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Strong Convergence Theorem for Generalized Mixed Equilibrium Problems and Bregman Nonexpansive Mapping in Banach Spaces

Vahid Darvish

Abstract. In this paper, we study an iterative method for a common fixed point of a Bregman strongly nonexpansive mapping in the frame work of reflexive real Banach spaces. Moreover, we prove the strong convergence theorem for finding common fixed points with the solutions of a generalized mixed equilibrium problem.

1. Introduction

Let E be a real reflexive Banach space and C a nonempty, closed and convex subset of E and E^* be the dual space of E. Let Θ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, $\Psi : C \to E^*$ be a nonlinear mapping and $\varphi : C \to \mathbb{R}$ be a real valued function. The *generalized mixed equilibrium problem* is to find $x \in C$ such that

$$\Theta(x, y) + \langle \Psi x, y - x\rangle + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $GMEP(\Theta)$, that is

$$GMEP(\Theta) = \{x \in C : \; \Theta(x, y) + \langle \Psi x, y - x\rangle + \varphi(y) \geq \varphi(x), \quad \forall y \in C\}.$$

In particular, if $\Psi \equiv 0$, the problem (1) is reduced into the *mixed equilibrium problem* [10] for finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{2}$$

The set of solutions of (2) is denoted by $MEP(\Theta, \varphi)$.

If $\varphi \equiv 0$, the problem (1) is reduced into the *generalized equilibrium problem* [29] for finding $x \in C$ such that

$$\Theta(x, y) + \langle \Psi x, y - x\rangle \geq 0, \quad \forall y \in C. \tag{3}$$

The set of solution (3) is denoted by $GEP(\Theta, \Psi)$.

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If $\Theta \equiv 0$, the problem (1) is reduced into the *mixed variational inequality of Browder type* [6] for finding $x \in C$ such that

$$\langle \Psi x, y - x \rangle + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{4}$$

The set of solution of (4) is denoted by $MVI(C, \varphi, \Psi)$.

If $\Psi \equiv 0$ and $\varphi \equiv 0$, the problem (1) is reduced into the *equilibrium problem* [2] for finding $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{5}$$

The set of solutions of (5) is denoted by $EP(\Theta)$. This problem contains fixed point problems, includes as special cases numerous problems in physics, optimization and economics. Some methods have been proposed to solve the equilibrium problem, (see [12, 14]).

The above formulation (5) was shown in [2] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games.

Equilibrium problems which were introduced by Blum and Oettli [2] and Noor and Oettli [3] in 1994 have had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization.

In [26], Reich and Sabach proposed an algorithm for finding a common fixed point of finitely many Bregman strongly nonexpansive mappings $T_i : C \to C (i = 1, 2, \dots, N)$ satisfying $\cap_{i=1}^{N} F(T_i) \neq \emptyset$ in a reflexive Banach space E as follows:

$$\begin{aligned}
x_0 &\in E, \text{chosen arbitrarily},\\
y_n^i &= T_i(x_n + e_n^i),\\
C_n^i &= \{z \in E : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\},\\
C_n &= \cap_{i=1}^{N} C_n^i,\\
Q_n^i &= \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\},\\
x_{n+1} &= proj_{C_n \cap Q_n}^{f}(x_0), \quad \forall n \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
x_0 &\in E,\\
C_0^i &= E, i = 1, 2, \dots, N,\\
y_n^i &= T_i(\nu_n + e_n^i),\\
C_{n+1}^i &= \{z \in C_n^i : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\},
\end{aligned}$$

$$\begin{aligned} C_{n+1} &= \cap_{i=1}^{N} C_{n+1}^{i}, \\ x_{n+1} &= proj_{C_{n+1}}(x_0), \quad \forall n \geq 0, \end{aligned}$$

where $proj_C^f$ is the Bregman projection with respect to f from E onto a closed and convex subset C of E. They proved that the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$.

The authors of [18] introduced the following algorithm:

$$\begin{aligned} x_1 &= x \in C \quad \text{chosen arbitrarily}, \\ z_n &= Res_H^f(x_n), \\ y_n &= \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(T_n(z_n))) \\ x_{n+1} &= \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T_n(y_n))), \end{aligned} \tag{6}$$

where H is an equilibrium bifunction and T_n is a Bregman strongly nonexpansive mapping for any $n \in \mathbb{N}$. They proved the sequence (6) converges strongly to the point $proj_{F(T)\cap EP(H)}x$.

Also, in [13] the following algorithm was considered:

$$\begin{aligned} x_1 &= x \in C \quad \text{chosen arbitrarily}, \\ z_n &= Res_{\Theta,\varphi}^f(x_n), \\ y_n &= proj_C^f \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(T(z_n))) \\ x_{n+1} &= proj_C^f \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n))), \end{aligned} \tag{7}$$

where $\varphi : C \to \mathbb{R}$ is a real-valued function, $\Theta : C \times C \to \mathbb{R}$ is an equilibrium bifunction and T is a Bregman strongly nonexpansive mapping. It was prove that the sequence $\{x_n\}$ defined in (7) converges strongly to the point $proj_{(\cap_{i=1}^N F(T_i))\cap MEP(\Theta)}x$.

In this paper, motivated by above algorithms, we present the following iterative scheme:

$$\begin{aligned} x_1 &= x \in C \quad \text{chosen arbitrarily}, \\ y_n &= Res_{\Theta,\varphi,\Psi}^f(x_n), \\ x_{n+1} &= \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n))), \end{aligned} \tag{8}$$

where $\varphi : C \to \mathbb{R}$ is a real-valued function, $\Psi : C \to E^*$ is a continuous monotone mapping, $\Theta : C \times C \to \mathbb{R}$ is an equilibrium bifunction and T is Bregman strongly nonexpansive mapping. We will prove that the sequence $\{x_n\}$ defined in (8) converges strongly to the point $proj_{F(T)\cap GMEP(\Theta,\varphi,\Psi)}x$.

2. PRELIMINARIES

Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. We denote by dom$f$, the domain of f, that is the set $\{x \in E :$

$f(x) < +\infty\}$. Let $x \in \text{int}(\text{dom}f)$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \forall y \in E\},$$

where the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

For any $x \in \text{int}(\text{dom}f)$, the right-hand derivative of f at x in the derivation $y \in E$ is defined by

$$f^{'}(x, y) := \lim_{t\searrow 0} \frac{f(x + ty) - f(x)}{t}.$$

The function f is called Gâteaux differentiable at x if $\lim_{t\searrow 0} \frac{f(x+ty)-f(x)}{t}$ exists for all $y \in E$. In this case, $f^{'}(x, y)$ coincides with $\nabla f(x)$, the value of the gradient (∇f) of f at x. The function f is called Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \text{int}(\text{dom}f)$ and f is called Fréchet differentiable at x if this limit is attain uniformly for all y which satisfies $\|y\| = 1$. The function f is uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for any $x \in C$ and $\|y\| = 1$. It is known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $\text{int}(\text{dom}f)$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp. continuous) on $\text{int}(\text{dom}f)$ (see [5]).

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function $D_f : \text{dom}f \times \text{int}(\text{dom}f) \to [0, +\infty)$ defined as follows:

$$D_f(x, y) := f(x) - f(y) - \langle\nabla f(y), x - y\rangle \tag{9}$$

is called the Bregman distance with respect to f, [11].

The Legendre function $f : E \to (-\infty, +\infty]$ is defined in [4]. It is well known that in reflexive spaces, f is Legendre function if and only if it satisfies the following conditions:

(L_1) The interior of the domain of f, $\text{int}(\text{dom}f)$, is nonempty, f is Gâteaux differentiable on $\text{int}(\text{dom}f)$ and $\text{dom}f = \text{int}(\text{dom}f)$;

(L_2) The interior of the domain of f^*, $\text{int}(\text{dom}f^*)$, is nonempty, f^* is Gâteaux differentiable on $\text{int}(\text{dom}f^*)$ and $\text{dom}f^* = \text{int}(\text{dom}f^*)$.

Since E is reflexive, we know that $(\partial f)^{-1} = \partial f^*$ (see [5]). This , with (L_1) and (L_2), imply the following equalities:

$$\nabla f = (\nabla f^*)^{-1}, \quad \text{ran}\nabla f = \text{dom}\nabla f^* = \text{int}(\text{dom}f^*)$$

and

$$\text{ran}\nabla f^* = \text{dom}(\nabla f) = \text{int}(\text{dom}f),$$

where $\text{ran}\nabla f$ denotes the range of ∇f.

When the subdifferential of f is single-valued, it coincides with the gradient $\partial f = \nabla f$, [22]. By Bauschke et al [4] the conditions (L_1) and (L_2) also yields that the function f and f^* are strictly convex on the interior of their

respective domains.
If E is a smooth and strictly convex Banach space, then an important and interesting Legendre function is $f(x) := \frac{1}{p}\|x\|^p (1 < p < \infty)$. In this case the gradient ∇f of f coincides with the generalized duality mapping of E, i.e., $\nabla f = J_p (1 < p < \infty)$. In particular, $\nabla f = I$, the identity mapping in Hilbert spaces. From now on we assume that the convex function $f : E \to (-\infty, \infty]$ is Legendre.

Definition 2.1. Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The Bregman projection of $x \in \text{int}(\text{dom} f)$ onto the nonempty, closed and convex subset $C \subset \text{dom} f$ is the necessary unique vector $proj_C^f(x) \in C$ satisfying

$$D_f(proj_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Remark 2.1. If E is a smooth and strictly convex Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then we have that $\nabla f(x) = 2Jx$ for all $x \in E$, where J is the normalized duality mapping from E in to 2^{E^*}, and hence $D_f(x, y)$ reduced to $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, for all $x, y \in E$, which is the Lyapunov function introduced by Alber [1] and Bregman projection $P_C^f(x)$ reduces to the generalized projection $\Pi_C(x)$ which is defined by

$$\phi(\Pi_C(x), x) = \min_{y \in C} \phi(y, x).$$

If $E = H$, a Hilbert space, J is the identity mapping and hence Bregman projection $P_C^f(x)$ reduced to the metric projection of H onto C, $P_C(x)$.

Definition 2.2. [9] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. f is called:

(1) *totally convex* at $x \in \text{int}(\text{dom} f)$ if its modulus of total convexity at x, that is, the function $\nu_f : \text{int}(\text{dom} f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in \text{dom} f, \|y - x\| = t\},$$

is positive whenever $t > 0$;

(2) totally convex if it is totally convex at every point $x \in \text{int}(\text{dom} f)$;

(3) totally convex on bounded sets if $\nu_f(B, t)$ is positive for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \text{int}(\text{dom} f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(B, t) := \inf\{\nu_f(x, t) : x \in B \cap \text{dom} f\}.$$

The set $lev_{\leq}^f(r) = \{x \in E : f(x) \leq r\}$ for some $r \in \mathbb{R}$ is called a sublevel of f.

Definition 2.3. [9, 26] The function $f : E \to (-\infty, +\infty]$ is called;

(1) *cofinite* if $\text{dom} f^* = E^*$;

(2) *coercive* [15] if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\|\to+\infty} f(x) = +\infty;$$

(3) *strongly coercive* if $\lim_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = +\infty$;
(4) *sequentially consistent* if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.1. [8] *The function f is totally convex on bounded subsets if and only if it is sequentially consistent.*

Lemma 2.2. [26, Proposition 2.3] *If $f : E \to (-\infty, +\infty]$ is Fréchet differentiable and totally convex, then f is cofinite.*

Lemma 2.3. [8] *Let $f : E \to (-\infty, +\infty]$ be a convex function whose domain contains at least two points.Then the following statements hold:*
(1) *f is sequentially consistent if and only if it is totally convex on bounded sets;*
(2) *If f is lower semicontinuous, then f is sequentially consistent if and only if it is uniformly convex on bounded sets;*
(3) *If f is uniformly strictly convex on bounded sets, then it is sequentially consistent and the converse implication holds when f is lower semicontinuous, Fréchet differentiable on its domain and Fréchet derivative ∇f is uniformly continuous on bounded sets.*

Lemma 2.4. [24, Proposition 2.1] *Let $f : E \to \mathbb{R}$ be uniformly Fréchet differentiable and bounded on bounded subsets of E. Then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.5. [26, Lemma 3.1] *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Let $T : C \to C$ be a nonlinear mapping. The fixed points set of T is denoted by $F(T)$, that is $F(T) = \{x \in C : Tx = x\}$. A mapping T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$. A point $p \in C$ is called an asymptotic fixed point of T (see [23, 28]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ the set of asymptotic fixed points of T.

A mapping $T : C \to \text{int}(\text{dom} f)$ with $F(T) \neq \emptyset$ is called:
(1) quasi-Bregman nonexpansive [26] with respect to f if

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T).$$

(2) Bregman relatively nonexpansive [26] with respect to f if,

$$D_f(p,Tx) \le D_f(p,x), \quad \forall x \in C, p \in F(T), \quad \text{and} \quad \widehat{F}(T) = F(T).$$

(3) Bregman strongly nonexpansive (see [7, 26]) with respect to f and $\widehat{F}(T)$ if,

$$D_f(p,Tx) \le D_f(p,x), \quad \forall x \in C, p \in \widehat{F}(T)$$

and, if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$, and

$$\lim_{z\to\infty} (D_f(p,x_n) - D_f(p,Tx_n)) = 0,$$

it follows that

$$\lim_{n\to\infty} D_f(x_n,Tx_n) = 0.$$

(4) Bregman firmly nonexpansive (for short BFNE) with respect to f if, for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \le \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle$$

equivalently,

$$\begin{aligned} &D_f(Tx,Ty) + D_f(Ty,Tx) + D_f(Tx,x) + D_f(Ty,y) \\ &\le D_f(Tx,y) + D_f(Ty,x). \end{aligned} \tag{10}$$

The existence and approximation of Bregman firmly nonexpansive mappings was studied in [23]. It is also known that if T is Bregman firmly nonexpansive and f is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E, then $F(T) = \widehat{F}(T)$ and $F(T)$ is closed and convex. It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $F(T) = \widehat{F}(T)$.

Let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Let $x \in E$ it is known from [8] that $z = proj_C^f(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z\rangle \le 0, \quad \forall y \in C.$$

We also know the following:

$$D_f(y, proj_C^f(x)) + D_f(proj_C^f(x), x) \le D_f(y,x), \quad \forall x \in E, y \in C. \tag{11}$$

Let $f : E \to \mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function. Following [1] and [11], we make use of the function $V_f : E \times E^* \to [0,\infty)$ associated with f, which is defined by

$$V_f(x,x^*) = f(x) - \langle x^*, x\rangle + f^*(x^*), \qquad \forall x \in E, x^* \in E^*.$$

Then V_f is nonexpansive and $V_f(x,x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. Moreover, by the subdifferential inequality,

$$V_f(x,x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \le V_f(x, x^* + y^*) \tag{12}$$

for all $x \in E$ and $x^*, y^* \in E^*$ [17]. In addition, if $f : E \to (-\infty, +\infty]$ is a proper lower semicontinuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous and convex function (see [19]). Hence, V_f is convex in the second variable. Thus, for all $z \in E$,

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i),$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.6. [25] *Let C be a nonempty, closed and convex subset of $int(domf)$ and $T : C \to C$ be a quasi-Bregman nonexpansive mappings with respect to f. Then $F(T)$ is closed and convex.*

For solving the generalized mixed equilibrium problem, let us assume that the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following conditions:
(A_1) $\Theta(x,x) = 0$ for all $x \in C$;
(A_2) Θ is monotone, i.e., $\Theta(x,y) + \Theta(y,x) \leq 0$ for any $x, y \in C$;
(A_3) for each $y \in C, x \mapsto \Theta(x,y)$ is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1-t)x, y) \leq \Theta(x,y);$$

(A_4) for each $x \in C, y \mapsto \Theta(x,y)$ is convex and lower semicontinuous (see [21]).

Definition 2.4. Let C be a nonempty, closed and convex subsets of a real reflexive Banach space and let φ be a lower semicontinuous and convex functional from C to $\mathbb{R}$ and $\Psi : C \to E^*$ be a continuous monotone mapping. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunctional satisfying (A_1)-(A_4). The *mixed resolvent* of Θ is the operator $Res^f_{\Theta,\varphi,\Psi} : E \to 2^C$

$$\begin{aligned} Res^f_{\Theta,\varphi,\Psi}(x) &= \{z \in C : \Theta(z,y) + \varphi(y) + \langle \Psi x, y - z\rangle \\ &\quad + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq \varphi(z), \quad \forall y \in C\}. \end{aligned} \tag{13}$$

Lemma 2.7. *Let $f : E \to (-\infty, +\infty]$ be a coercive and Gâteaux differentiable function. Let C be a closed and convex subset of E. Assume that $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional, $\Psi : C \to E^*$ be a continuous monotone mapping and the bifunctional $\Theta : C \times C \to \mathbb{R}$ satisfies conditions (A_1)-(A_4), then $dom(Res^f_{\Theta,\varphi,\Psi}) = E$.*

Proof. Since f is a coercive function, the function $h : E \times E \to (-\infty, +\infty]$ defined by

$$h(x,y) = f(y) - f(x) - \langle x^*, y - x\rangle,$$

satisfies the following for all $x^* \in E^*$ and $y \in C$

$$\lim_{\|x-y\| \to +\infty} \frac{h(x,y)}{\|x-y\|} = +\infty.$$

Then from [2, Theorem 1], there exists $\bar{x} \in C$ such that

$$\Theta(\bar{x}, y) + \langle \Psi x, y - \bar{x} \rangle + \varphi(y) - \varphi(\bar{x}) + f(y) - f(\bar{x}) - \langle x^*, y - \bar{x} \rangle \geq 0,$$

for any $y \in C$. So, we have

$$\Theta(\bar{x}, y) + \langle \Psi x, y - \bar{x} \rangle + \varphi(y) + f(y) - f(\bar{x}) - \langle x^*, y - \bar{x} \rangle \geq \varphi(\bar{x}). \tag{14}$$

We know that inequality (14) holds for $y = t\bar{x} + (1-t)\bar{y}$ where $\bar{y} \in C$ and $t \in (0, 1)$. Therefore,

$$\begin{aligned} \Theta(\bar{x}, t\bar{x} + (1-t)\bar{y}) \quad + \quad & \langle \Psi x, t\bar{x} + (1-t)\bar{y} - \bar{x} \rangle + \varphi(t\bar{x} + (1-t)\bar{y}) \\ & + f(t\bar{x} + (1-t)\bar{y}) - f(\bar{x}) - \langle x^*, t\bar{x} + (1-t)\bar{y} - \bar{x} \rangle \\ & \geq \varphi(\bar{x}) \end{aligned}$$

for all $\bar{y} \in C$. By convexity of φ we have

$$\begin{aligned} \Theta(\bar{x}, t\bar{x} + (1-t)\bar{y}) \quad + \quad & (1-t)\langle \Psi x, \bar{y} - \bar{x} \rangle + (1-t)\varphi(\bar{y}) \\ & + f(t\bar{x} + (1-t)\bar{y}) - f(\bar{x}) \\ & - \langle x^*, t\bar{x} + (1-t)\bar{y} - \bar{x} \rangle \\ & \geq (1-t)\varphi(\bar{x}). \end{aligned} \tag{15}$$

Since

$$f(t\bar{x} + (1-t)\bar{y}) - f(\bar{x}) \leq \langle \nabla f(t\bar{x} + (1-t)\bar{y}), t\bar{x} + (1-t)\bar{y} - \bar{x} \rangle,$$

we have from (15) and (A_4) that

$$\begin{aligned} t\Theta(\bar{x}, \bar{x}) + (1-t)\Theta(\bar{x}, \bar{y}) \quad + \quad & (1-t)\langle \Psi x, \bar{y} + \varphi(\bar{y}) - \bar{x} \rangle \\ & + \langle \nabla f(t\bar{x} + (1-t)\bar{y}), t\bar{x} + (1-t)\bar{y} - \bar{x} \rangle \\ & - \langle x^*, t\bar{x} + (1-t)\bar{y} - \bar{x} \rangle \geq (1-t)\varphi(\bar{x}) \end{aligned}$$

for all $\bar{y} \in C$. From (A_1) we have

$$\begin{aligned} (1-t)\Theta(\bar{x}, \bar{y}) \quad + \quad & (1-t)\langle \Psi x, \bar{y} - \bar{x} \rangle + (1-t)\varphi(\bar{y}) \\ & + \langle \nabla f(t\bar{x} + (1-t)\bar{y}), (1-t)(\bar{y} - \bar{x}) \rangle \\ & - \langle x^*, (1-t)(\bar{y} - \bar{x}) \rangle \geq (1-t)\varphi(\bar{x}). \end{aligned}$$

Equivalently

$$\begin{aligned} & (1-t)[\Theta(\bar{x}, \bar{y}) + \langle \Psi x, \bar{y} - \bar{x} \rangle + \varphi(\bar{y}) + \langle \nabla f(t\bar{x} + (1-t)\bar{y}), \bar{y} - \bar{x} \rangle \\ & - \langle x^*, \bar{y} - \bar{x} \rangle] \geq (1-t)\varphi(\bar{x}). \end{aligned}$$

So, we have

$$\Theta(\bar{x}, \bar{y}) + \langle \Psi x, \bar{y} - \bar{x} \rangle + \varphi(\bar{y}) + \langle \nabla f(t\bar{x} + (1-t)\bar{y}), \bar{y} - \bar{x} \rangle - \langle x^*, \bar{y} - \bar{x} \rangle \geq \varphi(\bar{x}),$$

for all $\bar{y} \in C$. Since f is Gâteaux differentiable function, it follows that ∇f is norm-to-weak* continuous (see [22, Proposition 2.8]. Hence, letting $t \to 1^{-1}$ we then get

$$\Theta(\bar{x}, \bar{y}) + \langle \Psi x, \bar{y} - \bar{x} \rangle + \varphi(\bar{y}) + \langle \nabla f(\bar{x}), \bar{y} - \bar{x} \rangle - \langle x^*, \bar{y} - \bar{x} \rangle \geq \varphi(\bar{x}).$$

By taking $x^* = \nabla f(x)$ we obtain $\bar{x} \in C$ such that

$$\Theta(\bar{x}, \bar{y}) + \langle \Psi x, \bar{y} - \bar{x} \rangle + \varphi(\bar{y}) + \langle \nabla f(\bar{x}) - \nabla f(x), \bar{y} - \bar{x} \rangle \geq \varphi(\bar{x}),$$

for all $\bar{y} \in C$, i.e., $\bar{x} \in Res^f_{\Theta,\varphi,\Psi}(x)$. So, $\mathrm{dom}(Res^f_{\Theta,\varphi,\Psi}) = E$.

□

Lemma 2.8. *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let C be a closed and convex subset of E. If the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies conditions (A_1)-(A_4), then*

(1) *$Res^f_{\Theta,\varphi,\Psi}$ is single-valued;*
(2) *$Res^f_{\Theta,\varphi,\Psi}$ is a BFNE operator;*
(3) *$F\left(Res^f_{\Theta,\varphi,\Psi}\right) = GMEP(\Theta)$;*
(4) *$GMEP(\Theta)$ is closed and convex;*
(5) *$D_f\left(p, Res^f_{\Theta,\varphi,\Psi}(x)\right) + D_f\left(Res^f_{\Theta,\varphi,\Psi}(x), x\right) \leq D_f(p, x)$, $\forall p \in F\left(Res^f_{\Theta,\varphi,\Psi}\right), x \in E$.*

Proof. (1) Let $z_1, z_2 \in Res^f_{\Theta,\varphi,\Psi}(x)$ then by definition of the resolvent we have

$$\Theta(z_1, z_2) + \langle \Psi x, z_1 - z_2 \rangle + \varphi(z_2) + \langle \nabla f(z_1) - \nabla f(x), z_2 - z_1 \rangle \geq \varphi(z_1)$$

and

$$\Theta(z_2, z_1) + \langle \Psi x, z_2 - z_1 \rangle + \varphi(z_1) + \langle \nabla f(z_2 - \nabla f(x), z_1 - z_2 \rangle \geq \varphi(z_2).$$

Adding these two inequalities, we obtain

$$\begin{aligned}&\Theta(z_1, z_2) + \Theta(z_2, z_1) + \langle \Psi x, z_1 - z_2 \rangle + \langle \Psi x, z_2 - z_1 \rangle + \varphi(z_1) + \varphi(z_2)\\&+\langle \nabla f(z_2) - \nabla f(z_1), z_1 - z_2 \rangle\\&\geq \varphi(z_1) + \varphi(z_2).\end{aligned}$$

So,

$$\Theta(z_1, z_2) + \Theta(z_2, z_1) + \langle \nabla f(z_2) - \nabla f(z_1), z_1 - z_2 \rangle \geq 0.$$

By (A_2), we have

$$\langle \nabla f(z_2) - \nabla f(z_1), z_1 - z_2 \rangle \geq 0.$$

Since f is Legendre then it is strictly convex. So, ∇f is strictly monotone and hence $z_1 = z_2$. It follows that $Res^f_{\Theta,\varphi,\Psi}$ is single-valued.

(2) Let $x, y \in E$, we then have

$$\begin{aligned}\Theta(Res^f_{\Theta,\varphi,\Psi}(x), Res^f_{\Theta,\varphi,\Psi}(y)) \quad + \quad &\langle \Psi x, Res^f_{\Theta,\varphi,\Psi}(y) - Res^f_{\Theta,\varphi,\Psi}(x) \rangle\\&+\varphi(Res^f_{\Theta,\varphi,\Psi}(y))\end{aligned}$$

$$+\langle \nabla f(Res^f_{\Theta,\varphi,\Psi}(x)) - \nabla f(x), Res^f_{\Theta,\varphi,\Psi}(y)$$
$$-Res^f_{\Theta,\varphi,\Psi}(x)\rangle$$
$$\geq \varphi(Res^f_{\Theta,\varphi,\Psi}(x)) \tag{16}$$

and

$$\begin{aligned}\Theta(Res^f_{\Theta,\varphi,\Psi}(y), Res^f_{\Theta,\varphi,\Psi}(x)) \quad + \quad & \langle \Psi x, Res^f_{\Theta,\varphi,\Psi}(x) - Res^f_{\Theta,\varphi,\Psi}(y)\rangle \\ & +\varphi(Res^f_{\Theta,\varphi,\Psi}(x)) \\ & +\langle \nabla f(Res^f_{\Theta,\varphi,\Psi}(y) - \nabla f(y), Res^f_{\Theta,\varphi,\Psi}(x) \\ & -Res^f_{\Theta,\varphi,\Psi}(y)\rangle \\ & \geq \varphi(Res^f_{\Theta,\varphi,\Psi}(y)). \end{aligned} \tag{17}$$

Adding the inequalities (16) and (17), we have

$$\begin{aligned}&\Theta(Res^f_{\Theta,\varphi,\Psi}(x), Res^f_{\Theta,\varphi,\Psi}(y)) + \Theta(Res^f_{\Theta,\varphi,\Psi}(y), Res^f_{\Theta,\varphi,\Psi}(x)) \\ &+\langle \nabla f(Res^f_{\Theta,\varphi,\Psi}(x)) \\ &-\nabla f(x) + \nabla f(y) - \nabla f(Res^f_{\Theta,\varphi,\Psi}(y)), Res^f_{\Theta,\varphi,\Psi}(y) - Res^f_{\Theta,\varphi,\Psi}(x)\rangle \geq 0.\end{aligned}$$

By (A_2), we obtain

$$\begin{aligned}&\langle \nabla f(Res^f_{\Theta,\varphi,\Psi}(x)) - \nabla f(Res^f_{\Theta,\varphi,\Psi}(y)), Res^f_{\Theta,\varphi,\Psi}(x) - Res^f_{\Theta,\varphi,\Psi}(y)\rangle \\ &\leq \langle \nabla f(x) - \nabla f(y), Res^f_{\Theta,\varphi,\Psi}(x) - Res^f_{\Theta,\varphi,\Psi}(y)\rangle.\end{aligned}$$

It means $Res^f_{\Theta,\varphi,\Psi}$ is BFNE operator.

(3)

$$\begin{aligned} x \in F(Res^f_{\Theta,\varphi,\Psi}) \quad &\Leftrightarrow \quad x = Res^f_{\Theta,\varphi,\Psi}(x) \\ &\Leftrightarrow \quad \Theta(x,y) + \langle \Psi x, y - x\rangle + \varphi(y) \\ &\qquad +\langle \nabla f(x) - \nabla f(x), y - x\rangle \geq \varphi(x), \quad \forall y \in C \\ &\Leftrightarrow \quad \Theta(x,y) + \langle \Psi x, y - x\rangle + \varphi(y) \geq \varphi(x), \quad \forall y \in C \\ &\Leftrightarrow \quad x \in GMEP(\Theta). \end{aligned}$$

(4) Since $Res^f_{\Theta,\varphi,\Psi}$ is a BFNE operator, it follows from [25, Lemma 1.3.1] that $F(Res^f_{\Theta,\varphi,\Psi})$ is a closed and convex subset of C. So, from (3) we have $GMEP(\Theta) = F(Res^f_{\Theta,\varphi,\Psi})$ is a closed and convex subset of C.

(5) Since $Res^f_{\Theta,\varphi,\Psi}$ is a BFNE operator, we have from (10) that for all

$x, y \in E$

$$
\begin{aligned}
&D_f(Res^f_{\Theta,\varphi,\Psi}(x), Res^f_{\Theta,\varphi,\Psi}(y)) + D_f(Res^f_{\Theta,\varphi,\Psi}(y), Res^f_{\Theta,\varphi,\Psi}(x)) \\
&\leq D_f(Res^f_{\Theta,\varphi,\Psi}(x), y) - D_f(Res^f_{\Theta,\varphi,\Psi}(x), x) + D_f(Res^f_{\Theta,\varphi,\Psi}(y), x) \\
&-D_f(Res^f_{\Theta,\varphi,\Psi}(y), y).
\end{aligned}
$$

Let $y = p \in F(Res^f_{\Theta,\varphi,\Psi})$, we then get

$$
\begin{aligned}
&D_f(Res^f_{\Theta,\varphi,\Psi}(x), p) + D_f(p, Res^f_{\Theta,\varphi,\Psi}(x)) \\
&\leq D_f(Res^f_{\Theta,\varphi,\Psi}(x), p) - D_f(Res^f_{\Theta,\varphi,\Psi}(x), x) + D_f(p, x) - D_f(p, p).
\end{aligned}
$$

Hence,

$$
D_f(p, Res^f_{\Theta,\varphi,\Psi}(x)) + D_f(Res^f_{\Theta,\varphi,\Psi}(x), x) \leq D_f(p, x).
$$

□

Lemma 2.9. [30] *Assume that $\{x_n\}$ is a sequence of nonnegative real numbers such that*

$$
x_{n+1} \leq (1 - \alpha_n)x_n + \beta_n, \qquad \forall n \geq 1,
$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\beta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
(2) $\limsup_{n\to\infty} \frac{\beta_n}{x_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < +\infty$.

Then $\lim_{n\to\infty} x_n = 0$.

3. MAIN RESULT

Theorem 3.1. *Let E be a real reflexive Banach space, C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let T be a Bregman strongly nonexpansive mappings with respect to f such that $F(T) = \widehat{F}(T)$ and T is uniformly continuous. Let $\Theta : C \times C \to \mathbb{R}$ satisfying conditions (A_1)-(A_4) and $F(T) \cap GMEP(\Theta)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by*

$$
\begin{aligned}
x_1 &= x \in C \quad \textit{chosen arbitrarily}, \\
y_n &= Res^f_{\Theta,\varphi,\Psi}(x_n), \\
x_{n+1} &= \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(T(y_n))),
\end{aligned} \tag{18}
$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $proj_{F(T)\cap GMEP(\Theta)}x$.

Proof. We note from Lemma 2.6 that $F(T)$ is closed and convex. Let $p = proj_{F(T)\cap GMEP(\Theta)}x \in F(T) \cap GMEP(\Theta)$. Then $p \in F(T)$ and $p \in$

$GMEP(\Theta)$. By (18) and Lemma 2.8, we have $D_f(p, y_n) = D_f(p, Res^f_{\Theta,\varphi,\Psi}(x_n)) \leq D_f(p, x_n)$, so

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n)))) \\ &\leq \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, T(y_n)) \\ &\leq \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, y_n) \\ &\leq \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, x_n) \\ &\leq D_f(p, x_n). \end{aligned}$$

Hence $\{D_f(p, x_n)\}$ is bounded. Using [16, Proposition 5] we obtain that $\{x_n\}$ is also bounded. Since $\{D_f(p, x_n)\}$ is bounded, there exists $M > 0$ such that

$$f(p) - \langle \nabla f(x_n), p\rangle + f^*(\nabla f(x_n)) = V_f(p, \nabla f(x_n)) = D_f(p, x_n) \leq M.$$

Therefore, $\{\nabla f(x_n)\}$ is contained in the sublevel set $lev^{\psi}_{\leq}(M - f(p)$, where $\psi = f^* - \langle \cdot, p\rangle$. Since f is lower semicontinuous, f^* is weak* lower semicontinuous. Hence the function ψ is coercive by Moreau-Rockafellar Theorem (see [27, Theorem 7A] and [20]). This shows that $\{\nabla f(x_n)\}$ is bounded. Since f is strongly coercive, f^* is bounded on bounded sets (see [31, Lemma 3.6.1] and [4, Theorem 3.3]). Hence, ∇f^* is also bounded on bounded subset of E^* (see [8, Proposition 1.1.11]). Since f is a Legendre function, it follows that $x_n = \nabla f^*(\nabla f(x_n))$ is bounded for all $n \in \mathbb{N}$.
By (18) we have

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(T(y_n))\|_* = \lim_{n\to\infty} \alpha_n \|\nabla f(x_n) - \nabla f(T(y_n))\|_*.$$

Since $\alpha_n \to 0$ when $n \to \infty$ we have

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(T(y_n))\|_* = 0.$$

Since f is strongly coercive and uniformly convex on bounded subsets of E, then f^* is uniformly Fréchet differentiable on bounded subsets of E^*. Moreover, f^* is bounded on bounded subsets. Since f is Legendre function we have

$$(19)\quad \lim_{n\to\infty} \|x_{n+1} - T(y_n)\| = \lim_{n\to\infty} \|\nabla f^* \nabla f(x_{n+1}) - \nabla f^* \nabla f(T(y_n))\| = 0.$$

On the other hand, since f is uniformly Fréchet differentiable on bounded subsets of E, f is uniformly continuous on bounded subsets of E. It follows that

$$\lim_{n\to\infty} \|f(x_{n+1}) - f(T(y_n))\| = 0.$$

From (11) and (18), we have

$$\begin{aligned} \lim_{n\to\infty} D_f(x_n, y_n) &= \lim_{n\to\infty} D_f(x_n, Res^f_{\Theta,\varphi,\Psi} x_n) \\ &\leq \lim_{n\to\infty} [D_f(p, Res^f_{\Theta,\varphi,\Psi} x_n) - D_f(p, x_n)] \end{aligned}$$

$$\begin{aligned} &\leq \lim_{n\to\infty}[D_f(p,x_n) - D_f(p,x_n)] \\ &= 0. \end{aligned}$$

By Lemma 2.1, we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{20}$$

Now, we claim that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{21}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, by Lemma 2.4, ∇f is norm-to-norm uniformly continuous on bounded subsets of E. So,

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_n)\|_* = 0. \tag{22}$$

Since f is uniformly Fréchet differentiable, it is also uniformly continuous, we get

$$\lim_{n\to\infty} \|f(x_n) - f(y_n)\| = 0. \tag{23}$$

By Bregman distance we have

$$\begin{aligned} &D_f(p,x_n) - D_f(p,y_n) \\ &= f(p) - f(x_n) - \langle \nabla f(x_n), p - x_n\rangle - f(p) + f(y_n) + \langle \nabla f(y_n), p - y_n\rangle \\ &= f(y_n) - f(x_n) + \langle \nabla f(y_n), p - y_n\rangle - \langle \nabla f(x_n), p - x_n\rangle \\ &= f(y_n) - f(x_n) + \langle \nabla f(y_n), x_n - y_n\rangle - \langle \nabla f(y_n) - \nabla f(x_n), p - x_n\rangle, \end{aligned}$$

for each $p \in F(T)$. By (20)-(23), we obtain

$$\lim_{n\to\infty} (D_f(p,x_n) - D_f(p,y_n)) = 0. \tag{24}$$

By above equation, we have

$$\begin{aligned} D_f(y_n, x_{n+1}) &= D_f(p, x_{n+1}) - D_f(p,y_n) \\ &= D_f(p, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(x_n)) - D_f(p,y_n)) \\ &\leq \alpha_n D_f(p,x_n) + (1-\alpha_n) D_f(p, T(y_n) - D_f(p,y_n) \\ &\leq \alpha_n D_f(p,x_n) + (1-\alpha_n) D_f(p,y_n) - D_f(p,y_n) \\ &= \alpha_n (D_f(p,x_n) - D_f(p,y_n)). \end{aligned}$$

By Lemma 2.1, we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0.$$

From above equation and (19), we can write

$$\begin{aligned} \|y_n - T(y_n)\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - T(y_n)\| \\ &= 0 \end{aligned} \tag{25}$$

when $n \to \infty$. By applying the triangle inequality, we get

$$\|x_n - T(x_n)\| \leq \|x_n - y_n\| + \|y_n - T(y_n)\| + \|T(y_n) - T(x_n)\|.$$

By (20), (25) and since T is uniformly continuous, we have

$$\lim_{n\to\infty} \|x_n - T(x_n)\| = 0.$$

As claimed in (21).
Since $\|x_{n_k} - T(x_{n_k})\| \to 0$ as $k \to \infty$, we have $q \in F(T)$.
From (23) we can write

$$\lim_{n\to\infty} \|Jy_n - Jx_n\| = 0.$$

Here, we prove that $q \in GMEP(\Theta)$. For this reason, consider that $y_n = Res^f_{\Theta,\varphi,\Psi}(x_n)$, so we have

$$\Theta(y_n, y) + \langle \Psi x_n, y - y_n\rangle + \varphi(y) + \langle Jy_n - Jx_n, y - y_n\rangle \geq \varphi(y_n), \quad \forall y \in C.$$

From (A_2), we have

$$\begin{aligned} \Theta(y, y_n) &\leq -\Theta(y_n, y) \\ &\leq \langle \Psi x_n, y - y_n\rangle + \varphi(y) - \varphi(y_n) \\ &\quad + \langle Jy_n - Jx_n, y - y_n\rangle, \quad \forall y \in C. \end{aligned}$$

Hence,

$$\Theta(y, y_{n_i}) \leq \langle \Psi x_{n_i}, y - y_{n_i}\rangle + \varphi(y) - \varphi(y_{n_i}) + \langle Jy_{n_i} - Jx_{n_i}, y - y_{n_i}\rangle, \quad \forall y \in C.$$

Since $y_{n_i} \rightharpoonup q$ and from the weak lower semicontinuity of φ and $\Theta(x, y)$ in the second variable y, we also have

$$\Theta(y, q) + \langle \Psi q, q - y\rangle + \varphi(q) - \varphi(y) \leq 0, \quad \forall y \in C.$$

For t with $0 \leq t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)q$. Since $y \in C$ and $q \in C$ we have $y_t \in C$ and hence $\Theta(y_t, q) + \langle \Psi q, q - y_t\rangle + \varphi(q) - \varphi(y_t) \leq 0$. So, from the continuity of the equilibrium bifunction $\Theta(x, y)$ in the second variable y, we have

$$\begin{aligned} 0 &= \Theta(y_t, y_t) + \langle \Psi q, y_t - y_t\rangle + \varphi(y_t) - \varphi(y_t) \\ &\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, q) + t\langle \Psi q, y - y_t\rangle + (1-t)\langle \Psi q, q - y_t\rangle \\ &\quad + t\varphi(y) + (1-t)\varphi(q) - \varphi(y_t) \\ &\leq t[\Theta(y_t, y) + \langle \Psi q, y - y_t\rangle + \varphi(y) - \varphi(y_t)]. \end{aligned}$$

Therefore, $\Theta(y_t, y) + \langle \Psi q, y - y_t\rangle + \varphi(y) - \varphi(y_t) \geq 0$. Then, we have

$$\Theta(q, y) + \langle \Psi q, y - q\rangle + \varphi(y) - \varphi(q) \geq 0, \quad \forall y \in C.$$

Hence we have $q \in GMEP(\Theta)$. We showed that $q \in F(T) \cap GMEP(\Theta)$.
Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \rightharpoonup q \in C$ and

$$\limsup_{n\to\infty} \langle \nabla f(x_n) - \nabla f(p), x_n - p\rangle = \langle \nabla f(x_n) - \nabla f(p), q - p\rangle.$$

On the other hand, since $\|x_{n_k} - Tx_{n_k}\| \to 0$ as $k \to \infty$, we have $q \in F(T)$. It follows from the definition of the Bregman projection that

$$\limsup_{n\to\infty} \langle \nabla f(x_n) - \nabla f(p), x_n - p\rangle = \langle \nabla f(x_n) - \nabla f(p), q - p\rangle \leq 0. \tag{26}$$

From (12), we obtain

$$\begin{aligned} D_f(p, x_{n+1}) &= V_f(p, \alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n)) \\ &\leq V_f(p, \alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n)) \\ &\quad -\alpha_n(\nabla f(x_n) - \nabla f(p))) \\ &\quad +\langle \alpha_n(\nabla f(x_n) - \nabla f(p)), x_{n+1} - p\rangle \\ &= V_f(p, \alpha_n \nabla f(p) + (1-\alpha_n)\nabla f(T(y_n)) \\ &\quad +\alpha_n\langle \nabla f(x_n) - \nabla f(p), x_{n+1} - p\rangle \\ &\leq \alpha_n V_f(p, \nabla f(p)) + (1-\alpha_n) V_f(p, \nabla f(T(y_n))) \\ &\quad +\alpha_n\langle \nabla f(x_n) - \nabla f(p), x_{n+1} - p\rangle \\ &= (1-\alpha_n) D_f(p, T(y_n) + \alpha_n\langle \nabla f(x_n) - \nabla f(p), x_{n+1} - p\rangle \\ &\leq (1-\alpha_n) D_f(p, x_n) + \alpha_n\langle \nabla f(x_n) - \nabla f(p), x_{n+1} - p\rangle. \end{aligned}$$

By Lemma 2.9 and (26), we can conclude that $\lim_{n\to\infty} D_f(p, x_n) = 0$. Therefore, by Lemma 2.1, $x_n \to p$. This completes the proof. □

If in Theorem 3.1, we consider $\Theta \equiv 0$, we have the following corollary.

Corollary 3.1. *Let E be a real reflexive Banach space, C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let T be a Bregman strongly nonexpansive mappings with respect to f such that $F(T) = \widehat{F}(T)$ and T is uniformly continuous. Let $F(T) \cap MVI(C, \varphi, \Psi)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_1 &= x \in C \quad \textit{chosen arbitrarily}, \\ y_n &= Res^f_{\varphi,\Psi}(x_n), \\ x_{n+1} &= \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n))), \end{aligned}$$

where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $proj_{F(T)\cap MVI(C,\varphi,\Psi)} x$.

If in Theorem 3.1, we consider $\Psi \equiv 0$, we have the following corollary.

Corollary 3.2. *Let E be a real reflexive Banach space, C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let T be a Bregman strongly nonexpansive mappings with respect to f such that $F(T) = \widehat{F}(T)$ and T is uniformly continuous.*

Let $\Theta : C \times C \to \mathbb{R}$ satisfying conditions (A_1)-(A_4) and $F(T) \cap MEP(\Theta, \varphi)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_1 &= x \in C \quad \text{chosen arbitrarily}, \\ y_n &= Res^f_{\Theta,\varphi}(x_n), \\ x_{n+1} &= \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n))), \end{aligned}$$

where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $proj_{F(T)\cap MEP(\Theta,\varphi)}x$.

If in Theorem 3.1, we consider $\varphi \equiv 0$, we have the following corollary.

Corollary 3.3. *Let E be a real reflexive Banach space, C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let T be a Bregman strongly nonexpansive mappings with respect to f such that $F(T) = \widehat{F}(T)$ and T is uniformly continuous. Let $\Theta : C \times C \to \mathbb{R}$ satisfying conditions (A_1)-(A_4) and $F(T) \cap GEP(\Theta, \Psi)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_1 &= x \in C \quad \text{chosen arbitrarily}, \\ y_n &= Res^f_{\Theta,\Psi}(x_n), \\ x_{n+1} &= \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(y_n))), \end{aligned}$$

where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $proj_{F(T)\cap GEP(\Theta,\Psi)}x$.

If in Theorem 3.1, we assume that E is a uniformly smooth and uniformly convex Banach space and $f(x) := \frac{1}{p}\|x\|^p \quad (1 < p < \infty)$, we have that $\nabla f = J_p$, where J_p is the generalization duality mapping from E onto E^*. Thus, we get the following corollary.

Corollary 3.4. *Let E be a uniformly smooth and uniformly convex Banach space and $f(x) := \frac{1}{p}\|x\|^p \quad (1 < p < \infty)$. Let C be a nonempty, closed and convex subset of $int(domf)$ and T be a finite family of Bregman strongly nonexpansive mappings with respect to f such that $F(T) = \widehat{F}(T)$ and T is uniformly continuous. Let $\Theta : C \times C \to \mathbb{R}$ satisfying conditions (A_1)-(A_4) and $F(T) \cap GMEP(\Theta)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_1 &= x \in C \quad \text{chosen arbitrarily}, \\ y_n &= Res^f_{\Theta,\varphi,\Psi}(x_n), \\ x_{n+1} &= J_p^{-1}(\alpha_n J_p(x_n) + (1-\alpha_n) J_p(T(y_n))), \end{aligned}$$

where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $proj_{F(T)\cap GMEP(\Theta)}x$.

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VAHID DARVISH
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
AMIRKABIR UNIVERSITY OF TECHNOLOGY
HAFEZ AVE., P.O. BOX 15875-4413
TEHRAN, IRAN
E-mail address: vahid.darvish@mail.com