

Note on “Common fixed point theorem for weakly compatible multi-valued mappings satisfying strongly tangential property”

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ABSTRACT. In our paper [1], the existence and uniqueness of a common fixed point for multi valued mappings was given, hence in this note we point out to no uniqueness of common fixed point in theorem 3.1, corollary 3.1 and corollary 3.2. We provide an example to demonstrate our purpose.

1. MAIN RESULTS

Theorem 1. *Let $f, g : X \rightarrow X$, be single valued mappings and $S, T : X \rightarrow CB(X)$ multi-valued mappings of a metric space (X, d) , such for all x, y in X we have:*

$$(1) \quad \int_0^{H(Sx, Ty)} \varphi(t) \leq \phi \left(\int_0^{d(fx, gy)} \varphi(t) dt, \int_0^{d(fx, Sx)} \varphi(t) dt, \int_0^{d(gy, Ty)} \varphi(t) dt, \int_0^{d(fx, Ty)} \varphi(t) dt, \int_0^{d(gy, Sx)} \varphi(t) dt \right),$$

where $\phi : \mathbb{R}_+^5 \rightarrow \mathbb{R}_+$ is an upper semi continuous, non decreasing function satisfying:

$$\psi(t) = \max \left(\phi(0, t, 0, 0, t), \phi(0, 0, t, t, 0), \phi(t, 0, 0, t, t) \right) < t,$$

for any $t > 0$ and $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a Lebesgue-integrable function which is summable on each compact subset of \mathbb{R}_+ , non-negative, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t) dt > 0$. Suppose that the two pairs (f, S) , (g, T) are weakly compatible, moreover if (f, g) is strongly tangential w.r.t (S, T) , then f, g, S and T have a common fixed point in X .

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Remark 1. In paper [1], the proof of uniqueness of the common fixed point is false, since the inequality $d(a, b) \leq H(A, B)$, for all $a \in A$ and $b \in B$ is not true in general. For example $A = [0, 2], B = [3, 5]$ and $H(A, B) = 3$, but $d(0, 5) = 5$ (with euclidian metric).

If $S = T$ and $f = g$, we obtain the following corollary:

Corollary 1. *Let $f : X \rightarrow X$, and $S : X \rightarrow CB(X)$ be single and set valued mappings of metric space (X, d) such:*

$$\int_0^{H(Sx, Sy)} \varphi(t) \leq \phi \left(\int_0^{d(fx, fy)} \varphi(t) dt, \int_0^{d(fx, Sx)} \varphi(t) dt, \int_0^{d(fy, Sy)} \varphi(t) dt, \int_0^{d(fx, Sy)} \varphi(t) dt, \int_0^{d(fy, Sx)} \varphi(t) dt \right),$$

where $\phi \in \Phi$ and $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a Lebesgue-integrable function which is summable on each compact subset of \mathbb{R}_+ , non-negative, and such that for each $\varepsilon > 0, \int_0^\varepsilon \varphi(t) dt > 0$, if f is strongly tangential w.r.t S and (f, S) is weakly compatible, then f and S have a common fixed point.

Corollary 2. *Let $f, g : X \rightarrow X$, and $S, T : X \rightarrow CB(X)$ be single and set valued mappings of metric space (X, d) such:*

$$\left(\int_0^{H(Sx, Ty)} \varphi(t) dt \right)^p \leq a \left(\int_0^{d(fx, gy)} \varphi(t) dt \right)^p + b \left(\int_0^{d(fx, Sx)} \varphi(t) dt \right)^p + c \left(\int_0^{d(gy, Ty)} \varphi(t) dt \right)^p,$$

where $\phi \in \Phi$ and $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a Lebesgue-integrable function which is summable on each compact subset of \mathbb{R}^+ , non-negative, and such that for each $\varepsilon > 0, \int_0^\varepsilon \varphi(t) dt > 0$. and a, b, c are nonnegative real numbers such $a + b + c < 1$ and $p \in \mathbb{N}^*$, if $\{f, g\}$ is strongly tangential w.r.t (S, T) and the two pairs $(f, g), (S, T)$ are weakly compatible, then f, g, S and T have a common fixed point.

Example 1. Let $X = [0, 4]$ endowed with the euclidian metric. Define the mappings f, g, S and T as follows:

$$x = gx = \begin{cases} \frac{x+1}{2}, & 0 \leq x \leq 1; \\ 2, & 1 < x \leq 2; \\ \frac{7}{2}, & 2 < x \leq 4; \end{cases} \quad Sx = Tx = \begin{cases} \{1\}, & 0 \leq x < 2; \\ \{2\}, & 2 \leq x \leq 4. \end{cases}$$

Let $\{x_n\}, \{y_n\}$ be two sequences which defined by: $x_n = 1 - e^{-n}$ and $y_n = 1 - \frac{1}{n}$. Clearly that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Sy_n = [0, 1]$$

and

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} fy_n = 1 \in [0, 1],$$

then f is strongly tangential w.r.t S . We will utilize Corollary 1, with

$$\phi(t_1, t_2, t_3, t_4, t_5) = \frac{2}{5} \max\{t_1, t_2, t_3, t_4, t_5\} \quad \text{and} \quad \varphi(t) = 1.$$

For the contractive condition (1), we have the following cases:

- (1) For $x, y \in [0, 2) \cup [2, 4]$, we have $H(Sx, Ty) = 0$, so obviously that (1) is satisfied.
- (2) For $x \in [0, 2)$ and $y \in [2, 4]$ we have

$$H(Sx, Ty) = 1 \leq \frac{7}{5} = \frac{2}{5}d(fy, Sy).$$

- (3) For $2 \leq x \leq 4$ and $y \in [0, 2)$, we have

$$H(Sx, Ty) = 1 \leq \frac{7}{5} = \frac{2}{5}d(fx, Sx).$$

Also 1 and 2 are two coincidence points satisfying $fS1 = [0, 1] = Sf1$, $fS2 = \{2\} = Sf2$, consequently all hypotheses of Corollary 1 are satisfied, then f and S have at least a common fixed point, therefore 1 and 2 are two common fixed point.

REFERENCES

- [1] S. Beloul, *A Common Fixed Point Theorem for Weakly Compatible Multi-Valued Mappings Satisfying Strongly Tangential Property*, *Mathematica Moravica*, **18-2** (2014), 63-72.

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