Thuan [9] generalized the results of Choudhury and Maity [5] by proving some coupled fixed point theorems in partially ordered G-metric spaces under a nonlinear contractive condition of the form

$$G(F(x, y), F(u, v), F(w, z)) \leq \frac{G(x, u, w) + G(y, v, z)}{2} - \theta(G(x, u, w), G(y, v, z))$$

for $w \preceq u \preceq x$ and $y \preceq v \preceq z$, where $\theta \in \Theta$.

2. MAIN RESULTS

Our first result is the following coupled coincidence point theorem.

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a G-metric G on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X and there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose there exist $\phi \in \Phi$ and $\theta \in \Theta$ such that*

$$\begin{aligned} &\phi(G(F(x, y), F(u, v), F(w, z)) \\ (1) \qquad &\leq \frac{1}{2}\phi(G(g(x), g(u), g(w)) + G(g(y), g(v), g(z))) \\ &\qquad - \theta(G(g(x), g(u), g(w)), G(g(y), g(v), g(z))) \end{aligned}$$

for all $(x, y), (u, v), (w, z) \in X \times X$ with $g(w) \preceq g(u) \preceq g(x)$ and $g(y) \preceq g(v) \preceq g(z)$. Further suppose that F is continuous, $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F. Then, there exist $x, y \in X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$.

Analogously, there exist $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$.

Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n) \ \forall \ n \geq 0. \tag{2}$$

Now we prove that for all $n \geq 0$,

$$g(x_n) \preceq g(x_{n+1}) \text{ and } g(y_n) \succeq g(y_{n+1}). \tag{3}$$

We shall use the mathematical induction. Let $n = 0$, since $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$, in view of $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$, we have $g(x_0) \preceq g(x_1)$ and $g(y_0) \succeq g(y_1)$, that is, (3) holds for $n = 0$. We assume that (3) holds for some $n > 0$. As F has the mixed g-monotone property and $g(x_n) \preceq g(x_{n+1})$, $g(y_n) \succeq g(y_{n+1})$, from (2), we get

$$g(x_{n+1}) = F(x_n, y_n) \preceq F(x_{n+1}, y_n), \ \ F(y_{n+1}, x_n) \preceq F(y_n, x_n) = g(y_{n+1}).$$

Also for the same reason we have

$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \succeq F(x_{n+1}, y_n), \ F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = g(y_{n+2}).$$

Merging the above results, we obtain $g(x_{n+1}) \preceq g(x_{n+2})$ and $g(y_{n+1}) \succeq g(y_{n+2})$.

Thus by the mathematical induction, we conclude that (3) holds for all $n \geq 0$.

Since $g(x_n) \succeq g(x_{n-1})$ and $g(y_n) \preceq g(y_{n-1})$, from (1) and (2), we have

$$\begin{aligned}
&\phi(G(g(x_{n+1}), g(x_{n+1}), g(x_n)))\\
&= \phi(G(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1}))\\
&\leq \frac{1}{2}\phi(G(g(x_n), g(x_n), g(x_{n-1})) + G(g(y_n), g(y_n), g(y_{n-1})))\\
&\qquad - \theta(G(g(x_n), g(x_n), g(x_{n-1})), G(g(y_n), g(y_n), g(y_{n-1}))).
\end{aligned}$$

As $\theta(t_1, t_2) \geq 0$ for all $t_1, t_2 \in [0, +\infty)$, we have

$$\begin{aligned}
&\phi(G(g(x_{n+1}), g(x_{n+1}), g(x_n)))\\
&\leq \frac{1}{2}\phi(G(g(x_n), g(x_n), g(x_{n-1})) + G(g(y_n), g(y_n), g(y_{n-1}))).
\end{aligned} \tag{4}$$

Similarly, since $g(y_{n-1}) \succeq g(y_n)$ and $g(x_{n-1}) \preceq g(x_n)$, from (1) and (2), we have

$$\begin{aligned}
&\phi(G(g(y_{n+1}), g(y_{n+1}), g(y_n)))\\
&\leq \frac{1}{2}\phi(G(g(y_n), g(y_n), g(y_{n-1})) + G(g(x_n), g(x_n), g(x_{n-1}))).
\end{aligned} \tag{5}$$

From (4), (5) and a property of ϕ, we have

$$\begin{aligned}
&\phi(G(g(x_{n+1}), g(x_{n+1}), g(x_n)) + G(g(y_{n+1}), g(y_{n+1}), g(y_n))) \\
&\leq \frac{1}{2}\phi(G(g(x_n), g(x_n), g(x_{n-1})) + G(g(y_n), g(y_n), g(y_{n-1}))) \\
&\qquad + \frac{1}{2}\phi(G(g(y_n), g(y_n), g(y_{n-1})) + G(g(x_n), g(x_n), g(x_{n-1}))) \\
&= \phi(G(g(x_n), g(x_n), g(x_{n-1})) + G(g(y_n), g(y_n), g(y_{n-1}))).
\end{aligned}$$

Set $\rho_n = G(g(x_{n+1}), g(x_{n+1}), g(x_n)) + G(g(y_{n+1}), g(y_{n+1}), g(y_n))$, then using the monotonicity of ϕ the sequence $\{\rho_n\}$ is non-increasing and so there exists $\rho \geq 0$ such that

$$\begin{aligned}
\lim_{n\to+\infty} \rho_n &= \lim_{n\to+\infty} \Big[G(g(x_{n+1}), g(x_{n+1}), g(x_n)) \\
&\qquad + G(g(y_{n+1}), g(y_{n+1}), g(y_n))\Big] \\
&= \rho.
\end{aligned} \tag{6}$$

We shall show that $\rho = 0$. Suppose, on the contrary, that $\rho > 0$. By (6), the sequences $\{G(g(x_{n+1}), g(x_{n+1}), g(x_n))\}$ and $\{G(g(y_{n+1}), g(y_{n+1}), g(y_n))\}$ have convergent subsequences that are still denoted by $\{G(g(x_{n+1}), g(x_{n+1}), g(x_n))\}$ and $\{G(g(y_{n+1}), g(y_{n+1}), g(y_n))\}$ respectively. Suppose that

$$\begin{aligned}
&\lim_{n\to+\infty} G(g(x_{n+1}), g(x_{n+1}), g(x_n)) = \rho_1 \quad \text{and} \\
&\lim_{n\to+\infty} G(g(y_{n+1}), g(y_{n+1}), g(y_n)) = \rho_2.
\end{aligned}$$

Then $\rho_1 + \rho_2 = \rho > 0$. Reasoning as above and using a property of ϕ, we have

$$\begin{aligned}
&\phi(G(g(x_{n+1}), g(x_{n+1}), g(x_n)) + G(g(y_{n+1}), g(y_{n+1}), g(y_n))) \\
&\leq \phi(G(g(x_{n+1}), g(x_{n+1}), g(x_n))) + \phi(G(g(y_{n+1}), g(y_{n+1}), g(y_n))) \\
&\leq \phi(G(g(x_n), g(x_n), g(x_{n-1})) + G(g(y_n), g(y_n), g(y_{n-1}))) \\
&\qquad - \theta(G(g(x_n), g(x_n), g(x_{n-1})), G(g(y_n), g(y_n), g(y_{n-1}))) \\
&\qquad - \theta(G(g(y_n), g(y_n), g(y_{n-1})), G(g(x_n), g(x_n), g(x_{n-1}))).
\end{aligned}$$

Taking the limit as $n \to +\infty$ in the last inequality, using (6), the continuity of ϕ and the property of θ, we have

$$\begin{aligned}
\phi(\rho) &\leq \phi(\rho) - \lim_{n\to+\infty} \theta(G(g(x_n), g(x_n), g(x_{n-1})), G(g(y_n), g(y_n), g(y_{n-1}))) \\
&\qquad - \lim_{n\to+\infty} \theta(G(g(y_n), g(y_n), g(y_{n-1})), G(g(x_n), g(x_n), g(x_{n-1}))) \\
&< \phi(\rho),
\end{aligned}$$

which is a contradiction. Thus $\rho = 0$, that is,

$$\begin{aligned} &\lim_{n\to+\infty} G(g(x_{n+1}), g(x_{n+1}), g(x_n)) = 0, \\ &\lim_{n\to+\infty} G(g(y_{n+1}), g(y_{n+1}), g(y_n)) = 0. \end{aligned} \tag{7}$$

In what follows, we shall prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{g(x_n)\}$ and $\{g(y_n)\}$ is not a G-Cauchy sequence, that is,

$$\lim_{m,n\to+\infty} G(g(x_m), g(x_n), g(x_n)) \neq 0.$$

Then, there exists $\varepsilon > 0$ for which we can find two subsequences $\{g(x_{m(i)})\}$ and $\{g(x_{n(i)})\}$ of $\{x_n\}$ such that $n(i)$ is the smallest index for which $n(i) > m(i) > i$,

$$r_i = G(g(x_{n(i)}), g(x_{n(i)}), g(x_{m(i)})) + G(g(y_{n(i)}), g(y_{n(i)}), g(y_{m(i)})) \geq \varepsilon.$$

This means that

$$G(g(x_{n(i)-1}), g(x_{n(i)-1}), g(x_{m(i)})) + G(g(y_{n(i)-1}), g(y_{n(i)-1}), g(y_{m(i)})) < \varepsilon.$$

By rectangle inequality, we get

$$\begin{aligned} &G(g(x_{n(i)}), g(x_{n(i)}), g(x_{m(i)})) \\ &\leq G(g(x_{n(i)}), g(x_{n(i)}), g(x_{n(i)-1})) + G(g(x_{n(i)-1}), g(x_{n(i)-1}), g(x_{m(i)})) \end{aligned}$$

and

$$\begin{aligned} &G(g(y_{n(i)}), g(y_{n(i)}), g(y_{m(i)})) \\ &\leq G(g(y_{n(i)}), g(y_{n(i)}), g(y_{n(i)-1})) + G(g(y_{n(i)-1}), g(y_{n(i)-1}), g(y_{m(i)})). \end{aligned}$$

Using the above inequalities, we get

$$\begin{aligned} \varepsilon \leq r_i &= G(g(x_{n(i)}), g(x_{n(i)}), g(x_{m(i)})) + G(g(y_{n(i)}), g(y_{n(i)}), g(y_{m(i)})) \\ &\leq G(g(x_{n(i)}), g(x_{n(i)}), g(x_{n(i)-1})) + G(g(y_{n(i)}), g(y_{n(i)}), g(y_{n(i)-1})) + \varepsilon. \end{aligned}$$

Letting $i \to +\infty$ and using (7) we have

(8)

$$\lim_{i\to+\infty} r_i = \lim_{i\to+\infty} \Big[G(g(x_{n(i)}), g(x_{n(i)}), g(x_{m(i)})) + G(g(y_{n(i)}), g(y_{n(i)}), g(y_{m(i)})) \Big]$$

Using the fact that $G(x,x,y) \leq 2G(x,y,y)$ for any $x,y \in X$, we obtain from $(G5)$ that

$$\begin{aligned}
&G(g(x_{n(i)}), g(x_{n(i)}), g(x_{m(i)})) \\
&\leq G(g(x_{n(i)}), g(x_{n(i)}), g(x_{n(i)+1})) \\
&\qquad + G(g(x_{n(i)+1}), g(x_{n(i)+1}), g(x_{m(i)})) \\
&\leq 2G(g(x_{n(i)+1}), g(x_{n(i)+1}), g(x_{n(i)})) \\
&\qquad + G(g(x_{n(i)+1}), g(x_{n(i)+1}), g(x_{m(i)+1})) \\
&\qquad + G(g(x_{m(i)+1}), g(x_{m(i)+1}), g(x_{m(i)}))
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
&G(g(y_{n(i)}), g(y_{n(i)}), g(y_{m(i)})) \\
&\leq 2G(g(y_{n(i)+1}), g(y_{n(i)+1}), g(y_{n(i)})) \\
&\qquad + G(g(y_{n(i)+1}), g(y_{n(i)+1}), g(y_{m(i)+1})) \\
&\qquad + G(g(y_{m(i)+1}), g(y_{m(i)+1}), g(y_{m(i)})).
\end{aligned} \tag{10}$$

By (9) and (10), we have

$$\begin{aligned}
r_i &= G(g(x_{n(i)}), g(x_{n(i)}), g(x_{m(i)})) + G(g(y_{n(i)}), g(y_{n(i)}), g(y_{m(i)})) \\
&\leq 2\rho_{n(i)} + \rho_{m(i)} + G(g(x_{n(i)+1}), g(x_{n(i)+1}), g(x_{m(i)+1})) \\
&\qquad + G(g(y_{n(i)+1}), g(y_{n(i)+1}), g(y_{m(i)+1})).
\end{aligned} \tag{11}$$

By (11), using the properties of ϕ, we get

$$\begin{aligned}
\phi(r_i) &\leq \phi(2\rho_{n(i)} + \rho_{m(i)} + G(g(x_{n(i)+1}), g(x_{n(i)+1}), g(x_{m(i)+1})) \\
&\qquad + G(g(y_{n(i)+1}), g(y_{n(i)+1}), g(y_{m(i)+1}))) \\
&\leq 2\phi(\rho_{n(i)}) + \phi(\rho_{m(i)}) + \phi(G(g(x_{n(i)+1}), g(x_{n(i)+1}), g(x_{m(i)+1}))) \\
&\qquad + \phi(G(g(y_{n(i)+1}), g(y_{n(i)+1}), g(y_{m(i)+1}))).
\end{aligned} \tag{12}$$

Since $n(i) > m(i)$, $g(x_{n(i)}) \succeq g(x_{m(i)})$ and $g(y_{m(i)}) \succeq g(y_{n(i)})$, by (1) we deduce

$$\begin{aligned}
&\phi(G(g(x_{n(i)+1}), g(x_{n(i)+1}), g(x_{m(i)+1}))) \\
&= \phi(G(F(x_{n(i)}, y_{n(i)}), F(x_{n(i)}, y_{n(i)}), F(x_{m(i)}, y_{m(i)})) \\
&\leq \frac{1}{2}\phi(G(g(x_{n(i)}), g(x_{n(i)}), g(x_{m(i)})) + G(g(y_{n(i)}), g(y_{n(i)}), g(y_{m(i)}))) \\
&\qquad - \theta(G(g(x_{n(i)}), g(x_{n(i)}), g(x_{m(i)})), G(g(y_{n(i)}), g(y_{n(i)}), g(y_{m(i)}))) \\
&\leq \frac{1}{2}\phi(r_i) - \theta(G(g(x_{n(i)}), g(x_{n(i)}), g(x_{m(i)})), G(g(y_{n(i)}), g(y_{n(i)}), g(y_{m(i)})))
\end{aligned} \tag{13}$$

Similarly, we have

$$\begin{aligned}\phi(G(g(y_{n(i)+1}),g(y_{n(i)+1}),g(y_{m(i)+1}))) \leq \frac{1}{2}\phi(r_i) \\ - \theta(G(g(y_{n(i)}),g(y_{n(i)}),g(y_{m(i)})),G(g(x_{n(i)}),g(x_{n(i)}),g(x_{m(i)}))).\end{aligned} \tag{14}$$

Inserting (13) and (14) in (12), we get

$$\begin{aligned}&\phi(r_i) \leq 2\phi(\rho_{n(i)}) + \phi(\rho_{m(i)}) + \phi(r_i) \\ &\quad - \theta(G(g(x_{n(i)}),g(x_{n(i)}),g(x_{m(i)})),G(g(y_{n(i)}),g(y_{n(i)}),g(y_{m(i)}))) \\ &\quad - \theta(G(g(y_{n(i)}),g(y_{n(i)}),g(y_{m(i)})),G(g(x_{n(i)}),g(x_{n(i)}),g(x_{m(i)}))).\end{aligned} \tag{15}$$

By (8), the sequences

$$\{G(g(x_{n(i)}),g(x_{n(i)}),g(x_{m(i)}))\} \quad \text{and} \quad \{G(g(y_{n(i)}),g(y_{n(i)}),g(y_{m(i)}))\}$$

have subsequences converging to ε_1 and ε_2 (say) respectively and $\varepsilon_1 + \varepsilon_2 = \varepsilon > 0$. By passing to subsequences, we may assume that

$$\lim_{i\to+\infty} G(g(x_{n(i)}),g(x_{n(i)}),g(x_{m(i)})) = \varepsilon_1$$

and

$$\lim_{i\to+\infty} G(g(y_{n(i)}),g(y_{n(i)}),g(y_{m(i)})) = \varepsilon_2.$$

Taking $i \to +\infty$ in (15) and using (7), (8), the properties of ϕ and θ, we have

$$\begin{aligned}\phi(\varepsilon) &\leq 2\phi(0) + \phi(0) + \phi(\varepsilon) - \lim_{i\to+\infty}\theta\Big(G(g(x_{n(i)}),g(x_{n(i)}),g(x_{m(i)})), \\ &\qquad\qquad G(g(y_{n(i)}),g(y_{n(i)}),g(y_{m(i)}))\Big) \\ &\quad - \lim_{i\to+\infty}\theta\Big(G(g(y_{n(i)}),g(y_{n(i)}),g(y_{m(i)})),G(g(x_{n(i)}),g(x_{n(i)}),g(x_{m(i)}))\Big) \\ &= \phi(\varepsilon) - \lim_{i\to+\infty}\theta\Big(G(g(x_{n(i)}),g(x_{n(i)}),g(x_{m(i)})), \\ &\qquad\qquad G(g(y_{n(i)}),g(y_{n(i)}),g(y_{m(i)}))\Big) \\ &\quad - \lim_{i\to+\infty}\theta\Big(G(g(y_{n(i)}),g(y_{n(i)}),g(y_{m(i)})),G(g(x_{n(i)}),g(x_{n(i)}),g(x_{m(i)}))\Big) \\ &< \phi(\varepsilon),\end{aligned}$$

which is a contradiction. Thus, $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Since the G-metric space (X, G) is complete, there exist $x, y \in X$ such that $\{g(x_n)\}$ and $\{g(y_n)\}$ are convergent to x and y respectively, that is from Proposition 1.1, we have

$$\lim_{i\to+\infty} G(g(x_n),g(x_n),x) = \lim_{i\to+\infty} G(g(x_n),x,x) = 0$$

and

$$\lim_{i\to+\infty} G(g(y_n),g(y_n),y) = \lim_{i\to+\infty} G(g(y_n),y,y) = 0.$$

Using the continuity of g and Proposition 1.3, we get

$$\lim_{i\to+\infty} G(g(g(x_n)), g(g(x_n)), g(x)) = \lim_{i\to+\infty} G(g(g(x_n)), g(x), g(x)) = 0 \tag{16}$$

and

$$\lim_{i\to+\infty} G(g(g(y_n)), g(g(y_n)), g(y)) = \lim_{i\to+\infty} G(g(g(y_n)), g(y), g(y)) = 0.$$

Since, $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$, hence the commutativity of F and g yields that

$$\begin{cases} g(g(x_{n+1})) = g(F(x_n, y_n)) = F(g(x_n), g(y_n)), \\ g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g(y_n), g(x_n)). \end{cases} \tag{17}$$

Next we show that $F(x, y) = g(x)$ and $F(y, x) = g(y)$. The mapping F is continuous and since the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are respectively G-convergent to x and y, hence using Definition 1.3 the sequence $\{F(g(x_n), g(y_n))\}$ is G-convergent to $F(x, y)$. Therefore, by (17), $\{g(g(x_{n+1}))\}$ is G-convergent to $F(x, y)$. By uniqueness of limit and using (16), we have $F(x, y) = g(x)$. Similarly, we can show that $F(y, x) = g(y)$. Hence (x, y) is a coupled coincidence point of F and g. □

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X and there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose there exist $\phi \in \Phi$ and $\theta \in \Theta$ such that*

$$\begin{aligned} &\phi(G(F(x, y), F(u, v), F(w, z)) \\ &\leq \frac{1}{2}\phi(G(g(x), g(u), g(w)) + G(g(y), g(v), g(z))) \\ &\qquad - \theta(G(g(x), g(u), g(w)), G(g(y), g(v), g(z))) \end{aligned} \tag{18}$$

for all $(x, y), (u, v), (w, z) \in X \times X$ with $g(w) \preceq g(u) \preceq g(x)$ and $g(y) \preceq g(v) \preceq g(z)$. Further suppose that $(g(X), G)$ or $(F(X \times X), G)$ is complete, $F(X \times X) \subseteq g(X)$ and the following conditions hold:

(i) *if a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,*

(ii) *if a non-increasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n.*

Then, there exist $x, y \in X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Following the proof of Theorem 2.1, it follows that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Now, we distinguish the following two cases.
Case 1. If $(g(X), G)$ is complete, then there exist $x, y \in X$ such that $g(x_n) \to g(x)$ and $g(y_n) \to g(y)$ as $n \to +\infty$. Since $\{g(x_n)\}$ is non-decreasing and $\{g(y_n)\}$ is non-increasing, by given hypotheses, we have

$g(x_n) \preceq g(x)$ and $g(y) \preceq g(y_n)$ for all $n \geq 0$. Then using (18) and the properties of ϕ and θ, we have

$$\begin{aligned} &\phi(G(F(x,y), g(x_{n+1}), g(x_{n+1}))) \\ &= \phi(G(F(x,y), F(x_n,y_n), F(x_n,y_n))) \\ &\leq \frac{1}{2}\phi(G(g(x), g(x_n), g(x_n)) + G(g(y), g(y_n), g(y_n))) \\ &\qquad - \theta(G(g(x), g(x_n), g(x_n)), G(g(y), g(y_n), g(y_n))) \\ &\leq \frac{1}{2}\phi(G(g(x), g(x_n), g(x_n)) + G(g(y), g(y_n), g(y_n))). \end{aligned}$$

Letting $n \to +\infty$ in the last inequality and using the properties of ϕ, we obtain

$$\phi(G(F(x,y), g(x), g(x))) \leq 0,$$

which implies that $G(F(x,y), g(x), g(x)) = 0$, that is, $F(x,y) = g(x)$. Similarly, it can be shown that $F(y,x) = g(y)$.
Case 2. If $(F(X \times X), G)$ is complete, then there exist $p, q \in F(X \times X)$ such that $F(x_n, y_n) \to p$ and $F(y_n, x_n) \to q$ as $n \to +\infty$. Since $F(X \times X) \subseteq g(X)$, so there exist $x, y \in X$ such that $p = g(x)$ and $q = g(y)$ and from here onwards the proof follows as in Case 1. □

If $g = I$, the identity mapping in Theorem 2.1, then we deduce the following result of coupled fixed point.

Corollary 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a G-metric G on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ be a continuous mapping such that F has the mixed monotone property on X and there exist two elements $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Also suppose there exist $\phi \in \Phi$ and $\theta \in \Theta$ such that*

$$\begin{aligned} &\phi(G(F(x,y), F(u,v), F(w,z))) \\ &\leq \frac{1}{2}\phi(G(x,u,w) + G(y,v,z)) - \theta(G(x,u,w), G(y,v,z)) \end{aligned} \tag{19}$$

for all $(x,y), (u,v), (w,z) \in X \times X$ with $w \preceq u \preceq x$ and $y \preceq v \preceq z$ Then, there exist $x, y \in X$ such that $F(x,y) = x$ and $F(y,x) = y$, that is, F has a coupled fixed point $(x,y) \in X \times X$.

If $\phi = g = I$, the identity mapping in Theorem 2.1, then we obtain the result of Luong and Thuan [9] in the form of following corollary.

Corollary 2.2 ([9])**.** *Let $(X, \preceq)$ be a partially ordered set and suppose there is a G-metric G on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ be a mapping such that F is continuous and has the mixed monotone property on X and there exist two elements $x_0, y_0 \in X$ with*

$x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Also suppose there exists $\theta \in \Theta$ such that

$$\begin{aligned}&G(F(x,y),F(u,v),F(w,z)\\&\leq \frac{1}{2}(G(x,u,w)+G(y,v,z))-\theta(G(x,u,w),G(y,v,z))\end{aligned}$$

for all $(x, y), (u, v), (w, z) \in X \times X$ with $w \preceq u \preceq x$ and $y \preceq v \preceq z$. Then, there exist $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$, that is, F has a coupled fixed point $(x, y) \in X \times X$.

Let Ψ denote the class of functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying $\lim_{t \to r} \psi(t) > 0$ for each $r > 0$. Now, considering $\theta(t_1, t_2) = \psi(\max\{t_1, t_2\})$ for all $t_1, t_2 \in [0, +\infty)$ with $\psi \in \Psi$ in Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a G-metric G on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ and $g : X \times X$ be mappings such that F is continuous, F has the g-mixed monotone property on X and there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Also suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned}&\phi(G(F(x,y),F(u,v),F(w,z))\\&\leq \frac{1}{2}\phi(G(g(x),g(u),g(w))+G(g(y),g(v),g(z)))\\&-\psi(\max\{G(g(x),g(u),g(w)),G(g(y),g(v),g(z))\})\end{aligned}$$

for all $(x, y), (u, v), (w, z) \in X \times X$ with $g(w) \preceq g(u) \preceq g(x)$ and $g(y) \preceq g(v) \preceq g(z)$. Then, there exist $x, y \in X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Moreover, if $\phi = g = I$, the identity mapping in Theorem 2.1, $\theta(t_1, t_2) = \frac{1-k}{2}(t_1 + t_2)$ for all $t_1, t_2 \in [0, +\infty)$, where $\theta \in \Theta$ with $k \in [0, 1)$ then we obtain the main result of Choudhury and Maity [5] in the form of the following corollary.

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a G-metric G on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ be a mapping such that F is continuous and has the mixed monotone property on X and there exist two elements $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Also suppose there exists $k \in [0, 1)$ such that*

$$G(F(x,y),F(u,v),F(w,z)\leq \frac{k}{2}(G(x,u,w)+G(y,v,z))$$

for all $(x, y), (u, v), (w, z) \in X \times X$ with $w \preceq u \preceq x$ and $y \preceq v \preceq z$. Then, there exist $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$, that is, F has a coupled fixed point $(x, y) \in X \times X$.

Now, we give sufficient conditions for uniqueness of the coupled fixed point. If $(X, \preceq)$ is a partially ordered set, then we endow the product space $X \times X$ with the following partial order:

$$\text{for } (x,y),(u,v) \in X \times X, \quad (u,v) \preceq (x,y) \Leftrightarrow x \succeq u,\ y \preceq v.$$

Theorem 2.3. *In addition to the hypotheses of Theorem 2.1, suppose that for every $(x,y),(x^*,y^*) \in X \times X$ there exists $(u,v) \in X \times X$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x,y) \in X \times X$ such that $x = g(x) = F(x,y)$ and $y = g(y) = F(y,x)$.*

Proof. From Theorem 2.1, the set of coupled coincidence points of F and g is non-empty. Suppose that (x,y) and (x^*,y^*) are coupled coincidence points of F and g, that is, $g(x) = F(x,y)$, $g(y) = F(y,x)$, $g(x^*) = F(x^*,y^*)$ and $g(y^*) = F(y^*,x^*)$, then we show that

$$g(x) = g(x^*) \text{ and } g(y) = g(y^*). \tag{20}$$

By assumption, there exists $(u,v) \in X \times X$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. Put $u_0 = u$, $v_0 = v$, and choose $u_1, v_1 \in X$ so that $g(u_1) = F(u_0,v_0)$ and $g(v_1) = F(v_0,u_0)$. Then, proceeding as in the proof of Theorem 2.1, we can inductively define sequences $\{g(u_n)\}$, $\{g(v_n)\}$ such that

$$g(u_{n+1}) = F(u_n,v_n) \text{ and } g(v_{n+1}) = F(v_n,u_n)\ \ \forall\, n \geq 0.$$

Further, set $x_0 = x, y_0 = y$, $x_0^* = x^*, y_0^* = y^*$ and, on the same way, define the sequences $\{g(x_n)\}$, $\{g(y_n)\}$, $\{g(x_n^*)\}$ and $\{g(y_n^*)\}$. Then it is easy to show that

$$\begin{aligned} g(x_n) &\to F(x,y), & g(y_n) &\to F(y,x),\\ g(x_n^*) &\to F(x^*,y^*), & g(y_n^*) &\to F(y^*,x^*) \end{aligned}$$

as $n \to +\infty$.

Since $(F(x,y),F(y,x)) = (g(x_1),g(y_1)) = (g(x),g(y))$ and $(F(u,v),F(v,u)) = (g(u_1),g(v_1))$ are comparable, then $g(x) \preceq g(u_1)$ and $g(y) \succeq g(v_1)$. It is easy to show that $(g(x),g(y))$ and $(g(u_n),g(v_n))$ are comparable, that is, $g(x) \preceq g(u_n)$ and $g(y) \succeq g(v_n)$ for all $n \geq 1$. Thus from (1), we have

$$\begin{aligned} &\phi(G(g(u_{n+1}),g(x),g(x)))\\ &= \phi(G(F(u_n,v_n),F(x,y),F(x,y)))\\ &\leq \frac{1}{2}\phi(G(g(u_n),g(x),g(x)) + G(gv_n,g(y),g(y)))\\ &\qquad - \theta(G(g(u_n),g(x),g(x)),G(g(v_n),g(y),g(y))). \end{aligned}$$

Similarly,

$$\begin{aligned}
&\phi(G(g(v_{n+1}), g(y), g(y))) \\
&= \phi(G(F(v_n, u_n), F(y, x), F(y, x))) \\
&\le \frac{1}{2}\phi(G(g(v_n), g(y), g(y)) + G(g(u_n), g(x), g(x))) \\
&\qquad - \theta(G(g(v_n), g(y), g(y)), G(g(u_n), g(x), g(x))).
\end{aligned}$$

Using the above inequalities and a property of ϕ, we have

$$\begin{aligned}
&\phi(G(g(u_{n+1}), g(x), g(x)) + G(g(v_{n+1}), g(y), g(y))) \\
&\le \phi(G(g(u_{n+1}), g(x), g(x))) + \phi(G(g(v_{n+1}), g(y), g(y))) \\
&\le \phi(G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))) \\
&\qquad - \theta(G(g(u_n), g(x), g(x)), G(g(v_n), g(y), g(y))) \\
&\qquad - \theta(G(g(v_n), g(y), g(y)), G(g(u_n), g(x), g(x))) \\
&\le \phi(G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))).
\end{aligned} \tag{21}$$

By monotonicity of ϕ, it follows that

$$\begin{aligned}
&G(g(u_{n+1}), g(x), g(x)) + G(g(v_{n+1}), g(y), g(y)) \\
&\le G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y)).
\end{aligned}$$

Let $\alpha_n = G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))$, then the sequence $\{\alpha_n\}$ is a non-increasing sequence, so there exists some $\alpha \ge 0$ such that

$$\lim_{n\to+\infty} \alpha_n = \lim_{n\to+\infty}[G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))] = \alpha.$$

We shall show that $\alpha = 0$. Suppose, to the contrary, that $\alpha > 0$. Therefore, $\{G(g(u_n), g(x), g(x))\}$ and $\{G(g(v_n), g(y), g(y))\}$ have subsequences converging to α_1 and α_2 (say) respectively. Taking the limit, up to subsequences, as $n \to +\infty$ in (21) and using the continuity of ϕ, we have

$$\begin{aligned}
\phi(\alpha) &\le \phi(\alpha) - \lim_{n\to+\infty} \theta(G(g(u_n), g(x), g(x)), G(g(v_n), g(y), g(y))) \\
&\qquad - \lim_{n\to+\infty} \theta(G(g(v_n), g(y), g(y)), G(g(u_n), g(x), g(x))) \\
&< \phi(\alpha),
\end{aligned}$$

a contradiction. Thus, $\alpha = 0$, that is

$$\lim_{n\to+\infty}[G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))] = 0.$$

Hence, it follows immediately that $g(u_n) \to g(x)$ and $g(v_n) \to g(y)$. Similarly, one can prove that $g(u_n) \to g(x^*)$ and $g(v_n) \to g(y^*)$. By uniqueness of limit, it follows that $g(x) = g(x^*)$ and $g(y) = g(y^*)$. Thus we proved (20).

Now, since $F(x, y) = g(x)$, $F(y, x) = g(y)$ and the pair (F, g) is commuting, it follows that

$$\begin{cases} g(g(x)) = g(F(x, y)) = F(g(x), g(y)), \\ g(g(y)) = g(F(y, x)) = F(g(y), g(x)). \end{cases} \tag{22}$$

Denote $g(x) = z$ and $g(y) = w$. Then by (22), we deduce

$$g(z) = F(z,w) \text{ and } g(w) = F(w,z).$$

Thus (z,w) is a coupled coincidence point. Then by (20) with $x^* = z$ and $y^* = w$, it follows that $g(z) = g(x)$ and $g(w) = g(y)$, that is $g(z) = z$ and $g(w) = w$. It follows $z = g(z) = F(z,w)$ and $w = g(w) = F(w,z)$. Therefore (z,w) is a coupled common fixed point of F and g. To prove the uniqueness, assume that (p,q) is another coupled common fixed point. Then by (20), we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. □

If $g = I$, the identity mapping in Theorem 2.3, then we deduce the following corollary.

Corollary 2.5. *In addition to the hypotheses of Corollary 2.1, suppose that for every $(x,y),(x^*,y^*) \in X \times X$ there exists a $(u,v) \in X \times X$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. Then F has a unique coupled fixed point, that is, there exists a unique $(x,y) \in X \times X$ such that $x = F(x,y)$ and $y = F(y,x)$.*

Now, we give a simple illustrative example.

Example 2.1. Let $X = [0,+\infty)$. Then $(X,\preceq)$ is a partially ordered set with the partial ordering given by

$$x \preceq y \quad \Longleftrightarrow \quad (x = y \text{ or } x,y \in [0,1] \text{ and } x \le y).$$

Let $G(x,y,z) = |x-y| + |y-z| + |z-x|$ for $x,y,z \in X$. Define $g : X \to X$ by $g(x) = x$ for all $x \in X$ and $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} \frac{x}{8} & \text{if } x \in [0,1],\ y \in X \\ x - \frac{7}{8} & \text{if } x > 1,\ y \in X. \end{cases}$$

Define also $\phi : [0,+\infty) \to [0,+\infty)$ by $\phi(t) = \frac{t}{2}$ for all $t \in [0,+\infty)$ and $\theta : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ by $\theta(t_1,t_2) = \frac{t_1+t_2}{12}$ for all $t_1,t_2 \in [0,+\infty)$. By routine calculations, the reader can easily verify that the following assumptions hold:

(i) (X,G) is a complete G-metric space;
(ii) F has the mixed monotone property;
(iii) $(x_0,y_0) = (0,1) \Rightarrow x_0 = 0 = F(x_0,y_0)$ and $y_0 = 1 > \frac{1}{8} = F(y_0,x_0)$;
(iv) F is continuous.

Here, we show only that condition (19) in Corollary 2.1 holds for all $(x,y),(u,v),(w,z) \in X \times X$ with $w \preceq u \preceq x$ and $y \preceq v \preceq z$.
We distinguish the following four cases:

Case 1. If $(x, y), (u, v), (w, z) \in [0, 1] \times [0, 1]$, we have

$$\begin{aligned}
&\phi(G(F(x, y), F(u, v), F(w, z))\\
&= \frac{1}{2}\frac{|x-u| + |u-w| + |w-x|}{8}\\
&\le \frac{1}{6}(|x-u| + |u-w| + |w-x| + |y-v| + |v-z| + |z-y|)\\
&= \frac{1}{4}(|x-u| + |u-w| + |w-x| + |y-v| + |v-z| + |z-y|)\\
&\qquad - \frac{1}{12}(|x-u| + |u-w| + |w-x| + |y-v| + |v-z| + |z-y|)\\
&= \frac{1}{2}\phi(G(x, u, w) + G(y, v, z)) - \theta(G(x, u, w), G(y, v, z)).
\end{aligned}$$

Case 2. If $(x, y), (u, v), (w, z) \in [0, 1] \times (1, +\infty)$, we have $y = v = z$ and

$$\begin{aligned}
&\phi(G(F(x, y), F(u, v), F(w, z))\\
&= \frac{1}{16}(|x-u| + |u-w| + |w-x|)\\
&\le \frac{1}{6}(|x-u| + |u-w| + |w-x|)\\
&= \frac{|x-u| + |u-w| + |w-x|}{4} - \frac{|x-u| + |u-w| + |w-x|}{12}\\
&= \frac{1}{2}\phi(G(x, u, w) + G(y, v, z)) - \theta(G(x, u, w), G(y, v, z)).
\end{aligned}$$

Case 3. If $(x, y), (u, v), (w, z) \in (1, +\infty) \times (1, +\infty)$, we have $x = u = w$, $y = v = z$ and hence

$$\phi(G(F(x, y), F(u, v), F(w, z)) = 0.$$

Case 4. If $(x, y), (u, v), (w, z) \in (1, +\infty) \times [0, 1]$, we have again

$$\phi(G(F(x, y), F(u, v), F(w, z)) = 0.$$

Thus condition (19) holds in all the cases. Hence by Corollary 2.1, F has a coupled fixed point $(0, 0) \in X \times X$.

Note that Corollary 2.1 is not applicable in respect of the usual order of real numbers because condition (19) does not hold. In fact, in this case, from condition (19) with $(x, y) = (4, 2), (u, v) = (3, 3), (w, z) = (2, 4)$, we have

$$\begin{aligned}
&\phi(G(F(x, y), F(u, v), F(w, z)) = 2\\
&\not\le \frac{4}{3} = \frac{1}{2}\phi(G(x, u, w) + G(y, v, z)) - \theta(G(x, u, w), G(y, v, z)).
\end{aligned}$$

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