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Optimal harvesting policy for the Beverton–Holt quantum difference model

Martin Bohner and Sabrina Streipert

Abstract. In this paper, we introduce exploitation to the Beverton–Holt equation in the quantum calculus time setting. We first give a brief introduction to quantum calculus and to the Beverton–Holt q-difference equation before formulating the harvested Beverton–Holt q-difference equation. Under the assumption of a periodic carrying capacity and periodic inherent growth rate, we derive its unique periodic solution, which globally attracts all solutions. We further derive the optimal harvest effort for the Beverton–Holt q-difference equation under the catch-per-effort hypothesis. Examples are provided and discussed in the last section.

1. Introduction

Beverton and Holt introduced their population model in the context of fisheries [3] in 1957 as

$$x_{n+1} = \frac{\nu K x_n}{K + (\nu - 1)x_n}, \quad n \in \mathbb{N}_0, \tag{1}$$

where $x_0 > 0$, $\nu > 1$ is the inherent growth rate, and $K > 0$ is the carrying capacity.

The model is applied in various fields such as biology, economy and social science, see [2, 3, 18, 15]. To achieve a more realistic presentation of population dynamics, additional assumptions have been added to the traditional model such as contest competition [12], within-year resource limited competition [14], and survivor-rates [13]. In [17], the authors considered modifications of the Beverton–Holt model and the authors of [16] discussed the sigmoid Beverton–Holt model.

In [11], the authors investigated (1) on time scales. Recently, assuming a periodically forced environment and periodic growth rate, (1) was analyzed and the Cushing–Henson conjectures for the case of periodic coefficients were discussed in [7].

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In [10], the authors discussed the Beverton–Holt equation with exploitation that reads as

$$(1 + h_n)x_{n+1} = \frac{K_n \nu_n x_n}{K_n + (\nu_n - 1)x_n}, \tag{2}$$

where h represents the harvest effort.

The following theorems were proved in [10].

Theorem 1.1 (See [10]). *Assume*

$$\begin{cases} K\colon \mathbb{N}_0 \to \mathbb{R}^+ \text{ is } \omega\text{-periodic}, \\ \alpha : \mathbb{N}_0 \to \mathbb{R}^+ \text{ is } \omega\text{-periodic and } 0 < \alpha_n < 1 \text{ for all } n \in \mathbb{N}_0, \\ h : \mathbb{N}_0 \to \mathbb{R}^+ \text{ is } \omega\text{-periodic and } 0 < h_n < \frac{\alpha_n}{1-\alpha_n} \text{ for all } n \in \mathbb{N}_0. \end{cases} \tag{3}$$

Then (2) *has a unique ω-periodic solution which globally attracts all its solutions.*

Theorem 1.2 (See [10]). *Assume* (3) *and (in order to guarantee a nonnegative harvest effort)*

$$\frac{K_n^{\Delta}}{K_n} \leq \frac{1 + \sqrt{1 - \alpha_{n+1}}}{(1 + \sqrt{1 - \alpha_n})\sqrt{1 - \alpha_{n+1}}} - 1.$$

The optimal harvest effort for (2) *is*

$$h^* = \ominus \left(\frac{1}{2} \odot (-\alpha) \right)^{\sigma} \ominus \frac{\left(\frac{\frac{1}{2} \odot (-\alpha)}{\alpha} \right)^{\Delta}}{\frac{\frac{1}{2} \odot (-\alpha)}{\alpha}} \ominus \frac{K^{\Delta}}{K},$$

and the maximal harvest yield over one period is

$$Y(h^*) = \sum_{j=0}^{\omega-1} \frac{\left(\frac{1}{2} \odot (-\alpha_j) \right)^2}{\alpha_j} K_j = \sum_{j=0}^{\omega-1} \frac{(1 - \sqrt{1 - \alpha_j})^2}{\alpha_j} K_j.$$

In this paper, we include exploitation to the periodically forced Beverton–Holt equation in the quantum calculus setting, which is classically defined as

$$x(qt) = \frac{\nu(t)K(t)x(t)}{K(t) + (\nu(t) - 1)x(t)},$$

where $x_0 > 0$, and $\nu, K : q^{\mathbb{N}_0} \to \mathbb{R}$ are the inherent growth rate and carrying capacity, respectively.

In [4], the authors analyzed the solution of classical quantum Beverton–Holt model for one-periodic growth rate ν and also discussed the Cushing–Henson conjectures for the case of a one-periodic inherent growth rate. The case of a one-periodic inherent growth rate for the q-difference equation corresponds to a constant inherent growth rate in the classical Beverton–Holt differential/difference equation. In [8, 9], the Beverton–Holt q-difference equation, assuming periodic growth rate and periodic carrying capacity, as investigated and formulations related to the Cushing–Henson conjectures were presented. In this work, we continue the discussion of the Beverton–Holt q-difference equation from an economical perspective by including exploitation by a catch-per-effort hypothesis. We formulate the model and derive its periodic solution, which is shown to be globally asymptotically stable. Further, the maximum sustainable yield for the harvested Beverton–Holt q-difference equation is derived.

2. Some Quantum Calculus Essentials

In this section, we provide some quantum calculus prerequisites. Throughout, let $q > 1$.

Definition 2.1 (See [5, Definition 1.1])**.** The forward jump operator $\sigma : q^{\mathbb{N}_0} \to q^{\mathbb{N}_0}$ is defined by

$$\sigma(t) := qt, \quad t \in q^{\mathbb{N}_0}.$$

Definition 2.2 (See [5, Definition 2.25])**.** A function $p : q^{\mathbb{N}_0} \to \mathbb{R}$ is called regressive provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in q^{\mathbb{N}_0}, \text{ where } \mu(t) = \sigma(t) - t = (q-1)t.$$

The set of all regressive functions is denoted by $\mathcal{R}$. Moreover, $p \in \mathcal{R}$ is called positively regressive, denoted by $p \in \mathcal{R}^+$, if

$$1 + \mu(t)p(t) > 0 \quad \text{for all } t \in q^{\mathbb{N}_0}.$$

Using the introduced function μ, the derivative can be defined as follows.

Definition 2.3. The derivative of a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is given by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(qt) - f(t)}{(q-1)t} \quad \text{for all } t \in q^{\mathbb{N}_0}.$$

Definition 2.4 (See [4])**.** Let $p \in \mathcal{R}$ and $s \in q^{\mathbb{N}_0}$. The exponential function is defined by

$$e_p(t,s) = \prod_{k \in [s,t) \cap q^{\mathbb{N}_0}} (1 + (q-1)kp(k)) \quad \text{for all } t \in q^{\mathbb{N}_0} \text{with } t > s,$$

$e_p(s,s) = 1$, and $e_p(t,s) = \frac{1}{e_p(s,t)}$ for $t < s$.

It is not hard to show that the following property holds.

Theorem 2.1. *If $p \in \mathcal{R}$, then $e_p(t,s) = e_p(t,r)e_p(r,s)$ for all $s,t,r \in q^{\mathbb{N}_0}$.*

Theorem 2.2 (See [5, Theorem 2.44])**.** *If $p \in \mathcal{R}^+$ and $t_0 \in q^{\mathbb{N}_0}$, then $e_p(t,t_0) > 0$ for all $t \in q^{\mathbb{N}_0}$.*

Theorem 2.3 (See [5, Theorem 2.62])**.** *Suppose $p \in \mathcal{R}$. Let $t_0 \in q^{\mathbb{N}_0}$ and $y_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$y^{\Delta} = p(t)y, \quad y(t_0) = y_0$$

is given by

$$y = e_p(\cdot, t_0)y_0.$$

The integral in quantum calculus is defined in the following way.

Definition 2.5 (See [4, Definition 2.6])**.** Let m, $n \in \mathbb{N}_0$ with $m < n$. For $f : q^{\mathbb{N}_0} \to \mathbb{R}$, we define

$$\int_{q^m}^{q^n} f(t)\,\Delta t := (q-1)\sum_{k=m}^{n-1} q^k f(q^k).$$

Theorem 2.4 (See [5, Theorem 2.36 and 2.39])**.** *If $p \in \mathcal{R}$ and $a, b, c \in q^{\mathbb{N}_0}$, then*

(4) $$\int_a^b p(t)e_p(t,c)\Delta t = e_p(b,c) - e_p(a,c),$$

(5) $$\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b).$$

In particular in the last section, the following operations will be used.

Definition 2.6 (See [6, p. 10])**.** Define the "circle plus" addition on $\mathcal{R}$ as

$$(p \oplus q)(t) = p(t) + q(t) + (q-1)tp(t)q(t),$$

and the "circle minus" subtraction as

$$(p \ominus q)(t) = \frac{p(t) - q(t)}{1 + (q-1)tq(t)}.$$

Theorem 2.5 (See [6, Theorem 1.39])**.** *If $p, q \in \mathcal{R}$, then*

(6) $$e_{p\oplus q}(t,s) = e_p(t,s)e_q(t,s),$$

(7) $$e_{\ominus p}(t,s) = e_p(s,t) = \frac{1}{e_p(t,s)}.$$

Besides the circle plus and circle minus operation, a circle dot operation is defined.

Definition 2.7 (See [6, p. 18])**.** The circle dot multiplication $\odot$ of a constant value α and a function $p \in \mathcal{R}^+$ is defined as

$$(\alpha \odot p)(t) = \alpha p(t)\int_0^1 (1 + \mu(t)hp(t))^{\alpha-1}\mathrm{d}h,$$

if it exists.

Example 2.1. Let $p \in \mathcal{R}^+$ and $\alpha = \frac{1}{2}$. Then

$$\left(\frac{1}{2} \odot p\right)(t) = \frac{1}{2}\int_0^1 \frac{p(t)}{\sqrt{1 + \mu(t)hp(t)}}\mathrm{d}h$$

$$= \frac{1}{\mu(t)}\left(\sqrt{1+\mu(t)p(t)}-1\right) = \frac{p(t)}{1+\sqrt{1+\mu(t)p(t)}}.$$

Note that by the definition of the dot multiplication,

$$\left(\frac{1}{2}\odot(-\alpha)\right)\oplus\left(\frac{1}{2}\odot(-\alpha)\right) = -\alpha.$$

We furthermore need the definition of periodicity for functions $f : q^{\mathbb{N}_0} \to \mathbb{R}$.

Definition 2.8 (See [4]). A function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is called ω-periodic provided

$$f(t) = q^{\omega} f(q^{\omega} t) \text{ for all } t \in q^{\mathbb{N}_0}.$$

3. The Beverton–Holt q-difference equation with exploitation

The Beverton–Holt q-difference equation was presented in [4] as

$$x(qt) = \frac{v(t)K(t)x(t)}{K(t)+(v(t)-1)x(t)}, \tag{8}$$

where $K : q^{\mathbb{N}_0} \to \mathbb{R}^+$ is the carrying capacity, $v : q^{\mathbb{N}_0} \to (1,\infty)$ is the intrinsic growth rate, and $x : q^{\mathbb{N}_0} \to \mathbb{R}^+$ represents the population density. Using the substitution $\alpha = \frac{v-1}{\mu v}$, we obtain the logistic dynamic equation

$$x^{\Delta}(t) = \alpha(t)x(qt)\left(1-\frac{x(t)}{K(t)}\right),$$

that is well studied in [5].

We introduce exploitation to (8) by the catch-per-effort hypothesis, which yields

$$(1+H(t))\,x(qt) = \frac{v(t)K(t)x(t)}{K(t)+(v(t)-1)x(t)}, \tag{9}$$

where $H : q^{\mathbb{N}_0} \to \mathbb{R}^+$ represents the harvest effort. When studying $H(t)$, we should be aware that the time intervals in the quantum calculus setting are increasing. We can therefore express H more explicitly as $H(t) = \mu(t)h(t)$. This allows us to investigate the harvest effort reduced by the time stretching property.

Applying the substitution $\alpha = \frac{v-1}{\mu v}$ to (9), we obtain

$$x(qt) = \frac{K(t)x(t)}{(1-\mu(t)\alpha(t))K(t)+\mu(t)\alpha(t)x(t)} - \mu(t)h(t)x(qt), \tag{10}$$

which is equivalent to

$$\begin{aligned} x(qt)K(t) - \mu(t)x(qt)\alpha(t)K(t) + \mu(t)x(qt)\alpha(t)x(t) \\ = K(t)x(t) - \mu(t)h(t)x(qt)K(t) + \mu^2(t)h(t)x(qt)\alpha(t)K(t) \\ - \mu^2(t)h(t)x(qt)\alpha(t)x(t), \end{aligned}$$

i.e.,

$$x^{\Delta}(t) = x(qt)\alpha(t)\left(1+\mu(t)h(t)\right) \\ - x(qt)\frac{\alpha(t)}{K(t)}x(t)\left(1+\mu(t)h(t)\right) - h(t)\left(x(t)+\mu(t)x^{\Delta}(t)\right),$$

i.e.,

$$x^{\Delta}(t) = x(qt)\alpha(t)\left(1-\frac{x(t)}{K(t)}\right) - E(t)x(t), \tag{11}$$

where $E(t) = \frac{h(t)}{1+\mu(t)h(t)} = -(\ominus h)(t)$. Recall that the logistic differential equation including exploitation is given in the similar form

$$x'(t) = x(t)\alpha(t)\left(1-\frac{x(t)}{K(t)}\right) - E(t)x(t).$$

The q-difference equation (11) is solved by using the transformation $u = 1/x$, which yields

$$u^{\Delta}(t) = -\alpha(t)u(t) + \frac{\alpha(t)}{K(t)} + \frac{h(t)}{1+\mu(t)h(t)}u(qt), \tag{12}$$

i.e.,

$$u^{\Delta}(t) = \left(h\oplus(-\alpha)\right)(t)u(t) + \frac{\alpha(t)(1+\mu(t)h(t))}{K(t)}.$$

This is in the form of a first-order q-difference equation with the solution given in [5] by

$$u(t) = e_{h\oplus(-\alpha)}(t,t_0)u(t_0) + \int_{t_0}^{t} e_{h\oplus(-\alpha)}(t,qs)\frac{\alpha(s)(1+\mu(s)h(s))}{K(s)}\Delta s. \tag{13}$$

3.1. **Existence and uniqueness Theorem.** In this section, we are interested in providing conditions for the existence and uniqueness of a periodic solution. We aim to prove the following theorem.

Theorem 3.1. *Assume*

$$\begin{cases} K\colon q^{\mathbb{N}_0} \to \mathbb{R}^+ \text{ is } \omega\text{-periodic}, \\ \alpha : q^{\mathbb{N}_0} \to \mathbb{R}^+ \text{ is } \omega\text{-periodic and } 0 < \mu(t)\alpha(t) < 1 \text{ for all } t \in q^{\mathbb{N}_0}, \\ h : q^{\mathbb{N}_0} \to \mathbb{R}^+ \text{ is } \omega\text{-periodic and } 0 < h(t) < \ominus(-\alpha(t)) \text{ for all } t \in q^{\mathbb{N}_0}. \end{cases} \tag{14}$$

Then (10) *has a unique ω-periodic solution which globally attracts all positive solutions.*

Let us first present the following lemmas that will assist us in the analysis.

Lemma 3.1. *If $f, g \in \mathcal{R}$ are ω-periodic, then $f \oplus g$ and $f \ominus g$ is ω-periodic.*

Proof. We have

$$\begin{aligned} q^{\omega}\left(f \oplus g\right)(q^{\omega}t) &= q^{\omega}(f(q^{\omega}t) + g(q^{\omega}t) + \mu(q^{\omega}t)f(q^{\omega}t)g(q^{\omega}t)) \\ &= q^{\omega}(q^{-\omega}f(t) + q^{-\omega}g(t) + q^{\omega}\mu(t)q^{-\omega}f(t)q^{-\omega}g(t)) = (f \oplus g)(t) \end{aligned}$$

for all $t \in q^{\mathbb{N}_0}$. Also,

$$\begin{aligned} q^{\omega}\left(\ominus g\right)(q^{\omega}t) &= q^{\omega}\frac{-g(q^{\omega}t)}{1+\mu(q^{\omega}t)g(q^{\omega}t)} \\ &= q^{\omega}\frac{-q^{-\omega}g(t)}{1+q^{\omega}\mu(t)q^{-\omega}g(t)} \\ &= \frac{-g(t)}{1+\mu(t)g(t)} = (\ominus g)(t), \end{aligned}$$

which completes the proof since $f \ominus g = f \oplus (\ominus g)$. □

Lemma 3.2. *If $f \in \mathcal{R}$ is ω-periodic, then*

$$e_f(q^{\omega}t, q^{\omega}t_0) = e_f(t, t_0) \quad \textit{for all } t \in q^{\mathbb{N}_0} \tag{15}$$

and

$$e_f(q^{\omega}t, t) = e_f(q^{\omega}t_0, t_0) \quad \textit{for all } t \in q^{\mathbb{N}_0}. \tag{16}$$

Proof. Let $a, b \in \mathbb{N}_0$ such that $t_0 = q^a$ and $t = q^b$, and assume w.l.o.g. $t > t_0$. Then

$$\begin{aligned} e_f(q^{\omega}t, q^{\omega}t_0) &= \prod_{i=a+\omega}^{b+\omega-1}\left[1+\mu(q^i)f(q^i)\right] = \prod_{i=a}^{b-1}\left[1+\mu(q^{i+\omega})f(q^{i+\omega})\right] \\ &= \prod_{i=a}^{b-1}\left[1+q^{\omega}\mu(q^i)q^{-\omega}f(q^i)\right] = \prod_{i=a}^{b-1}\left[1+\mu(q^i)f(q^i)\right] \\ &= e_f(t, t_0). \end{aligned}$$

For the second equation, note that

$$\begin{aligned} e_f(q^{\omega}t, t) &= e_f(q^{\omega}t, q^{\omega}t_0)e_f(q^{\omega}t_0, t) \\ &\overset{(15)}{=} e_f(t, t_0)e_f(q^{\omega}t_0, t) = e_f(q^{\omega}t_0, t_0), \end{aligned}$$

which completes the proof. □

Lemma 3.3. *If* (14) *holds, then $h \oplus (-\alpha) \in \mathcal{R}^+$.*

Proof. We have

$$1 + \mu(t)(h \oplus (-\alpha))(t) = (1+\mu(t)h(t))(1-\mu(t)\alpha(t)) > 0$$

since $\mu\alpha \in (0, 1)$. □

Lemma 3.4. *Assume* (14)*. If $\beta(t) := \frac{\alpha(t)}{K(t)}(1+\mu(t)h(t))$, then $\beta(q^{\omega}t) = \beta(t)$ for all $t \in q^{\mathbb{N}_0}$.*

Proof. We have

$$\beta(q^\omega t) = \frac{\alpha(q^\omega t)}{K(q^\omega t)}(1+\mu(q^\omega t)h(q^\omega t)) = \frac{q^{-\omega}\alpha(t)}{q^{-\omega}K(t)}(1+q^\omega\mu(t)q^{-\omega}h(t))$$
$$= \frac{\alpha(t)}{K(t)}(1+\mu(t)h(t)) = \beta(t),$$

which shows the claim. □

Proof of Theorem 14. If $\bar{x}$ is any ω-periodic solution of (10), then the corresponding periodic solution $\bar{u}$ of (12) satisfies $\bar{u}(t) = q^{-\omega}\bar{u}(q^\omega t)$. Then

$$\begin{aligned}
\bar{u}(t) &= q^{-\omega}\bar{u}(q^\omega t) = q^{-\omega}e_{h\oplus(-\alpha)}(q^\omega t, t_0)\bar{u}(t_0) \\
&\qquad + q^{-\omega}\int_{t_0}^{q^\omega t} e_{h\oplus(-\alpha)}(q^\omega t, qs)\beta(s)\Delta s \\
&= q^{-\omega}e_{h\oplus(-\alpha)}(q^\omega t, t)e_{h\oplus(-\alpha)}(t, t_0)\bar{u}(t_0) \\
&\qquad + q^{-\omega}\int_{t_0}^{t} e_{h\oplus(-\alpha)}(q^\omega t, t)e_{h\oplus(-\alpha)}(t, qs)\beta(s)\Delta s \\
&\qquad + q^{-\omega}\int_{t}^{q^\omega t} e_{h\oplus(-\alpha)}(q^\omega t, t)e_{h\oplus(-\alpha)}(t, qs)\beta(s)\Delta s \\
&\overset{(16)}{=} q^{-\omega}e_{h\oplus(-\alpha)}(q^\omega t_0, t_0)\bar{u}(t) \\
&\qquad + q^{-\omega}e_{h\oplus(-\alpha)}(q^\omega t_0, t_0)\int_{t}^{q^\omega t} e_{h\oplus(-\alpha)}(t, qs)\beta(s)\Delta s,
\end{aligned}$$

where $\beta(s) := \frac{\alpha(s)(1+\mu(s)h(s))}{K(s)}$. We have

$$\bar{u}(t) = \frac{1}{q^\omega e_{h\oplus(-\alpha)}(t_0, q^\omega t_0) - 1}\int_{t}^{q^\omega t} e_{h\oplus(-\alpha)}(t, qs)\beta(s)\Delta s. \tag{17}$$

Conversely, the solution (17) satisfies $\bar{u}(t) = q^{-\omega}\bar{u}(q^\omega t)$. To realize that, let $\lambda := q^\omega e_{h\oplus(-\alpha)}(t_0, q^\omega t_0) - 1 \neq 0$. Then we have

$$\begin{aligned}
q^{-\omega}\bar{u}(q^\omega t) &= \frac{q^{-\omega}}{\lambda}\int_{q^\omega t}^{q^{2\omega} t} e_{h\oplus(-\alpha)}(q^\omega t, qs)\beta(s)\Delta s \\
&= \frac{q^{-\omega}}{\lambda}\sum_{i=\omega}^{2\omega-1} \mu(tq^i)e_{h\oplus(-\alpha)}(q^\omega t, q^{i+1}t)\beta(q^i t) \\
&= \frac{q^{-\omega}}{\lambda}\sum_{i=0}^{\omega-1} \mu(tq^{i+\omega})e_{h\oplus(-\alpha)}(q^\omega t, q^{i+\omega+1}t)\beta(q^{i+\omega} t) \\
&\overset{(15)}{=} \frac{1}{\lambda}\sum_{i=0}^{\omega-1} \mu(tq^i)e_{h\oplus(-\alpha)}(t, q^{i+1}t)\beta(q^i t)
\end{aligned}$$

$$= \frac{1}{\lambda} \int_t^{q^\omega t} e_{h\oplus(-\alpha)}(t, qs)\beta(s)\Delta s = \bar{u}(t).$$

Therefore the unique ω-periodic solution of (10) is given by

$$\bar{x}(t) = \lambda \left(\int_t^{q^\omega t} e_{h\oplus(-\alpha)}(t, qs)\beta(s)\Delta s \right)^{-1},$$

where $\lambda = q^\omega e_{h\oplus(-\alpha)}(t_0, q^\omega t_0) - 1 \neq 0$ and $\beta(s) = \frac{\alpha(s)(1+\mu(s)h(s))}{K(s)}$.

It is left to show that the ω-periodic solution is globally attractive. Let therefore x be any solution of (10) with $x_0 > 0$. Then

$$\begin{aligned} |x(t) - \bar{x}(t)| &= \Bigg| \frac{1}{e_{h\oplus(-\alpha)}(t, t_0)\frac{1}{x(t_0)} + \int_{t_0}^t e_{h\oplus(-\alpha)}(t, qs)\beta(s)\Delta s} \\ &\quad - \frac{1}{e_{h\oplus(-\alpha)}(t, t_0)\frac{1}{\bar{x}(t_0)} + \int_{t_0}^t e_{h\oplus(-\alpha)}(t, qs)\beta(s)\Delta s} \Bigg| \\ &\leq \frac{e_{h\oplus(-\alpha)}(t, t_0)\left|\frac{1}{x(t_0)} - \frac{1}{\bar{x}(t_0)}\right|}{\left(\int_{t_0}^t e_{h\oplus(-\alpha)}(t, qs)\beta(s)\Delta s\right)^2} \\ &\leq \|K\|_\infty^2 \frac{e_{h\oplus(-\alpha)}(t, t_0)\left|\frac{1}{x(t_0)} - \frac{1}{\bar{x}(t_0)}\right|}{\left(\int_{t_0}^t -e_{h\oplus(-\alpha)}(t, qs)\left(h \oplus (-\alpha)\right)\Delta s\right)^2} \\ &\leq \|K\|_\infty^2 \frac{e_{h\oplus(-\alpha)}(t, t_0)\left|\frac{1}{x(t_0)} - \frac{1}{\bar{x}(t_0)}\right|}{\left(1 - e_{h\oplus(-\alpha)}(t, t_0)\right)^2}. \end{aligned}$$

The last term tends to zero as $t \to \infty$ because $-1 < \mu(t)(h \oplus (-\alpha))(t) < 0$. ☐

4. The optimal sustainable yield

In order to discuss the maximum sustainable yield, let us recall that in quantum calculus, the time steps increase as time passes. To take this change of time intervals into consideration, we analyze the average of the harvest at each time step. This yields the formulation of the average of the sustainable yield

$$Y(h) = \frac{1}{\omega(q-1)} \int_{t_0}^{q^\omega t_0} \mu(t)h(t)x^\sigma(t)\,\Delta t.$$

Theorem 4.1. *Assume* (14). *Then the sustainable yield over one period*

$$Y(h) = \frac{1}{\omega(q-1)} \int_{t_0}^{t_0 q^\omega} \mu(t)h(t)\bar{x}(qt)\Delta t$$

is maximal for

$$h^* = \ominus q l^\sigma \ominus \frac{\left(\frac{m}{\alpha}\right)^\Delta}{\left(\frac{m}{\alpha}\right)} \ominus P \ominus \frac{K^\Delta}{K}, \tag{18}$$

where $l = \frac{1}{2} \odot (-\alpha)$, $m = l \ominus P$, $P = \frac{p^\Delta}{p}$ *with* $p(t) = \sqrt{t}$. *The maximum sustainable yield is then*

$$Y(h^*) = \int_{t_0}^{q^\omega t_0} \mu(s) \left(m^2 \frac{K}{\alpha}\right)(s) \Delta s.$$

Remark 4.1. The harvest yield has the property

$$Y(h) = \frac{1}{\omega(q-1)} \int_{t_0}^{t_0 q^\omega} \mu(t) h(t) \bar{x}(t) \Delta t = \frac{1}{\omega(q-1)} \int_{t^*}^{t^* q^\omega} \mu(t) h(t) \bar{x}(t) \Delta t$$

for any $t^* \in q^{\mathbb{N}_0}$.

In order to prove Theorem 4.1, the following lemmas will be useful.

Lemma 4.1. *If* $f \in \mathcal{R}$, *then*

$$e_{f^\Delta/f}(t,s) = \frac{f(t)}{f(s)} \tag{19}$$

for $s, t \in q^{\mathbb{N}_0}$.

Proof. Assume first $t > s$. Then

$$e_{f^\Delta/f}(t,s) = \prod_{\tau \in [s,t) \cap q^{\mathbb{N}_0}} \left[1 + \mu(\tau) \frac{f^\Delta(\tau)}{f(\tau)}\right] = \prod_{\tau \in [s,t) \cap q^{\mathbb{N}_0}} \frac{f^\sigma(\tau)}{f(\tau)} = \frac{f(t)}{f(s)}.$$

If $t < s$, then

$$e_{f^\Delta/f}(t,s) = \frac{1}{e_{f^\Delta/f}(s,t)} = \frac{1}{f(s)/f(t)} = \frac{f(t)}{f(s)},$$

and if $t = s$, then $e_{f^\Delta/f}(t,s) = 1$. □

Lemma 4.2. *Let* $p : q^{\mathbb{N}_0} \to \mathbb{R}$, $p(s) = \sqrt{s}$. *Then the function* $\frac{p^\Delta}{p} : q^{\mathbb{N}_0} \to \mathbb{R}$ *is* ω*-periodic for any* $\omega \geq 1$.

Proof. Let $\omega \geq 1$. Then

$$\begin{aligned} q^\omega \frac{p^\Delta(q^\omega t)}{p(q^\omega t)} &= q^\omega \left(\frac{p^\sigma(q^\omega t)}{\mu(q^\omega t) p(q^\omega t)} - \frac{1}{\mu(q^\omega t)} \right) \\ &= q^\omega \left(\frac{\sqrt{q^{\omega+1} t}}{q^\omega \mu(t) \sqrt{q^\omega t}} - \frac{1}{q^\omega \mu(t)} \right) = \left(\frac{\sqrt{qt}}{\mu(t)\sqrt{t}} - \frac{1}{\mu(t)} \right) = \frac{p^\Delta(t)}{p(t)}, \end{aligned}$$

which completes the proof. □

Lemma 4.3. *If $l \in \mathcal{R}$, then*

$$e_{ql^\sigma}(t,s) = e_l(qt,qs). \tag{20}$$

Proof. Let $i,n \in \mathbb{N}_0$ such that $t = q^i$ and $s = q^n$. Then for $t > s$

$$e_{ql^\sigma}(t,s) = \prod_{j=n}^{i-1}\left[1+\mu(q^j)ql(q^{j+1})\right] = \prod_{j=n+1}^{i}\left[1+\mu(q^j)l(q^j)\right] = e_l(qt,qs).$$

For $t < s$,

$$e_{ql^\sigma}(t,s) = \frac{1}{e_{ql^\sigma}(s,t)} = \frac{1}{e_l(qs,qt)} = e_l(qt,qs)$$

and if $t = s$, then $e_{ql^\sigma}(t,s) = 1 = e_l(qt,qs)$. □

Lemma 4.4. *Let m, l, P be defined as in Theorem 4.1. Then*

$$e_m(t_0, q^\omega t_0) = \sqrt{q^\omega}\, e_l(t_0, q^\omega t_0). \tag{21}$$

Proof. We have

$$e_m(t_0,q^\omega t_0) \overset{(6)}{=} e_l(t_0,q^\omega t_0)e_{\ominus P}(t_0,q^\omega t_0) \overset{(19)}{=} e_l(t_0,q^\omega t_0)\frac{\sqrt{q^\omega t_0}}{\sqrt{t_0}},$$

which completes the proof. □

Lemma 4.5. *Let m, l, P be defined as in Theorem 4.1. Then*

$$(1+\mu(t)l(t)) = \sqrt{q}(1+\mu(t)m(t)). \tag{22}$$

Proof. We have

$$\begin{aligned} 1+\mu(t)m(t) = 1+\mu(t)\frac{l(t)-P(t)}{1+\mu(t)P(t)} &= \frac{1+\mu(t)l(t)}{1+\mu(t)P(t)} \\ &= \frac{1+\mu(t)l(t)}{\frac{p^\sigma(t)}{p(t)}} = \frac{1+\mu(t)l(t)}{\sqrt{q}}, \end{aligned}$$

which shows the claim. □

Lemma 4.6. *Let m, h^* be defined as in Theorem 4.1 and* (14)*. Then*

$$\lambda^* = \lambda(h^*) = e_m(t_0, q^\omega t_0) - 1. \tag{23}$$

Proof. We have

$$\begin{aligned} \lambda(h^*) &= q^\omega e_{-\alpha\oplus h^*}(t_0,q^\omega t_0) - 1 \\ &= q^\omega e_{l\oplus l\ominus ql^\sigma}(t_0,q^\omega t_0)\frac{p(q^\omega t_0)}{p(t_0)}\frac{K(q^\omega t_0)}{K(t_0)}\frac{m(q^\omega t_0)\alpha(t_0)}{\alpha(q^\omega t_0)m(t_0)} - 1 \\ &= q^\omega e_{l\oplus l\ominus ql^\sigma}(t_0,q^\omega t_0)\sqrt{q^\omega}q^{-\omega} - 1 \\ &= q^\omega e_l(t_0,q^\omega t_0)\sqrt{q^\omega}q^{-\omega} - 1 \\ &= q^\omega e_l(t_0,q^\omega t_0)\sqrt{q^\omega}q^{-\omega} - 1 \overset{(21)}{=} e_m(t_0,q^\omega t_0) - 1. \end{aligned}$$

This completes the proof. □

Lemma 4.7. *Let m, h^* be defined as in Theorem 4.1. Then*

$$e_{h^*}(q^\omega t_0, t_0) = e_m(t_0, q^\omega t_0). \tag{24}$$

Proof. We have

$$\begin{aligned} e_{h^*}(q^\omega t_0, t_0) &\overset{(19)}{=} e_{\ominus q l^\sigma}(q^\omega t_0, t_0) \frac{p(t_0)}{p(q^\omega t_0)} \frac{K(t_0)}{K(q^\omega t_0)} \frac{\alpha(q^\omega t_0) m(t_0)}{m(q^\omega t_0)\alpha(t_0)} \\ &= e_l(t_0, q^\omega t_0) \frac{1}{\sqrt{q^\omega}} \frac{1}{q^{-\omega}} = e_l(t_0, q^\omega t_0)\sqrt{q^\omega} \\ &\overset{(21)}{=} e_m(t_0, q^\omega t_0). \end{aligned}$$

□

Lemma 4.8. *If $F : q^{\mathbb{N}_0} \to \mathbb{R}$ satisfies $F(q^\omega t) = q^{-2\omega} F(t)$, then*

$$\int_{qt_0}^{q^{\omega+1}t_0} tF(t)\Delta t = \int_{t_0}^{q^\omega t_0} tF(t)\Delta t. \tag{25}$$

Proof. W.l.o.g., let $t_0 = q^0 = 1$. Then

$$\begin{aligned} \int_{qt_0}^{q^{\omega+1}t_0} tF(t)\Delta t &= \sum_{n=1}^{\omega} \mu(q^n) q^n F(q^n) \\ &= \sum_{n=1}^{\omega-1} \mu(q^n) q^n F(q^n) + \mu(q^\omega t_0) q^\omega F(q^\omega t_0) \\ &= \sum_{n=1}^{\omega-1} \mu(q^n) q^n F(q^n) + \mu(t_0) F(t_0) \\ &= \sum_{n=0}^{\omega-1} \mu(q^n) q^n F(q^n) = \int_{t_0}^{q^\omega t_0} tF(t)\Delta t, \end{aligned}$$

which proves the statement. □

Lemma 4.9. *Let $G \in \mathcal{R}$ be ω-periodic. Then*

$$\int_{t_0}^{q^\omega t_0} qG^\sigma(t)\Delta t = \int_{t_0}^{q^\omega t_0} G(t)\Delta t. \tag{26}$$

Proof. W.l.o.g., let $t_0 = q^0 = 1$. Then

$$\begin{aligned} \int_{t_0}^{q^\omega t_0} qG^\sigma(t)\Delta t &= \sum_{n=0}^{\omega-1} \mu(q^n) q G(q^{n+1}) \\ &= \sum_{n=1}^{\omega} \mu(q^n) G(q^n) = \sum_{n=1}^{\omega-1} \mu(q^n) G(q^n) + \mu(q^\omega t_0) G(q^\omega t_0) \\ &= \sum_{n=1}^{\omega-1} \mu(q^n) G(q^n) + \mu(t_0) G(t_0) = \sum_{n=0}^{\omega-1} \mu(q^n) G(q^n) = \int_{t_0}^{q^\omega t_0} G(t)\Delta t. \end{aligned}$$

The proof is complete. □

Proof of Theorem 4.1. We use the notation: $l = \frac{1}{2} \odot (-\alpha)$. Then l is ω-periodic and $l \oplus l = -\alpha$. Let $m = l \ominus P$ and $n = l \oplus P$. Then $m \oplus n = -\alpha$. We apply the weighted Jensen inequality [19] (see also [1]) in the following way

$$
\begin{aligned}
(q-1)\omega Y(h) &= \int_{t_0}^{q^\omega t_0} \mu(t)h(t) \frac{\lambda}{\int_{qt}^{q^{\omega+1}t} e_{-\alpha\oplus h}(qt,qs)\frac{\alpha(s)(1+\mu(s)h(s))}{K(s)}\Delta s}\Delta t \\
&\overset{(6)}{=} \int_{t_0}^{q^\omega t_0} \mu(t)h(t) \frac{\lambda}{\int_{qt}^{q^{\omega+1}t} e_m(qt,qs)e_n(qt,qs)e_h(qt,s)\frac{\alpha(s)m(s)}{K(s)m(s)}\Delta s}\Delta t \\
&\le \lambda \int_{t_0}^{q^\omega t_0} \mu(t)h(t) \frac{\int_{qt}^{q^{\omega+1}t} e_{m\ominus n}(qt,qs)e_h(s,qt)\frac{K(s)m^2(s)}{\alpha(s)}\Delta s}{(\int_{qt}^{q^{\omega+1}t} e_m(qt,qs)m(s)\Delta s)^2}\Delta t \\
&\overset{(5)}{=} \lambda \int_{t_0}^{q^\omega t_0} \mu(t)h(t) \frac{\int_{qt}^{q^{\omega+1}t} \frac{s}{t}e_h(s,qt)\frac{K(s)m^2(s)}{\alpha(s)}\Delta s}{(1-e_m(t_0,q^\omega t_0))^2}\Delta t \\
&= \frac{\lambda}{(1-e_m(t_0,q^\omega t_0))^2}\sum_{i=0}^{\omega-1}\mu^2(q^i)h(q^i)\sum_{j=i+1}^{i+\omega}\frac{\mu(q^j)e_h(q^j,q^{i+1})K(q^j)m^2(q^j)}{q^{i-j}\alpha(q^j)} \\
&= \frac{\lambda(q-1)}{(1-e_m(t_0,q^\omega t_0))^2}\left\{\sum_{j=1}^{\omega}\frac{\mu(q^j)q^jK(q^j)m^2(q^j)}{\alpha(q^j)}\sum_{i=0}^{j-1}\mu(q^i)h(q^i)e_h(q^j,q^{i+1})\right. \\
&\qquad\left. + \sum_{j=\omega+1}^{2\omega-1}\frac{\mu(q^j)q^jK(q^j)m^2(q^j)}{\alpha(q^j)}\sum_{i=j-\omega}^{\omega-1}\mu(q^i)h(q^i)e_h(q^j,q^{i+1})\right\} \\
&= \frac{\lambda(q-1)}{(1-e_m(t_0,q^\omega t_0))^2}\left\{\int_{qt_0}^{q^{\omega+1}t_0} s\frac{K(s)m^2(s)}{\alpha(s)}\int_{t_0}^{s} h(\tau)e_h(s,q\tau)\Delta\tau\Delta s\right. \\
&\qquad\left. + \int_{qt_0}^{q^\omega t_0} s\frac{K(s)m^2(s)}{\alpha(s)}\int_{s}^{q^\omega t_0} h(\tau)e_h(sq^\omega,q\tau)\Delta\tau\Delta s\right\} \\
&\overset{(5)}{=} \frac{\lambda(q-1)}{(1-e_m(t_0,q^\omega t_0))^2}\left\{\int_{qt_0}^{q^{\omega+1}t_0} s\frac{K(s)m^2(s)}{\alpha(s)}[e_h(s,t_0)-1]\Delta s\right. \\
&\qquad\left. + \int_{qt_0}^{q^\omega t_0} s\frac{K(s)m^2(s)}{\alpha(s)}[e_h(sq^\omega,s)-e_h(sq^\omega,t_0q^\omega)]\Delta s\right\} \\
&= \frac{\lambda(q-1)}{(1-e_m(t_0,q^\omega t_0))^2}\int_{qt_0}^{q^{\omega+1}t_0} s\frac{K(s)m^2(s)}{\alpha(s)}[e_h(q^\omega t_0,t_0)-1]\Delta s
\end{aligned}
$$

$$\overset{(6)}{\underset{(19)}{=}} \frac{\lambda[e_h(q^\omega t_0, t_0) - 1](q-1)}{(1 - q^{\omega/2} e_l(t_0, q^\omega t_0))^2} \int_{qt_0}^{q^{\omega+1} t_0} s \frac{K(s) m^2(s)}{\alpha(s)} \Delta s$$

$$\leq (q-1) \int_{qt_0}^{q^{\omega+1} t_0} s \frac{K(s) m^2(s)}{\alpha(s)} \Delta s \overset{(25)}{=} \int_{t_0}^{q^\omega t_0} \mu(s) \frac{K(s) m^2(s)}{\alpha(s)} \Delta s,$$

where we have used that

$$\frac{\lambda[e_h(q^\omega t_0, t_0) - 1]}{(1 - q^{\omega/2} e_l(t_0, q^\omega t_0))^2} \leq 1.$$

To realize this, note that

$$[q^\omega e_{l \oplus l \oplus h}(t_0, t_0 q^\omega) - 1][e_{l \ominus l \ominus h}(t_0, q^\omega t_0) - 1] \leq q^\omega e_l^2(t_0, q^\omega t_0) - 2 q^{\omega/2} e_l(t_0, q^\omega t_0) + 1,$$

i.e.,

$$q^\omega e_{l \oplus h}(t_0, t_0 q^\omega) + e_{\ominus l \ominus h}(t_0, q^\omega t_0) - 2 q^{\omega/2} \geq 0,$$

i.e.,

$$\left(q^{\omega/2} \sqrt{e_{l \oplus h}(t_0, t_0 q^\omega)} - \sqrt{e_{\ominus l \ominus h}(t_0, q^\omega t_0)} \right)^2 \geq 0.$$

Now, we show that the optimal harvest yield is obtained at h^*:

$$\omega(q-1) Y(h^*) = \int_{t_0}^{q^\omega t_0} \frac{\mu(t) h^*(t) \lambda^*}{\int_{qt}^{q^{\omega+1} t} e_{-\alpha \oplus h^*}(qt, qs) \frac{\alpha(s)(1 + \mu(s) h^*(s))}{K(s)} \Delta s} \Delta t$$

$$\overset{(6)}{\underset{(19)}{=}} \int_{t_0}^{q^\omega t_0} \frac{\mu(t) h^*(t) \lambda^*}{\int_{qt}^{q^{\omega+1} t} e_{-\alpha}(qt, qs) e_{\ominus q l^\sigma}(qt, s) e_{\ominus P}(qt, s) \frac{K(s)}{K(qt)} \frac{m(s) \alpha(qt)}{\alpha(s) m(qt)} \frac{\alpha(s)}{K(s)} \Delta s} \Delta t$$

$$\overset{(20)}{=} \lambda^* \int_{t_0}^{q^\omega t_0} \frac{\mu(t) h^*(t) m(qt) \frac{K(qt)}{\alpha(qt)}}{\int_{qt}^{q^{\omega+1} t} e_{l \oplus l}(qt, qs) e_{\ominus l}(q^2 t, qs) e_{\ominus P}(qt, qs) \frac{1}{\sqrt{q}} m(s) \Delta s} \Delta t$$

$$\overset{(21)}{=} \lambda^* \sqrt{q} \int_{t_0}^{q^\omega t_0} \mu(t) h^*(t) \frac{m(qt) \frac{K(qt)}{\alpha(qt)} (1 + \mu(qt) l(qt))}{\int_{qt}^{q^{\omega+1} t} e_m(qt, qs) m(s) \Delta s} \Delta t$$

$$\overset{(5)}{=} \lambda^* \sqrt{q} \int_{t_0}^{q^\omega t_0} \mu(t) h^*(t) \frac{m(qt) \frac{K(qt)}{\alpha(qt)} (1 + \mu(qt) l(qt))}{1 - e_m(qt, q^{\omega+1} t)} \Delta t$$

$$\overset{(23)}{=} -\sqrt{q} \int_{t_0}^{q^\omega t_0} \mu(t) h^*(t) m(qt) \frac{K(qt)}{\alpha(qt)} (1 + \mu(qt) l(qt)) \Delta t$$

$$= -\sqrt{q} \int_{t_0}^{q^\omega t_0} \mu(t) m(qt) \frac{K(qt)}{\alpha(qt)} \left\{ h^*(t)(1 + \mu(t) q l(qt)) + q l(qt) - q l(qt) \Delta t \right\}$$

$$
\begin{aligned}
&= -\sqrt{q}\int_{t_0}^{q^\omega t_0} \mu(t)m(qt)\frac{K(qt)}{\alpha(qt)}(h^* \oplus ql^\sigma)(t)\Delta t \\
&\qquad + \sqrt{q}\int_{t_0}^{q^\omega t_0} \mu(t)m(qt)\frac{K(qt)}{\alpha(qt)}ql(qt))\Delta t \\
&= -\sqrt{q}\int_{t_0}^{q^\omega t_0} \mu(t)m(qt)\frac{K(qt)}{\alpha(qt)}\left(\ominus\frac{p^\Delta}{p} \ominus \frac{K^\Delta}{K} \ominus \frac{\left(\frac{m}{\alpha}\right)^\Delta}{\left(\frac{m}{\alpha}\right)}\right)(t)\Delta t \\
&\qquad + \sqrt{q}\int_{t_0}^{q^\omega t_0} \mu(qt)l(qt)m(qt)\frac{K(qt)}{\alpha(qt)}\Delta t \\
&\overset{(22)}{=} \sqrt{q}\int_{t_0}^{q^\omega t_0} \mu(t)\left(m\frac{K}{\alpha}\right)(t)\frac{p^\Delta(t)}{p^\sigma(t)}\Delta t + \sqrt{q}\int_{t_0}^{q^\omega t_0} \mu(t)\left(K\frac{m}{\alpha}\right)^\Delta(t)\Delta t \\
&\qquad + \sqrt{q}\int_{t_0}^{q^\omega t_0} (\sqrt{q}(1+\mu(t)m(t))^\sigma - 1)\left(m\frac{K}{\alpha}\right)^\sigma(t)\Delta t \\
&= \sqrt{q}\int_{t_0}^{q^\omega t_0} \mu(t)\left(m\frac{K}{\alpha}\right)(t)\frac{p^\Delta(t)}{p^\sigma(t)}\Delta t \\
&\qquad + \sqrt{q}\int_{t_0}^{q^\omega t_0} \left(m\frac{K}{\alpha}\right)^\sigma(t) - \left(m\frac{K}{\alpha}\right)(t)\Delta t \\
&\qquad + \sqrt{q}\int_{t_0}^{q^\omega t_0} (\sqrt{q}(1+\mu(t)m(t))^\sigma - 1)\left(m\frac{K}{\alpha}\right)^\sigma(t)\Delta t \\
&= \sqrt{q}\int_{t_0}^{q^\omega t_0} \left(m\frac{K}{\alpha}\right)(t)\left\{-1+\mu(t)\frac{p^\Delta(t)}{p^\sigma(t)}\right\}\Delta t \\
&\qquad + \sqrt{q}\int_{t_0}^{q^\omega t_0} \sqrt{q}\mu(qt)\left(m^2\frac{K}{\alpha}\right)^\sigma(t)\Delta t \\
&\qquad + \sqrt{q}\int_{t_0}^{q^\omega t_0} \sqrt{q}\left(m\frac{K}{\alpha}\right)^\sigma(t)\Delta t \\
&= -\int_{t_0}^{q^\omega t_0} \left(m\frac{K}{\alpha}\right)(t)\Delta t + \int_{t_0}^{q^\omega t_0} q\mu(qt)\left(m^2\frac{K}{\alpha}\right)^\sigma(t)\Delta t \\
&\qquad + \int_{t_0}^{q^\omega t_0} q\left(m\frac{K}{\alpha}\right)^\sigma(t)\Delta t \\
&\overset{(26)}{=} \int_{t_0}^{q^\omega t_0} \mu(t)\left(m^2\frac{K}{\alpha}\right)(t)\Delta t,
\end{aligned}
$$

which completes the proof. □

Example 4.1. Let us consider the case of $\omega = 1$, i.e., $K(t) = \frac{\kappa}{t}$ and $\alpha(t) = \frac{a}{t}$ for some positive constants κ, α, with $0 < a < \frac{1}{q-1}$. Theorem 4.1 provides the optimal harvest effort that maximizes the average of the sustainable yield

$$Y(h) = \frac{1}{\omega(q-1)} \int_{t_0}^{t_0 q^\omega} \mu(t)h(t)\bar{x}(qt)\Delta t$$

as

$$h^* = \ominus q l^\sigma \ominus P \ominus \frac{\left(\frac{m}{\alpha}\right)^\Delta}{\frac{m}{\alpha}} \ominus \frac{K^\Delta}{K},$$

where $l = \frac{1}{2} \odot (-\alpha)$, $m = l \ominus P$, $P = \frac{p^\Delta}{p}$ with $p(t) = \sqrt{t}$. In the case of one-periodic coefficients, which refers to the constant coefficients case in the continuous and discrete model, h^* simplifies to

$$h^* = \ominus l \oplus P.$$

To realize that, observe first that

$$\ominus q l^\sigma(t) = \ominus l(t)$$

because

$$\begin{aligned} \ominus q l^\sigma(t) &= \frac{-q l^\sigma(t)}{1 + \mu(t) q l^\sigma(t)} = \frac{\frac{q\alpha(qt)}{1+\sqrt{1-\mu(qt)\alpha(qt)}}}{1 + \mu(qt)\frac{-\alpha(qt)}{1+\sqrt{1-\mu(qt)\alpha(qt)}}} \\ &= \frac{q\alpha(qt)}{1 + \sqrt{1-\mu(qt)\alpha(qt)} - \mu(qt)\alpha(qt)} \\ &= \frac{\frac{a}{t}}{1 + \sqrt{1-\mu(t)\frac{a}{t}} - \mu(t)\frac{a}{t}} = \ominus l. \end{aligned}$$

In the case of constant coefficients, we have

$$f \ominus \frac{K^\Delta}{K} = f \oplus \frac{1}{t}.$$

This is true because

$$\begin{aligned} \left(\ominus \frac{K^\Delta}{K}\right)(t) &= \frac{-K^\Delta(t)}{K(qt)} = \frac{-1}{\mu(t)} + \frac{K(t)}{\mu(t)K(qt)} = \frac{-1}{\mu(t)} + \frac{\frac{\kappa}{t}}{\mu(t)\frac{\kappa}{qt}} \\ &= \frac{-1}{\mu(t)} + \frac{q}{\mu(t)} = \frac{q-1}{\mu(t)} = \frac{q-1}{(q-1)t} = \frac{1}{t}. \end{aligned}$$

Finally, note that

$$h^* = \ominus l \oplus \frac{p^\Delta}{p}$$

because

$$\frac{p^\Delta(t)}{p(t)} \ominus \frac{1}{t} = \frac{p^\Delta(t)}{p(t)} \ominus \frac{1}{t} = \frac{\frac{p^\Delta(t)}{p(t)} - \frac{1}{t}}{1 + \mu(t)\frac{1}{t}} = \frac{\frac{1}{\mu(t)}\left(\frac{p(qt)}{p(t)} - 1\right) - \frac{1}{t}}{q}$$

$$= \frac{1}{q}\left(\frac{\sqrt{q}-1}{(q-1)t} - \frac{1}{t}\right) = \frac{1}{(q-1)t}\left(\frac{\sqrt{q}-1}{q} - \frac{q-1}{q}\right)$$
$$= \frac{1}{(q-1)t}\left(\frac{1}{\sqrt{q}} - 1\right) = \frac{-1}{\mu(t)}\left(\frac{p(qt)-p(t)}{p(qt)}\right)$$
$$= -\frac{p^{\Delta}(t)}{p(qt)} = \ominus\frac{p^{\Delta}(t)}{p(t)}.$$

For example, if we chose $\mathbb{T} = 2^{\mathbb{N}_0}$ and one-periodic coefficients, then

$$h^*(t) = \frac{P-l}{1+\mu l} = \frac{\frac{\sqrt{2}-1}{t} + \frac{\alpha}{1+\sqrt{1-\mu(t)\alpha(t)}}}{1-\mu(t)\frac{\alpha}{1+\sqrt{1-\mu(t)\alpha(t)}}}.$$

Note that h^* is a one-periodic function, i.e.,

$$h^*(t) = \frac{H}{t}, \quad H = \frac{\sqrt{2}-1+\frac{a}{1+\sqrt{1-(q-1)a}}}{1-(q-1)\frac{a}{1+\sqrt{1-(q-1)a}}},$$

where $\alpha = \frac{a}{t}$. Figure 1 shows the optimal harvest effort h^* and also h^* without the time-stretching factor, i.e., $h^*t = H$ for $\alpha = \frac{a}{t}$ and $a = 0.3$.

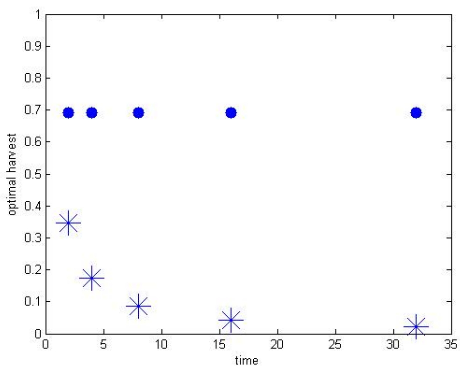

FIGURE 1. The optimal harvest h^* (stars) and $h^*t = H$ (dots).

Figure 2 shows the relation of h^* with respect to the growth rate α reduced by its time-stretching character, i.e., $\alpha t = a$. Note that this is a similar behavior as in the case $\mathbb{T} = \mathbb{Z}$ with constant coefficients, where the optimal harvest effort h^* had the behavior with respect to the one-periodic/constant growth rate as visualized in Figure 3. The difference in the values is caused by the fact that h^* is expressed without the time-stretching factor.

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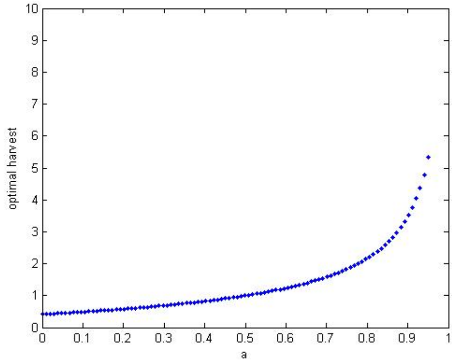

FIGURE 2. The optimal harvest $h^*t = H$.

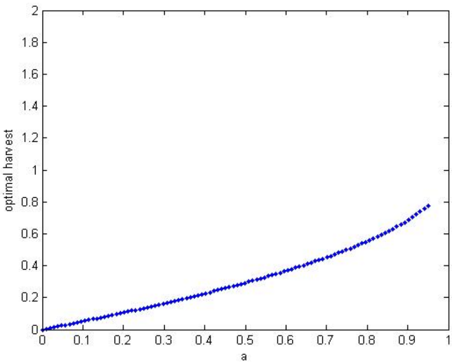

FIGURE 3. The optimal harvest $h^* = H$ for $\mathbb{T} = \mathbb{Z}$.

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Martin Bohner
Missouri University of Science and Technology
Department of Mathematics and Statistics
400 West, 12th Street
Rolla, MO, 65409-0020
USA
E-mail address: `bohner@mst.edu`

Sabrina Streipert
Missouri University of Science and Technology
Department of Mathematics and Statistics
400 West, 12th Street
Rolla, MO, 65409-0020
USA
E-mail address: `sstreipert@mst.edu`