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p^*-Closure Operator and p^*-Regularity in Fuzzy Setting

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Abstract. In this paper a new type of fuzzy regularity, viz. fuzzy p^*-regularity has been introduced and studied by a newly defined closure operator, viz., fuzzy p^*-closure operator. Also we have found the mutual relationship of this closure operator among other closure operators defined earlier. In p^*-regular space, p^*-closure operator is an idempotent operator. In the last section, p^*-closure operator has been characterized via p^*-convergence of a fuzzy net.

1. Introduction

Throughout the paper, by (X, τ) or simply by X we mean a fuzzy topological space (fts, for short) in the sense of Chang [3]. A fuzzy set [7] A is a mapping from a nonempty set X into a closed interval $I = [0, 1]$. The support [6] of a fuzzy set A in X will be denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. A fuzzy point [6] with the singleton support $x \in X$ and the value t $(0 < t \leq 1)$ at x will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 in X respectively. The complement [7] of a fuzzy set A in X will be denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for all $x \in X$. For two fuzzy sets A and B in X, we write $A \leq B$ if and only if $A(x) \leq B(x)$, for each $x \in X$, and AqB means A is quasi-coincident (q-coincident, for short) with B [6] if $A(x) + B(x) > 1$, for some $x \in X$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. clA and $intA$ of a fuzzy set A in X respectively stand for the fuzzy closure [3] and fuzzy interior [3] of A in X. A fuzzy set A in X is called fuzzy α-open [2] if $A \leq intclintA$. The complement of a fuzzy α-open set is called a fuzzy α-closed [2] set. The smallest fuzzy α-closed set containing a fuzzy set A is called fuzzy α-closure

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of A and is denoted by αclA [2], i.e.,

$$\alpha clA = \bigwedge \{U : A \leq U \text{ and } U \text{ is fuzzy } \alpha\text{-closed}\}$$

A fuzzy set A in X is fuzzy α-closed if $A = \alpha clA$ [2]. A fuzzy set B is called a quasi neighbourhood (q-nbd, for short) of a fuzzy set A in an fts X if there is a fuzzy open set U in X such that $AqU \leq B$. If, in addition, B is fuzzy open (resp., α-open) then B is called a fuzzy open (resp., α-open) q-nbd of A. In particular, a fuzzy set B in X is a fuzzy open (resp., α-open) q-nbd of a fuzzy point x_t in X if $x_tqU \leq B$, for some fuzzy open (resp., α-open) set U in X.

2. Fuzzy p^*-Closure Operator: Some Properties

In this section fuzzy p^*-closure operator has been introduced and studied. Let us recall a definition from [4] for ready reference.

Definition 2.1 ([4]). A fuzzy set A in an fts (X, τ) is called fuzzy preopen if $A \leq intclA$. The complement of a fuzzy preopen set is called a fuzzy preclosed set.

The union of all fuzzy preopen sets contained in a fuzzy set A is called fuzzy preinterior of A, to be denoted by $pintA$.

The intersection of all fuzzy preclosed sets containing a fuzzy set A is called fuzzy preclosure of A, to be denoted by $pclA$.

Definition 2.2. A fuzzy preopen set A in an fts (X, τ) is called a fuzzy pre-q-nbd of a fuzzy point x_t, if x_tqA.

Lemma 2.1. *For a fuzzy point x_t and a fuzzy set A in an fts (X, τ), $x_t \in pclA$ if and only if every fuzzy pre-q-nbd U of x_t, UqA.*

Proof. Let $x_t \in pclA$ and U be any fuzzy pre-q-nbd of x_t. Then $U(x) + t > 1 \Rightarrow t > 1 - U(x) \Rightarrow x_t \notin 1_X \setminus U$ which is fuzzy preclosed in X and hence by Definition 2.1, $A \not\leq 1_X \setminus U \Rightarrow$ there exists $y \in X$ such that $A(y) > (1_X \setminus U)(y) \Rightarrow A(y) + U(y) > 1 \Rightarrow AqU$.
Conversely, let for any fuzzy pre-q-nbd U of x_t, UqA. Let V be any fuzzy preclosed set containing A, i.e., $A \leq V$ ... (1). We have to show that $x_t \in V$. If possible, let $x_t \notin V$. Then $V(x) < t \Rightarrow 1 - V(x) > 1 - t \Rightarrow x_tq(1_X \setminus V)$. By assumption $(1_X \setminus V)qA \Rightarrow A > V$, contradicts (1). □

Lemma 2.2. *For any two fuzzy preopen sets A and B in an fts X, $A \not q B \Rightarrow pclA \not q B$ and $A \not q pclB$.*

Proof. If possible, let $pclAqB$. Then there exists $x \in X$ such that $pclA(x) + B(x) > 1$. Let $pclA(x) = t$. Then $B(x) + t > 1 \Rightarrow x_tqB$ and $x_t \in pclA$. By Lemma 2.1, BqA, a contradiction.
Similarly, we can prove that $A \not q pclB$. □

Definition 2.3. A fuzzy point x_t in an fts X is called fuzzy p^*-cluster point of a fuzzy set A in X if $pclUqA$ for every fuzzy pre-q-nbd U of x_t.
The union of all fuzzy p^*-cluster points of a fuzzy set A is called fuzzy p^*-closure of A, to be denoted by $[A]_p$. A fuzzy set A in X is called fuzzy p^*-closed if $A = [A]_p$. The complement of a fuzzy p^*-closed set is called fuzzy p^*-open.

Note 2.1. It is clear from Definition 2.1 and Definition 2.3 that $pclA \leq [A]_p$, for any fuzzy set A in an fts X. The converse is not true, in general, as seen from the following example.

Example 2.1. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, B\}$ where $B(a) = 0.7$, $B(b) = 0.5$. Then (X, τ) is an fts. The collection of all fuzzy preopen sets in (X, τ) is of the form $\{0_X, 1_X, B, U\}$ where $U \not\leq 1_X \setminus B$ and that of fuzzy preclosed sets is $\{0_X, 1_X, 1_X \setminus B, 1_X \setminus U\}$ where $1_X \setminus U \not\geq B$. Consider the fuzzy point $a_{0.2}$ and the fuzzy set V defined by $V(a) = V(b) = 0.1$. Then $a_{0.2} \notin pclV$ but $a_{0.2} \in [V]_p$. Indeed, $C(a) = 0.81, C(b) = 0.5$ is a fuzzy pre-q-nbd of $a_{0.2}$ but $C \not{q} V$. Although $pclC = 1_X qV$.

The following theorem shows that under which condition, the two closure operators pcl and p^* coincide.

Theorem 2.1. *For a fuzzy preopen set A in an fts (X, τ), $[A]_p = pclA$.*

Proof. By Note 2.1, it suffices to show that $[A]_p \leq pclA$, for any fuzzy preopen set A in X.
Let x_t be a fuzzy point in X such that $x_t \notin pclA$. Then there exists a fuzzy pre-q-nbd V of x_t such that $V \not{q} A$. Then $V(y) + A(y) \leq 1$, for all $y \in X \Rightarrow V(y) \leq 1 - A(y)$, for all $y \in X \Rightarrow pclV \leq pcl(1_X \setminus A) = 1_X \setminus A$ (since $1_X \setminus A$ is fuzzy preclosed in X). Thus $pclV \not{q} A$ and consequently, $x_t \notin [A]_p$. Hence $[A]_p \leq pclA$ for a fuzzy preopen set A in X. □

We now characterize fuzzy p^*-closure operator of a fuzzy set A in an fts X.

Theorem 2.2. *For any fuzzy set A in an fts (X, τ), $[A]_p = \bigcap\{[U]_p$: U is fuzzy preopen in X and $A \leq U\}$.*

Proof. Clearly, L.H.S. $\leq$ R.H.S.
If possible, let $x_t \in$ R.H.S, but $x_t \notin$ L.H.S. Then there exists a fuzzy pre-q-nbd V of x_t such that $pclV \not{q} A$ and so $A \leq 1_X \setminus pclV$ and $1_X \setminus pclV$ being fuzzy preopen set in X containing A, by our assumption, $x_t \in [1_X \setminus pclV]_p$. But $pclV \not{q} (1_X \setminus pclV)$ and so $x_t \notin [1_X \setminus pclV]_p$, a contradiction. This completes the proof. □

Remark 2.1. By Theorem 2.1 and Theorem 2.2, we can conclude that $[A]_p$ is fuzzy preclosed in X for a fuzzy set A in X.

Theorem 2.3. *In an fts (X, τ), the following hold:*

(a) *the fuzzy sets* 0_X *and* 1_X *are fuzzy* p^**-closed sets in* X*,*
(b) *for two fuzzy sets* A *and* B *in* X*, if* $A \leq B$*, then* $[A]_p \leq [B]_p$*,*
(c) *the intersection of any two fuzzy* p^**-closed sets in* X *is fuzzy* p^**-closed in* X*.*

Proof. (a) and (b) are obvious.
(c) Let A and B be any two fuzzy p^*-closed sets in X. Then $A = [A]_p$ and $B = [B]_p$. Now $A \bigwedge B \leq A$, $A \bigwedge B \leq B$. Then by (b), $[A \bigwedge B]_p \leq [A]_p$ and $[A \bigwedge B]_p \leq [B]_p$. Therefore, $[A \bigwedge B]_p \leq [A]_p \bigwedge [B]_p = A \bigwedge B$.

Conversely, let $x_t \in A \bigwedge B$. Then $x_t \in A = [A]_p$ and $x_t \in B = [B]_p$. Then $A(x) \geq t, B(x) \geq t$, i.e., $(A \bigwedge B)(x) = min\{A(x), B(x)\} \geq t$. Now for any fuzzy pre-q-nbd V of x_t, $pclVqA, pclVqB$. Then $V(x) + t > 1$. Therefore, $pclV(x) + (A \bigwedge B)(x) > 1 - t + t = 1$. Therefore, $pclVq(A \bigwedge B)$ for any fuzzy pre-q-nbd V of x_t and hence $x_t \in [A \bigwedge B]_p$. Consequently, $[A]_p \bigwedge [B]_p \leq [A \bigwedge B]_p$. □

Remark 2.2. In fact, the intersection of any collection of fuzzy p^*-closed sets is fuzzy p^*-closed. But the union of two fuzzy p^*-closed sets may not be fuzzy p^*-closed is clear from the following example.

Example 2.2. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.7$. Then (X, τ) is an fts. The collection of all fuzzy preopen sets in (X, τ) is $\{0_X, 1_X, A, U\}$ where $U \not\leq 1_X \setminus A$. Then the collection of all fuzzy preclosed sets is $\{0_X, 1_X, 1_X \setminus A, 1_X \setminus U\}$ where $1_X \setminus U \not\geq A$. Let C and D be two fuzzy sets given by $C(a) = 0.5, C(b) = 0.6$, $D(a) = 0.2, D(b) = 0.7$. Then $(C \bigvee D)(a) = 0.5, (C \bigvee D)(b) = 0.7$. Now $a_{0.6} \notin [C]_p$ as $a_{0.6}qU$ where $U(a) = 0.41, U(b) = 0.31$, but $pclU = U \not{q} C$. Again $a_{0.6} \notin [D]_p$ as $a_{0.6}qV$ where $V(a) = 0.7, V(b) = 0.2$, but $pclV = V \not{q} D$.
But for any fuzzy pre-q-nbd of $a_{0.6}$ is of the form U where $U \not\leq 1_X \setminus A$. Then $pclU = Uq(C \bigvee D)$ and consequently, $a_{0.6} \in [C \bigvee D]_p$. Therefore, $[C]_p \bigvee [D]_p < [C \bigvee D]_p$. Also $(C \bigvee D)(a) = 0.5 \not\geq 0.6$ and so $a_{0.6} \notin C \bigvee D$.

Note 2.2. It is clear from Remark 2.2 that fuzzy p^*-open sets in an fts (X, τ) may not form a base for a fuzzy topology.

Result 2.1. *We conclude that* $x_t \in [y_{t'}]_p$ *does not imply* $y_{t'} \in [x_t]_p$ *where* $x_t, y_{t'}$ $(0 < t, t' < 1)$ *are fuzzy points in* X *as shown from the following example.*

Example 2.3. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0, B(a) = 0.7, B(b) = 0$. Then (X, τ) is an fts. The collection of all fuzzy preopen sets in X is $\{0_X, 1_X, A, B, U, V\}$ where $0.3 < U(a) \leq 0.5, U(b) = 0$ and $V(a) > 0.5, 0 \leq V(b) \leq 1$. Then the collection of all fuzzy preclosed sets is $\{0_X, 1_X, 1_X \setminus A, 1_X \setminus B, 1_X \setminus U, 1_X \setminus V\}$ where $0.5 \leq 1 - U(a) < 0.7, U(b) = 1$ and $0 \leq 1 - V(a) < 0.5, 0 \leq 1 - V(b) \leq 1$. Consider the fuzzy points $a_{0.6}$ and $b_{0.1}$. We claim that $b_{0.1} \in [a_{0.6}]_p$, but $a_{0.6} \notin [b_{0.1}]_p$. Indeed, any fuzzy pre-q-nbd of $b_{0.1}$ is of the form V where $V(a) > 0.5, V(b) > 0.9$ and $pclV = W$

where $W(a) > 0.5, W(b) = 1$ and $Wqa_{0.6}$. But $D(a) = 0.41, D(b) = 0$ is a fuzzy pre-q-nbd of $a_{0.6}$ and $pclD = D \not{q} b_{0.1}$.

3. p^*-Closure Operator: Mutual Relationship with Other Closure Operators

In this section we have established some mutual relationship of p^*-closure operator with other closure operators, viz., α^*-closure operator, θ-closure operator.

First We recall some definitions for ready references.

Definition 3.1 ([5]). Let A be a fuzzy set and x_t, a fuzzy point in an fts X. x_t is called a fuzzy θ-cluster point of A if every closure of every fuzzy open q-nbd of x_t is q-coincident with A.

The union of all fuzzy θ-cluster points of A is called fuzzy θ-closure of A, to be denoted by $[A]_\theta$. A is called fuzzy θ-closed if $A = [A]_\theta$ and the complement of a fuzzy θ-closed set is called fuzzy θ-open.

Definition 3.2 ([1]). A fuzzy point x_t in an fts X is called a fuzzy α^*-cluster point of a fuzzy set A in X if $\alpha clUqA$ for every fuzzy α-open q-nbd U of x_t. The union of all fuzzy α^*-cluster points of A is called fuzzy α^*-closure of A, to be denoted by $[A]_\alpha$. A fuzzy set A is called fuzzy α^*-closed if $A = [A]_\alpha$ and the complement of fuzzy α^*-closed set is called fuzzy α^*-open.

Result 3.1. *$[A]_p \leq [A]_\theta$, for any fuzzy set A in an fts X.*

Proof. Let $x_t \in [A]_p$. Let V be any fuzzy open q-nbd of x_t. Then V is fuzzy pre-q-nbd of x_t also. As $x_t \in [A]_p$, $pclVqA \Rightarrow clVqA \Rightarrow x_t \in [A]_\theta$. □

Remark 3.1. It is clear from the following example that $[A]_p \neq [A]_\theta$, for any fuzzy set A in an fts X, in general.

Example 3.1. Consider Example 2.1. Consider the fuzzy point $a_{0.51}$ and a fuzzy set C given by $C(a) = C(b) = 0.1$. Then $U(a) = 0.5, U(b) = 0$ being a fuzzy pre-q-nbd of $a_{0.51}$, $pclU = U \not{q} C$ and so $a_{0.51} \notin [C]_p$. But other than 1_X, B is the only fuzzy open q-nbd of $a_{0.51}$ and $clB = 1_X qC$. Therefore, $a_{0.51} \in [C]_\theta$.

Result 3.2. *$[A]_p \leq [A]_\alpha$, for any fuzzy set A in an fts X.*

Proof. Let $x_t \in [A]_p$. Let U be a fuzzy α-open q-nbd of x_t. Then U is a fuzzy preopen set and hence $pclUqA \Rightarrow \alpha clUqA \Rightarrow x_t \in [A]_\alpha$. □

Remark 3.2. It is clear from the following example that $[A]_p \neq [A]_\alpha$, for any fuzzy set A in an fts X, in general.

Example 3.2. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4$, $B(a) = 0.7, B(b) = 0.5$. Then (X, τ) is an fts. The collection of all fuzzy α-open sets is $\{0_X, 1_X, A, B, V\}$ where $V \geq B$ and that of fuzzy preopen sets is $\{0_X, 1_X, A, B, U, V_1, W\}$ where $U \leq A, U \not\leq 1_X \setminus B, V_1 > 1_X \setminus A, W \geq B$.

Consider the fuzzy point $b_{0.71}$ and the fuzzy set D, defined by $D(a) = D(b) = 0.6$. Then $U_1(a) = 0.4, U_1(b) = 0.3$ is a fuzzy preopen set such that $b_{0.71} q U_1$. But $pclU_1 = U_1 \not{q} D$ and so $b_{0.71} \notin [D]_p$. All fuzzy α-open q-nbds of $b_{0.71}$ are $1_X, A, B, V$ where $V \geq B$. $\alpha clA = (1_X \setminus A)qD, \alpha clB = \alpha clV = \alpha cl 1_X = 1_X qD$ and so $b_{0.71} \in [D]_\alpha$.

Remark 3.3. The following two examples show that fuzzy p^*-closure operator and fuzzy closure operator are independent notions.

Example 3.3. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, B\}$ where $B(a) = 0.7, B(b) = 0.5$. Then (X, τ) is an fts. Consider the fuzzy point $a_{0.51}$ and the fuzzy set C given by $C(a) = C(b) = 0.4$. Then $a_{0.51} \notin [C]_p$ as U defined by $U(a) = 0.5, U(b) = 0$, being a fuzzy pre-q-nbd of $a_{0.51}$, $pclU = U \not{q} C$. But other than 1_X, B is the only fuzzy open q-nbd of $a_{0.51}$ such that BqC. Consequently, $a_{0.51} \in ClC$.

Example 3.4. Let $X = \{a\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.4, B(a) = 0.7$. Then (X, τ) is an fts. Then the collection of all fuzzy preopen sets is $\{0_X, 1_X, U, V\}$ where $U \leq A$, $V \geq B$. Consider the fuzzy point $a_{0.4}$ and the fuzzy set C given by $C(a) = 0.3$. Then B is a fuzzy open q-nbd of $a_{0.4}$, but $B \not{q} C$ and so $a_{0.4} \notin clC$. But any fuzzy pre-q-nbd of $a_{0.4}$ is of the form V and $pclV = 1_X qC$ and so $a_{0.4} \in [C]_p$.

4. Fuzzy p^*-Regular Space: Some Characterizations

In this section a new type of fuzzy regularity has been introduced and studied and shown that in this space p^*-closure operator and *pcl* operator coincide.

Definition 4.1. An fts (X, τ) is said to be fuzzy p^*-regular if for each fuzzy point x_t and each fuzzy pre-q-nbd U of x_t, there exists a fuzzy preopen set V in X such that $x_t qV \leq pclV \leq U$.

Theorem 4.1. *For an fts (X, τ), the following conditions are equivalent:*

(a) *X is fuzzy p^*-regular space.*
(b) *For any fuzzy set A in X, $[A]_p = pclA$,*
(c) *For each fuzzy point x_t and each fuzzy preclosed set F with $x_t \notin F$, there exists a fuzzy preopen set U such that $x_t \notin pclU$ and $F \leq U$.*
(d) *For each fuzzy point x_t and each fuzzy preclosed set F such that $x_t \notin F$, there exist fuzzy preopen sets U and V in X such that $x_t qU, F \leq V$ and $U \not{q} V$.*
(e) *For any fuzzy set A and any fuzzy preclosed set F with $A \not\leq F$, there exist fuzzy preopen sets U and V such that $AqU, F \leq V$ and $U \not{q} V$.*
(f) *For any fuzzy set A and any fuzzy preopen set U with AqU, there exists a fuzzy preopen set V such that $AqV \leq pclV \leq U$.*

Proof. (a) $\Rightarrow$ (b): By Note 2.1, it suffices to show that $[A]_p \leq pclA$, for any fuzzy set A in X.

Let $x_t \in [A]_p$ and V be any fuzzy pre-q-nbd of x_t. By (a), there exists a fuzzy preopen set W such that $x_t qW \leq pclW \leq V$. Since $x_t \in [A]_p$, $pclW qA$ and so VqA. Consequently, $x_t \in pclA \Rightarrow [A]_p \leq pclA$.

(b) $\Rightarrow$ (a): Let x_t be a fuzzy point in X and U be any fuzzy pre-q-nbd of x_t. Then $U(x)+t > 1 \Rightarrow x_t \notin (1_X \setminus U) = pcl(1_X \setminus U) = [1_X \setminus U]_p$ (by (b)). Then there exists a fuzzy pre-q-nbd V of x_t such that $pclV \not q(1_X \setminus U) \Rightarrow pclV \leq U$. Then $x_t qV \leq pclV \leq U \Rightarrow X$ is fuzzy p^*-regular.

(a) $\Rightarrow$ (c): Let x_t be a fuzzy point in X and F, a fuzzy preclosed set in X with $x_t \notin F$. Then $F(x) < t \Rightarrow 1 - F(x) + t > 1 \Rightarrow x_t q(1_X \setminus F)$. By (a), there exists a fuzzy preopen set W such that $x_t qW \leq pclW \leq 1_X \setminus F$. Therefore, $F \leq 1_X \setminus pclW = U$ (say) which is fuzzy preopen. Now $x_t qW \Rightarrow x_t q pintW \leq W \leq pint(pclW) \Rightarrow x_t q pint(pclW) \Rightarrow (pint(pclW))(x) + t > 1 \Rightarrow 1 - (pint(pclW))(x) < t \Rightarrow x_t \notin 1_X \setminus (pint(pclW)) \Rightarrow x_t \notin pcl(1_X \setminus pclW) \Rightarrow x_t \notin pclU$.

(c) $\Rightarrow$ (d): Let x_t be a fuzzy point in X and F, a fuzzy preclosed set in X with $x_t \notin F$. By (c), there exists a fuzzy preopen set U such that $x_t \notin pclU$ and $F \leq U$. Now $x_t \notin pclU \Rightarrow$ there exists a fuzzy pre-q-nbd W of x_t such that $W \not q U$.

(d) $\Rightarrow$ (e): Let A be any fuzzy set and F, any fuzzy preclosed set in X with $A \not\leq F$. Then there exists $x \in X$ such that $A(x) > F(x)$. Let $A(x) = t$. Then $x_t \notin F$. By (d), there exist fuzzy preopen sets U and V such that $x_t qU, F \leq V$ and $U \not q V$. Again, $U(x) + A(x) = U(x) + t > 1 \Rightarrow AqU$.

(e) $\Rightarrow$ (f): Let A be any fuzzy set and U, any fuzzy preopen set in X with AqU. Then $A \not\leq 1_X \setminus U$ which is fuzzy preclosed. By (e), there exist fuzzy preopen sets V and W such that $AqV, 1_X \setminus U \leq W$ and $V \not q W$. Then by Lemma 2.2, $pclV \not q W$. Thus $AqV \leq pclV \leq 1_X \setminus W \leq U$.

(f) $\Rightarrow$ (a): Obvious. □

Corollary 4.1. *An fts (X, τ) is fuzzy p^*-regular if and only if every fuzzy preclosed set in X is fuzzy p^*-closed in X.*

Proof. Let (X, τ) be fuzzy p^*-regular space and A, a fuzzy preclosed set in X. Then by Theorem 4.1 (a) $\Rightarrow$ (b), $A = pclA = [A]_p$ and hence A is fuzzy p^*-closed in X.

Conversely, let $A = [A]_p$ for any fuzzy preclosed set in X. Let B be any fuzzy set in X. Then $pclB = [pclB]_p$. Then $[B]_p \leq [pclB]_p = pclB$. Again from Note 2.1, $pclB \leq [B]_p$ and so $[B]_p = pclB$ for any fuzzy set B in X. Hence by Theorem 4.1 (b) $\Rightarrow$ (a), X is fuzzy p^*-regular space. □

Remark 4.1. In a fuzzy p^*-regular space (X, τ), $[[A]_p]_p = [A]_p$.

Proof. By Theorem 4.1 (a) $\Rightarrow$ (b), $[[A]_p]_p = [pclA]_p = pcl(pclA) = pclA = [A]_p$ (by Theorem 4.1 (a) $\Rightarrow$ (b)). □

5. Characterizations of Fuzzy p^*-Closure Operator Via Fuzzy Net

In this section fuzzy p^*-closure operator of a fuzzy set is characterized in terms of fuzzy p^*-cluster point of a fuzzy net and its fuzzy p^*-convergence.

Definition 5.1. A fuzzy point x_t in an fts (X, τ) is called a fuzzy p^*-cluster point of a fuzzy net $\{S_n : n \in (D, \geq)\}$ if for every fuzzy pre-q-nbd U of x_t and for any $n \in D$, there exists $m \in D$ with $m \geq n$ such that $S_m qpclU$.

Definition 5.2. A fuzzy net $\{S_n : n \in (D, \geq)\}$ in an fts (X, τ) is said to p^*-converge to a fuzzy point x_t if for any fuzzy pre-q-nbd U of x_t, there exists $m \in D$ such that $S_n qpclU$ for all $n \geq m$ $(n \in D)$. This is denoted by $S_n \overrightarrow{p^*} x_t$.

Theorem 5.1. *A fuzzy point x_t is a fuzzy p*-cluster point of a fuzzy net $\{S_n : n \in (D, \geq)\}$ in an fts (X, τ) if and only if there exists a fuzzy subnet of $\{S_n : n \in (D, \geq)\}$ which p*-converges to x_t.*

Proof. Let x_t be a fuzzy p^*-cluster point of the fuzzy net $\{S_n : n \in (D, \geq)\}$. Let $p(Q_{x_t})$ denote the set of fuzzy preclosures of all fuzzy pre-q-nbds of x_t. Then for any $A \in p(Q_{x_t})$, there exists $n \in D$ such that $S_n qA$. Let E denote the set of all ordered pairs (n, A) such that $n \in D$, $A \in p(Q_{x_t})$ and $S_n qA$. Then $(E, \gg)$ is a directed set, where $(m, A) \gg (n, B)$ $((m, A), (n, B) \in E)$ if and only if $m \geq n$ in D and $A \leq B$. Then $T : (E, \gg) \to (X, \tau)$ given by $T(m, A) = S_m$ is clearly a fuzzy subnet of $\{S_n : n \in (D, \geq)\}$.
We claim that $T \overrightarrow{p^*} x_t$. Let V be any fuzzy pre-q-nbd of x_t. Then there exists $n \in D$ such that $(n, pclV) \in E$ and so $S_n qpclV$. Now for any $(m, A) \gg (n, pclV)$, $T(m, A) = S_m qA \leq pclV \Rightarrow T(m, A) qpclV$. Consequently, $T \overrightarrow{p^*} x_t$.
Conversely, if x_t is not a fuzzy p^*-cluster point of the fuzzy net $\{S_n : n \in (D, \geq)\}$, then there exists a fuzzy pre-q-nbd U of x_t and an $n \in D$ such that $S_m \not{q} pclU$, for all $m \geq n$. Then clearly, no fuzzy subnet of the net $\{S_n : n \in (D, \geq)\}$ can p^*-converge to x_t. □

Theorem 5.2. *Let A be a fuzzy set in an fts (X, τ). A fuzzy point $x_t \in [A]_p$ if and only if there exists a fuzzy net $\{S_n : n \in (D, \geq)\}$ in A, which p*-converges to x_t.*

Proof. Let $x_t \in [A]_p$. Then for any fuzzy pre-q-nbd U of x_t, $pclUqA$, i.e., there exists $y^U \in suppA$ and a real number p_U with $0 < p_U \leq A(y^U)$ such that the fuzzy point $y^U_{p_U}$ with support y^U and value p_U belong to A and $y^U_{p_U} qpclU$. We choose and fix one such $y^U_{p_U}$ for each U. Let $\mathcal{D}$ denote the set of all fuzzy pre-q-nbds of x_t. Then $(\mathcal{D}, \succeq)$ is a directed set under inclusion relation, i.e., B, $C \in \mathcal{D}$, $B \succeq C$ iff $B \leq C$. Then $\{y^U_{p_U} \in A : y^U_{p_U} qpclU$ and $U \in \mathcal{D}\}$ is a fuzzy net in A such that it p^*-converges to x_t. Indeed, for any fuzzy pre-q-nbd U of x_t, if $V \in \mathcal{D}$ and $V \succeq U$ (i.e., $V \leq U$), then $y^V_{p_V} qpclV \leq pclU \Rightarrow y^V_{p_V} qpclU$.

Conversely, let $\{S_n : n \in (D, \geq)\}$ be a fuzzy net in A such that $S_n \overrightarrow{p^*} x_t$. Then for any fuzzy pre-q-nbd U of x_t, there exists $m \in D$ such that $n \geq m$ $\Rightarrow S_n q pclU \Rightarrow A q pclU$ (since $S_n \in A$). Hence $x_t \in [A]_p$. □

Remark 5.1. It is clear that an improved version of the converse of the last theorem can be written as "$x_t \in [A]_p$ if there exists a fuzzy net in A with x_t as a fuzzy p^*-cluster point".

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