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A general fixed point theorem for two hybrid pairs of mappings satisfying a mixed implicit relation and applications

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Abstract. The purpose of this paper is to extend Theorem 3.2 [16] for two hybrid pairs of mappings satisfying a mixed implicit relation and a new type of common limit range property without weak compatibility.

As applications, some fixed point results for pairs of mappings satisfying contractive conditions of integral type and φ-contractive maps are obtained.

1. Introduction

In 1969, Nadler [8] proved an analogue Banach principle with set - valued mappings employing Hausdorff-Pompeiu metric.

In 2011, Sintunavarat and Kumam [20] introduced the notion of common limit range property for single-valued mappings.

Imdad et al. [4] established common limit range property for a hybrid pair of mappings and obtained some fixed point results in symmetric spaces.

Quite recently, Imdad et al. [5] introduced the notion of joint common limit range property for two pairs of hybrid mappings.

The study of fixed points for mappings satisfying a contractive condition of integral type is introduced by Branciari [1].

It is proved in [15] that the study of fixed points of single-valued mappings and set-valued mappings satisfying integral condition is reduced to the study of fixed points for mappings involving altering distances.

Several classical fixed point theorems have been unified considering a general condition by an implicit relation in [9], [10] and in other papers. Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi-metric spaces, b-metric spaces, ultra-metric spaces, Hilbert spaces, reflexive spaces, compact metric spaces, in two and three metric spaces, for single-valued mappings, hybrid pairs of mappings and set-valued mappings.

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Quite recently, the method is used in the study of fixed points for mappings satisfying contractive/extensive conditions of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, G -metric spaces, G_p-metric spaces, and partial metric spaces.

With this method the proofs of existence of fixed points are more simple. Also, the method allows the study of local and global properties of fixed point structures.

In [11]-[13] and in other papers, the study of fixed points for hybrid pairs of mappings and set-valued mappings satisfying implicit relations is introduced.

A general fixed point theorem for a hybrid pair of mappings with common limit range property satisfying an implicit relation is proved in [2].

2. Preliminaries

Let (X, d) be a metric space. We denote by $CL(X)$ the family of all closed sets of X and by H the Hausdorff-Pompeiu metric, i.e.

$$H(A,B) = \max\{\sup_{x\in A}\{d(x,B), \sup_{x\in B} d(x,A)\}\},$$

where $A, B \in CL(X)$ and

$$d(x,A) = \inf_{y\in A}\{d(x,y)\}.$$

Definition 2.1. Let $f : X \to X$ be a single valued mapping and let $F : X \to 2^X$ be a multi-valued mapping.

1) A point $x \in X$ is said to be a coincidence point of f and F if $fx \in Fx$.
   The set of all coincidence points of f and F is denoted by $\mathcal{C}(f, F)$.
2) A point $x \in X$ is a common fixed point of f and F if $x = fx \in Fx$.

Definition 2.2 ([3])**.** Let $f : X \to X$ and $F : X \to 2^X$ be. The mapping f is said to be coincidentally idempotent with respect to F if $fx \in Fx$ implies $fx = ffx$, that is, f is idempotent at coincidence points of f and F.

Definition 2.3 ([4])**.** Let (X, d) be a metric space, $f : X \to X$ and $F : X \to CL(X)$. Then, (f, F) has a common limit range property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = fu \in A = \lim_{n\to\infty} Fx_n,$$

for some $u \in X$ and $A \in CL(X)$.

Definition 2.4 ([14])**.** Let A, S and T be self mappings of a metric space (X, d). The pair (A, S) is said to satisfy common limit range property with respect to T, denoted $CLR_{(A,S),T}$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in S(X) \cap T(X)$.

Definition 2.5 ([5]). Let (X, d) be a metric space, $f, g : X \to X$ and $F, G : X \to CL(X)$. Then, the pairs (f, F) and (g, G) are said to have joint common limit range property, denoted $(JCLR)$-property, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X and $A, B \in CL(X)$ such that

$$\lim_{n\to\infty} Fx_n = A, \lim_{n\to\infty} Gy_n = B,$$

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t, \text{ with } t \in f(X) \cap g(X) \cap A \cap B,$$

that is, there exist $u, v \in X$ such that $t = fu = gv \in A \cap B$.

Now we introduce a new type of common limit range property.

Definition 2.6. Let (X, d) be a metric space, $A : X \to CL(X)$ and $S, T : X \to X$. Then (A, S) satisfy a common limit range property with respect to T, denoted $CLR_{(A,S)T}$-property, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = z, \lim_{n\to\infty} Ax_n = D \in CL(X) \text{ and } z \in D \cap S(X) \cap T(X).$$

Example 2.1. Let $X = [0, \infty)$ be a metric space with the usual metric, and $Ax = \left[\frac{1}{4}, 1\right]$, $Sx = \frac{x^2+1}{2}$, $Tx = x + \frac{1}{4}$. Then $S(X) = \left[\frac{1}{2}, \infty\right)$, $T(X) = \left[\frac{1}{4}, \infty\right)$, $S(X) \cap T(X) = \left[\frac{1}{2}, \infty\right)$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = 0$. Then

$$\lim_{n\to\infty} Sx_n = t = \frac{1}{2}, \lim_{n\to\infty} Ax_n = \left[\frac{1}{4}, 1\right] = D \text{ and } t = \frac{1}{2} \in D \cap S(X) \cap T(X).$$

Remark 2.1. Let (X, d) be a metric space, $A, B : X \to CL(X)$ and $S, T : X \to X$. If (A, S) and (B, T) satisfy $(JCLR)$ - property, then (A, S) and T satisfy $CLR_{(A,S)T}$-property.

Definition 2.7 ([6]). An altering distance is a function $\psi : [0, \infty) \to [0, \infty)$ such that

(ψ_1): ψ is nondecreasing and continuous,
(ψ_2): $\psi(t) = 0$ if and only if $t = 0$.

3. Implicit relations

Definition 3.1. Let $\mathcal{F}_M$ be the set of all lower semi-continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ such that:

(F_1) : F is nondecreasing in variable t_1,
(F_2) : $F(t, 0, t, 0, 0, t) \geq 0$, for all $t > 0$,
(F_3) : $F(t, 0, 0, t, t, 0) \geq 0$, for all $t > 0$.

Example 3.1. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$, where $k \in [0, 1]$.

Example 3.2. $F(t_1, ..., t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{2}\right\}$, where $k \in [0, 1]$.

Example 3.3. $F(t_1, ..., t_6) = t_1 - k\max\left\{t_2, \dfrac{t_3 + t_4}{2}, \dfrac{t_5 + t_6}{2}\right\}$, where $k \in [0, 1]$.

Example 3.4. $F(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_5, t_6\}$, where $a, b, c \geq 0$ and $a + b + c \leq 1$.

Example 3.5. $F(t_1, ..., t_6) = t_1 - \alpha\max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, where $\alpha \in (0, 1)$, $a, b \geq 0$ and $a + b \leq 1$.

Example 3.6. $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c\max\{t_5, t_6\}$, where $a, b, c \geq 0$ and $a + b + c \leq 1$.

Example 3.7. $F(t_1, ..., t_6) = t_1 - at_2 - \dfrac{b\,(t_5 + t_6)}{1 + t_3 + t_4}$, where a, $b \geq 0$ and $a + 2b \leq 1$.

Example 3.8. $F(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, where $a, b, c \geq 0$ and $a + b + c \leq 1$.

Definition 3.2. Let $\mathcal{G}_M$ be the set of all lower semi-continuous functions $G : \mathbb{R}_+^5 \to \mathbb{R}$ such that $G\,(s_1, ..., s_5) > 0$ if one of $s_1, ..., s_5$ is greater than 0.

Example 3.9. $G(s_1, ..., s_5) = \max\{s_1, s_2, s_3, s_4, s_5\}$.

Example 3.10. $G(s_1, ..., s_5) = \max\left\{s_2, \dfrac{s_3 + s_4}{2}, \dfrac{s_5 + s_6}{2}\right\}$.

Example 3.11. $G(s_1, ..., s_5) = \alpha\max\{s_1, s_2, s_3\} - (1 - \alpha)(as_4 + bs_5)$, where $\alpha \in (0, 1)$, $a, b \geq 0$ and $a + b \leq 1$.

Example 3.12. $G(s_1, ..., s_5) = s_1^2 + s_2^2 + s_3^2 + s_4^2 + s_5^2$.

Example 3.13. $G(s_1, ..., s_5) = \dfrac{s_1}{1 + s_2} + \dfrac{s_2}{1 + s_3} + \dfrac{s_3}{1 + s_4} + \dfrac{s_4}{1 + s_5} + \dfrac{s_5}{1 + s_1}$.

Example 3.14. $G(s_1, ..., s_5) = \dfrac{s_1 + s_2 + s_3 + s_4 + s_5}{1 + s_1}$.

Example 3.15. $G(s_1, ..., s_5) = \dfrac{1}{s_1 + s_2 + s_3 + s_4 + s_5}$.

Example 3.16. $G(s_1, ..., s_5) = s_1 + \dfrac{s_2 s_5 + s_3 s_4}{1 + s_1}$.

Definition 3.3. A function $\phi\,(t_1, ..., t_6, s_1, ..., s_5) = F\,(t_1, ..., t_6) + G\,(s_1, ..., s_5)$ is called a mixed implicit relation.

Theorem 3.1 ([16]). *Let (X, d) be a metric space and $A, B, S, T : X \to X$ be self mappings of X satisfying*

$$F\Big(\psi\left(d\left(Ax, By\right)\right), \psi\left(d\left(Sx, Ty\right)\right), \psi\left(d\left(Sx, Ax\right)\right), \psi\left(d\left(Ty, By\right)\right), \psi\left(d\left(Sx, By\right)\right), \psi d\left(Ty, Ax\right)\Big) + G\Big(\psi(d\left(Sx, Ty\right)), \psi\left(d\left(Sx, Ax\right)\right), \psi\left(d\left(Ty, By\right)\right), \psi\left(d\left(Sx, By\right)\right), \psi\left(d\left(Ty, Ax\right)\right)\Big) \leq 0$$

for all $x, y \in X$, some $F \in \mathcal{F}_M$, $G \in \mathcal{G}_M$ and ψ is an altering distance.

If (A, S) and T satisfy $CLR_{(A,S)T}$ - property, then

1) $\mathcal{C}(A, S) \neq \emptyset$,
2) $\mathcal{C}(B, T) \neq \emptyset$.

Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

The purpose of this paper is to extend Theorem 3.1 for two hybrid pairs of mappings satisfying a mixed implicit relation and a new type of common limit property without weak compatibility. As applications, some fixed point results for mappings satisfying contractive conditions of integral type and φ-contractive maps.

4. Main results

Theorem 4.1. *Let (X, d) be a metric space, $A, B : X \to CL(X)$ and $S, T : X \to X$ such that*

$$F\Big(\psi\left(H\left(Ax, By\right)\right), \psi\left(d\left(Sx, Ty\right)\right), \psi\left(d\left(Sx, Ax\right)\right), \psi\left(d\left(Ty, By\right)\right), \psi\left(d\left(Sx, By\right)\right), \psi(d\left(Ty, Ax\right))\Big) + G\Big(\psi(d\left(Sx, Ty\right)), \psi\left(d\left(Sx, Ax\right)\right), \psi\left(d\left(Ty, By\right)\right), \psi\left(d\left(Sx, By\right)\right), \psi\left(d\left(Ty, Ax\right)\right)\Big) \leq 0 \tag{4.1}$$

for all $x, y \in X$, some $F \in \mathcal{F}_M$, $G \in \mathcal{G}_M$ and ψ is an altering distance.

If (A, S) and T satisfy $CLR_{(A,S)T}$ - property, then

1) $\mathcal{C}(A, S) \neq \emptyset$,
2) $\mathcal{C}(B, T) \neq \emptyset$.

Moreover,

a) *if S is coincidentally idempotent with respect to A, then S and A have a common fixed point,*
b) *if T is coincidentally idempotent with respect to B, then T and B have a common fixed point,*
c) *if the conditions of a) and b) hold, then S, T, A and B have a common fixed point.*

Proof. Since (A,S) and T satisfy $CLR_{(A,S)T}$ - property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = z,\ \lim_{n\to\infty} Ax_n = D,\ D \in CL(X) \text{ and } z \in D \cap S(X) \cap T(X).$$

Since $z \in T(X)$, there exists $u \in X$ such that $z = Tu$.

By (4.1) for $x = x_n$ and $y = u$ we obtain

$$\begin{aligned} &F\Big(\psi\left(H\left(Ax_n, Bu\right)\right), \psi\left(d\left(Sx_n, Tu\right)\right), \psi\left(d\left(Sx_n, Ax_n\right)\right),\\ &\qquad \psi\left(d\left(Tu, Bu\right)\right), \psi\left(d\left(Sx_n, Bu\right)\right), \psi(d\left(Tu, Ax_n\right))\Big)+\\ &+G\Big(\psi(d\left(Sx_n, Tu\right)), \psi\left(d\left(Sx_n, Ax_n\right)\right), \psi\left(d\left(Tu, Bu\right)\right),\\ &\qquad \psi\left(d\left(Sx_n, Bu\right)\right), \psi\left(d\left(Tu, Ax_n\right)\right)\Big) \leq 0 \end{aligned}$$

Letting n tend to infinity we obtain

$$\begin{aligned} &F\Big(\psi\left(H\left(D, Bu\right)\right), 0, 0, \psi\left(d\left(z, Bu\right)\right), \psi\left(d\left(z, Bu\right)\right), 0\Big)+\\ &+G\Big(0, 0, \psi\left(d\left(z, Bu\right)\right), \psi\left(d\left(z, Bu\right)\right), 0\Big) \leq 0. \end{aligned}$$

Since $z \in D$, then $d(z, Bu) \leq H(D, Bu)$, which implies by (ψ_1), $\psi(d(z, Bu)) \leq \psi(H(D, Bu))$. By (F_1) we obtain

$$\begin{aligned} &F\Big(\psi\left(d\left(z, Bu\right)\right), 0, 0, \psi\left(d\left(z, Bu\right)\right), \psi\left(d\left(z, Bu\right)\right), 0\Big)+\\ &+G\Big(0, 0, \psi\left(d\left(z, Bu\right)\right), \psi\left(d\left(z, Bu\right)\right), 0\Big) \leq 0. \end{aligned}$$

If $d(z, Bv) > 0$, then $\psi(d(z, Bu)) > 0$ and by $G \in \mathcal{G}_M$,

$$G\Big(0, 0, \psi\left(d\left(z, Bu\right)\right), \psi\left(d\left(z, Bu\right)\right), 0\Big) > 0.$$

Then,

$$F\Big(\psi\left(d\left(z, Bu\right)\right), 0, 0, \psi\left(d\left(z, Bu\right)\right), \psi\left(d\left(z, Bu\right)\right), 0\Big) < 0,$$

a contradiction of (F_2). Hence, $\psi(d(z, Bu)) = 0$, which implies $d(z, Bu) = 0$, i.e. $Tu = z \in Bu$. Therefore $\mathcal{C}(T, B) \neq \emptyset$.

On the other hand, $z \in S(X)$. Hence, there exists $v \in X$ such that $z = Sv$. By (4.1) for $x = v$ and $y = u$ we obtain

$$\begin{aligned} &F\Big(\psi\left(H\left(Av, Bu\right)\right), \psi\left(d\left(Sv, Tu\right)\right), \psi\left(d\left(Sv, Av\right)\right),\\ &\qquad \psi\left(d\left(Tu, Bu\right)\right), \psi\left(d\left(Sv, Bu\right)\right), \psi(d\left(Tu, Av\right))\Big)+\\ &+G\Big(\psi(d\left(Sv, Tu\right)), \psi\left(d\left(Sv, Av\right)\right), \psi\left(d\left(Tu, Bu\right)\right),\\ &\qquad \psi\left(d\left(Sv, Bu\right)\right), \psi\left(d\left(Tu, Av\right)\right)\Big) \leq 0. \end{aligned}$$

Since $z \in Bu$, then $d(z, Av) \leq H(Av, Bu)$ which implies by (ψ_1) that $\psi(d(z, A)) \leq \psi(H(Av, Bu))$. By (F_1) we have

$$\begin{aligned} &F\Big(\psi\left(d\left(z, Av\right)\right), 0, \psi\left(d\left(z, Av\right)\right), 0, 0, \psi(d\left(z, Av\right))\Big)+\\ &+G\Big(0, \psi\left(d\left(z, Av\right)\right), 0, 0, \psi\left(d\left(z, Av\right)\right)\Big) \leq 0. \end{aligned}$$

If $d(z, Au) > 0$, then by $G \in \mathcal{G}_M$, $G\big(0, \psi(d(z, Av)), 0, 0, \psi(d(z, Av))\big) > 0$, which implies,

$$F\Big(\psi(d(z, Av)), 0, \psi(d(z, Av)), 0, 0, \psi(d(z, Av))\Big) \leq 0,$$

a contradiction of (F_3). Hence, $d(z, Av) = 0$ which implies $Sv = z \in Av$. Therefore $\mathcal{C}(A, S) \neq \emptyset$.

Moreover,

a) If S is coincidentally idempotent with respect to A, then $Sz = SSz = Sv = z$ and z is a fixed point of S. By (4.1) for $x = z$ and $y = u$ we obtain

$$\begin{aligned} &F\Big(\psi(H(Az, Bu)), \psi(d(Sz, Tu)), \psi(d(Sz, Az)), \\ &\qquad \psi(d(Tu, Bu)), \psi(d(Sz, Bu)), \psi(d(Tu, Az))\Big) + \\ &+ G\Big(\psi(d(Sz, Tu)), \psi(d(Sz, Az)), \psi(d(Tu, Bu)), \\ &\qquad \psi(d(Sz, Bu)), \psi(d(Tu, Az))\Big) \leq 0. \end{aligned}$$

Since

$$d(Sz, Az) = d(Sv, Az) = d(Tu, Az) \leq H(Az, Bu),$$

then $\psi(d(Sz, Az)) \leq \psi(H(Az, Bu))$.

By (F_1) we have

$$\begin{aligned} &F\Big(\psi(d(z, Az)), 0, \psi(d(z, Az)), 0, 0, \psi(d(z, Az))\Big) + \\ &+ G\Big(0, \psi(d(z, Az)), 0, 0, \psi(d(z, Az))\Big) \leq 0. \end{aligned}$$

If $d(z, Az) > 0$, then $\psi(d(z, Az)) > 0$, which implies

$$G(0, \psi(d(z, Az)), 0, 0, \psi(d(z, Az))) > 0.$$

Hence,

$$F\Big(\psi(d(z, Az)), 0, \psi(d(z, Az)), 0, 0, \psi(d(z, Az))\Big) < 0,$$

a contradiction of (F_3). Hence, $d(z, Az) = 0$ which implies $Sz = z \in Az$. Therefore z is a common fixed point of A and S.

b) If T is coincidentally idempotent with respect to B, then $Tz = TTu = Tu = z$ and z is a fixed point of T. By (4.1) for $x = v$ and $y = z$ we have

$$\begin{aligned} &F\Big(\psi(H(Av, Bz)), \psi(d(Sv, Tz)), \psi(d(Sv, Av)), \\ &\qquad \psi(d(Tz, Bz)), \psi(d(Sv, Bz)), \psi(d(Tz, Av))\Big) + \\ &+ G\Big(\psi(d(Sv, Tz)), \psi(d(Sv, Av)), \psi(d(Tz, Bz)), \\ &\qquad \psi(d(Sv, Bz)), \psi(d(Tz, Av))\Big) \leq 0. \end{aligned}$$

Since

$$d(Tz, Bz) = d(Sv, Bz) \leq H(Av, Bz),$$

it follows that $\psi(d(Tz,Bz)) \leq \psi(d(z,Bz)) \leq \psi(H(Av,Bz))$.

By (F_1) we have

$$F\Big(\psi(d(z,Bz)),0,0,\psi(d(z,Bz)),\psi(d(z,Bz)),0\Big)+ \\ +G\Big(0,0,\psi(d(z,Bz)),\psi(d(z,Bz)),0\Big) \leq 0.$$

If $d(z,Bz)>0$, then $\psi(d(z,Bz))>0$ and

$$G\Big(0,0,\psi(d(z,Bz)),\psi(d(z,Bz)),0\Big) > 0.$$

Hence,

$$F\Big(\psi(d(z,Bz)),0,0,\psi(d(z,Bz)),\psi(d(z,Bz)),0\Big) < 0,$$

a contradiction of (F_3). Hence, $d(z,Bz)=0$ which implies $Tz = z \in Bz$ and z is a common fixed point of B and T.

c) If the conditions of a) and b) hold, then z is a common fixed point of A, B, S and T. □

If $\psi(t)=t$ by Theorem 4.1 we obtain

Theorem 4.2. *Let (X,d) be a metric space, $A,B: X \to CL(X)$ and $S,T: X \to X$ such that for all $x,y \in X$*

$$\begin{aligned} F\Big(&H(Ax,By), d(Sx,Ty), d(Sx,Ax), \\ &d(Ty,By), d(Sx,By), d(Ty,Ax)\Big) \\ &+G\Big((dSx,Ty), d(Sx,Ax), d(Ty,By), d(Sx,By), d(Ty,Ax)\Big) \leq 0 \end{aligned} \tag{4.2}$$

for some $F \in \mathcal{F}_M$ and $G \in \mathcal{G}_M$.

If (A,S) and T satisfy $CLR_{(A,S)T}$ - property, then

1) $\mathcal{C}(A,S) \neq \emptyset$,
2) $\mathcal{C}(B,T) \neq \emptyset$.

Moreover,

a) *if S is coincidentally idempotent with respect to A, then S and A have a common fixed point,*
b) *if T is coincidentally idempotent with respect to B, then T and B have a common fixed point,*
c) *if the conditions of a) and b) hold, then A, B, S and T have a common fixed point.*

5. Applications

5.1. **Fixed points for hybrid pairs of mappings satisfying contractive conditions of integral type.** In [1], Branciari established the following theorem, which opened the way to the study of fixed points for mappings satisfying a contractive condition of integral type.

Theorem 5.1 ([1]). *Let* (X,d) *be a metric space,* $c \in (0,1)$ *and* $f : X \to X$ *such that for all* $x, y \in X$

$$\int_0^{d(fx,fy)} h(t)\mathrm{d}t \le c \int_0^{d(x,y)} h(t)\mathrm{d}t,$$

where $h : [0,\infty) \to [0,\infty)$ *is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0,\infty)$*, such that* $\int_0^{\varepsilon} h(t)\mathrm{d}t > 0$*, for each* $\varepsilon > 0$*. Then,* f *has a unique fixed point* $z \in X$ *such that for all* $x \in X$*,* $z = \lim_{n\to\infty} f^n x$*.*

Some fixed point results for mappings satisfying contractive conditions of integral type are obtained in [15] and in other papers.

Lemma 5.1 ([15]). *Let* $h : [0,\infty) \to [0,\infty)$ *be as in Theorem 5.1. Then* $\psi(t) = \int_0^t h(x)\mathrm{d}x$ *is an altering distance.*

Theorem 5.2. *Let* (X,d) *be a metric space,* $A, B : X \to CL(X)$ *and* $S, T : X \to X$ *such that for all* $x, y \in X$

$$\begin{aligned} F\Big(&\int_0^{H(Ax,By)} h(t)\mathrm{d}t, \int_0^{d(Sx,Ty)} h(t)\mathrm{d}t, \int_0^{d(Sx,Ax)} h(t)\mathrm{d}t, \\ &\int_0^{d(Ty,By)} h(t)\mathrm{d}t, \int_0^{d(Sx,By)} h(t)\mathrm{d}t, \int_0^{d(Ty,Ax)} h(t)\mathrm{d}t\Big) + \\ + \ G\Big(&\int_0^{d(Sx,Ty)} h(t)\mathrm{d}t, \int_0^{d(Sx,Ax)} h(t)\mathrm{d}t, \int_0^{d(Ty,By)} h(t)\mathrm{d}t, \\ &\int_0^{d(Sx,By)} h(t)\mathrm{d}t, \int_0^{d(Ty,Ax)} h(t)\mathrm{d}t\Big) \le 0 \end{aligned} \tag{5.1}$$

for some $F \in \mathcal{F}_M$*,* $G \in \mathcal{G}_M$ *and* $h(t)$ *as in Theorem 5.1.*

If (A,S) *and* T *satisfy* $CLR_{(A,S)T}$ *- property, then*

1) $\mathcal{C}(A,S) \neq \emptyset$,
2) $\mathcal{C}(B,T) \neq \emptyset$.

Moreover,

a) *if* S *is coincidentally idempotent with respect to* A*, then* S *and* A *have a common fixed point,*
b) *if* T *is coincidentally idempotent with respect to* B*, then* T *and* B *have a common fixed point,*
c) *if the conditions of a) and b) hold, then* A, B, S *and* T *have a common fixed point.*

Proof. By Lemma 5.1, $\psi(t) = \int_0^t h(x)\,\mathrm{d}x$ is an altering distance. Then

$$\begin{aligned} &\int_0^{H(Ax,By)} h(t)\,\mathrm{d}t = \psi(H(Ax,By)), \int_0^{d(Sx,Ty)} h(t)\,\mathrm{d}t = \psi(d(Sx,Ty)), \\ &\int_0^{d(Sx,Ax)} h(t)\,\mathrm{d}t = \psi(d(Sx,Ax)), \int_0^{d(Ty,By)} h(t)\,\mathrm{d}t = \psi(d(Ty,By)), \\ &\int_0^{d(Sx,By)} h(t)\,\mathrm{d}t = \psi(d(Sx,By)), \int_0^{d(Ty,Ax)} h(t)\,\mathrm{d}t = \psi(d(Ty,Ax)). \end{aligned}$$

By (5.2) we obtain

$$\begin{aligned} &F\Big(\psi\left(H\left(Ax,By\right)\right),\psi\left(d\left(Sx,Ty\right)\right),\psi\left(d\left(Sx,Ax\right)\right),\\ &\quad\psi\left(d\left(Ty,By\right)\right),\psi\left(d\left(Sx,By\right)\right),\psi\left(d\left(Ty,Ax\right)\right)\Big)+\\ &+G\Big(\psi\left(d\left(Sx,Ty\right)\right),\psi\left(d\left(Sx,Ax\right)\right),\psi\left(d\left(Ty,By\right)\right),\\ &\quad\psi\left(d\left(Sx,By\right)\right),\psi\left(d\left(Ty,Ax\right)\right)\Big)\leq 0,\end{aligned}$$

which is inequality (4.1). Hence, the conditions of Theorem 4.1 are satisfied and the conclusions of Theorem 5.2 follows by Theorem 4.1. □

For example, by Theorem 4.1 and Examples 3.1 and 3.9 we obtain

Theorem 5.3. *Let (X,d) be a metric space, $A,B : X \to CL(X)$ and $S,T : X \to X$ such that for all $x,y \in X$*

$$\begin{aligned}\int_0^{H(Ax,By)} h(t)\mathtt{d}t \leq k\max\{&\int_0^{d(Sx,Ty)} h(t)\mathtt{d}t, \int_0^{d(Sx,Ax)} h(t)\mathtt{d}t,\\ &\int_0^{d(Ty,By)} h(t)\mathtt{d}t, \int_0^{d(Sx,By)} h(t)\mathtt{d}t, \int_0^{d(Ty,Ax)} h(t)\mathtt{d}t\}-\\ -\max\{&\int_0^{d(Sx,Ty)} h(t)\mathtt{d}t, \int_0^{d(Sx,Ax)} h(t)\mathtt{d}t, \int_0^{d(Ty,By)} h(t)\mathtt{d}t,\\ &\int_0^{d(Sx,By)} h(t)\mathtt{d}t, \int_0^{d(Ty,Ax)} h(t)\mathtt{d}t\}\end{aligned}$$

where $k \in [0,1)$ and h is as in Theorem 5.1.

If (A,S) and T satisfy $CLR_{(A,S)T}$ - property, then

1) $\mathcal{C}(A,S) \neq \emptyset$,
2) $\mathcal{C}(B,T) \neq \emptyset$.

Moreover,

a) *if S is coincidentally idempotent with respect to A, then S and A have a common fixed point,*
b) *if T is coincidentally idempotent with respect to B, then T and B have a common fixed point,*
c) *if the conditions of a) and b) hold, then A, B, S and T have a common fixed point.*

Remark 5.1. Combining Examples 3.2-3.8 and 3.10-3.16 with Theorem 4.1 we obtain new particular results.

5.2. **Fixed points for hybrid pair of mappings using φ-maps.** As in [7], let Φ be the set of all nondecreasing continuous functions $\varphi : [0,\infty) \to [0,\infty)$ such that

1) $\varphi(t) < t$, for all $t > 0$,
2) $\varphi(0) = 0$.

The following functions $F(t_1,...,t_6) \in \mathcal{F}_M$.

Example 5.1. $F(t_1,\ldots,t_6) = t_1 - \varphi(\max\{t_2,\ldots,t_6\})$.

Example 5.2. $F(t_1,\ldots,t_6) = t_1 - \varphi\Big(\max\Big\{t_2,t_3,t_4,\dfrac{t_5+t_6}{2}\Big\}\Big)$.

Example 5.3. $F(t_1,...,t_6) = t_1 - \varphi\Big(\max\Big\{t_2, \dfrac{t_3+t_4}{2}, \dfrac{t_5+t_6}{2}\Big\}\Big)$.

Example 5.4. $F(t_1,...,t_6) = t_1 - \varphi\left(\max\{t_2, \sqrt{t_3t_5}, \sqrt{t_4t_6}, \sqrt{t_5t_6}\}\right)$.

Example 5.5. $F(t_1,...,t_6) = t_1 - \varphi(at_2 + bt_3 + ct_4 + dt_5 + et_6)$, where $a,b,c,d,e \geq 0$ and $a+b+c+d+e \leq 1$.

Example 5.6. $F(t_1,...,t_6) = t_1 - \varphi\Big(at_2 + \dfrac{b\sqrt{t_5t_6}}{1+t_3+t_4}\Big)$, where $a,b \geq 0$ and $a+b \leq 1$.

Example 5.7. $F(t_1,...,t_6) = t_1 - \varphi\left(at_2 + b\max\{t_3,t_4\} + c\max\{\frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\}\right)$, where $a,b,c \geq 0$ and $a+b+c \leq 1$.

Example 5.8. $F(t_1,...,t_6) = t_1 - \varphi\left(at_2 + b\max\left\{\frac{2t_4+t_5}{3}, \frac{2t_4+t_6}{3}, \frac{t_3+t_5+t_6}{3}\right\}\right)$, where $a,b \geq 0$ and $a+b \leq 1$.

By Theorem 4.2 and Examples 5.1 and 3.9 we obtain

Theorem 5.4. *Let (X,d) be a metric space, $A,B : X \to CL(X)$ and $S,T : X \to X$ such that for all $x,y \in X$*

$$\begin{aligned} H(Ax,By) \leq \varphi(\max\{&d(Sx,Ty), d(Sx,Ax), d(Ty,By), \\ &d(Sx,By), d(Ty,Ax)\})+ \\ +G(\max\{&d(Sx,Ty), d(Sx,Ax), d(Ty,By), \\ &d(Sx,By), d(Ty,Ax)\})) \end{aligned}$$

for some $F \in \mathcal{F}_M$, $G \in \mathcal{G}_M$ and $\varphi \in \Phi$.

If (A,S) and T satisfy $CLR_{(A,S)T}$ - property, then

1) $\mathcal{C}(A,S) \neq \emptyset$,
2) $\mathcal{C}(B,T) \neq \emptyset$.

Moreover,

a) *if S is coincidentally idempotent with respect to A, then S and A have a common fixed point,*
b) *if T is coincidentally idempotent with respect to B, then T and B have a common fixed point,*
c) *if the conditions of a) and b) hold, then A,B,S and T have a common fixed point.*

Remark 5.2. Combining Examples 5.2-5.8 and 3.10-3.16 with Theorem 4.2 we obtain new particular results.

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