Differential sandwich results for Wanas operator of analytic functions

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Abstract. In the present article, we determine some subordination and superordination results involving Wanas operator for certain normalized analytic functions defined in the unit disk $\mathbb{U}$. These results are applied to establish sandwich results. Our results extend corresponding previously known results.

1. Introduction

Denote by $H = H(\mathbb{U})$ the collection of analytic functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and assume that $H[a,n]$ be the subclass of $H$ consisting of functions of the form:

$$f(z) = a + anz^n + an+1z^{n+1} + \ldots \quad (a \in \mathbb{C}, \ n \in \mathbb{N} = \{1,2,\ldots\}).$$

Also, let $\mathcal{A}$ be the subclass of $H$ consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$ (1)

Now we recall the principal of subordination between analytic functions, let the functions $f$ and $g$ be analytic in $\mathbb{U}$, we say that the function $f$ is subordinate to $g$, if there exists a Schwarz function $w$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1 \ (z \in \mathbb{U})$ such that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z) \ (z \in \mathbb{U})$. Furthermore, if the function $g$ is univalent in $\mathbb{U}$, then we have the following equivalent (see [8]),

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\xi, h \in H$ and $\psi (r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. If $\xi$ and

$$\psi (\xi (z), z\xi' (z), z^2\xi'' (z) ; z)$$

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are univalent functions in $\mathbb{U}$ and if $\xi$ satisfies the second-order differential superordination

\[(2) \quad h(z) \prec \psi(\xi(z), z\xi'(z), z^2\xi''(z); z),\]

then $\xi$ is called a solution of the differential superordination (2). (If $f$ is subordinate to $g$, then $g$ is superordinate to $f$). An analytic function $q$ is called a subordinant of (2), if $q \prec \xi$ for all $\xi$ satisfying (2). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinates $q$ of (2) is called the best subordinant.

For $\alpha \in \mathbb{R}$, $\beta \geq 0$ with $\alpha + \beta > 0$, $m, \delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}$, the Wanas operator $W_{\alpha,\beta}^{k,\delta}: \mathcal{A} \rightarrow \mathcal{A}$ (see [24]) is defined by

\[(3) \quad W_{\alpha,\beta}^{k,\delta} f(z) = z + \sum_{n=2}^{\infty} \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\delta} a_n z^n.\]

**Remark 1.** It should be remarked that the operator $W_{\alpha,\beta}^{k,\delta}$ generalizes some known operators considered earlier:

1. For $k = 1$, the operator $W_{\alpha,\beta}^{1,\delta} \equiv I_{\alpha,\beta}^\delta$ was introduced and studied by Swamy [22],
2. For $k = \beta = 1$, $\delta = -\mu$, $\text{Re}(\mu) > 1$ and $\alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the operator $W_{\alpha,1}^{1,-\mu} \equiv J_{\mu,\alpha}$ was investigated by Srivastava and Attiya [16]. The operator $J_{\mu,\alpha}$ is now popularly known in the literature as the Srivastava-Attiya operator. Various applications of the Srivastava-Attiya operator are found in [15, 17, 18, 19, 20] and in the references cited in each of these earlier works,
3. For $k = \beta = 1$ and $\alpha > -1$, the operator $W_{\alpha,1}^{1,\delta} \equiv I_{\alpha}^\delta$ was investigated by Cho and Srivastava [6],
4. For $k = \alpha = \beta = 1$, the operator $W_{1,1}^{1,\delta} \equiv I_{\alpha}^\delta$ was considered by Uralegaddi and Somanatha [23],
5. For $k = \alpha = \beta = 1$, $\delta = -\sigma$ and $\sigma > 0$, the operator $W_{1,1}^{1,-\sigma} \equiv I_{\sigma}$ was introduced by Jung et al. [7]. The operator $I_{\sigma}$ is the Jung-Kim-Srivastava integral operator,
6. For $k = \beta = 1$, $\delta = -1$ and $\alpha > -1$, the operator $W_{\alpha,1}^{1,-1} \equiv L_{\alpha}$ was studied by Bernardi [4],
7. For $\alpha = 0$, $k = \beta = 1$ and $\delta = -1$, the operator $W_{0,1}^{1,-1} \equiv u$ was investigated by Alexander [1],
8. For $k = 1$, $\alpha = 1 - \beta$ and $\beta \geq 0$, the operator $W_{1,1}^{1,\delta} \equiv D_{\beta}^\delta$ was given by Al-Oboudi [2],
9. For $k = 1$, $\alpha = 0$ and $\beta = 1$, the operator $W_{0,1}^{1,\delta} \equiv S_{\alpha}^\delta$ was considered by Sălăgean [13].
It is readily verified from (3) that

\[ z \left( W_{\alpha,\beta}^k f(z) \right)' = \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \right] W_{\alpha,\beta}^{k+1} f(z) \]

\[ - \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right] W_{\alpha,\beta}^k f(z). \]

Very recently, Rahrovi [12], Attiya and Yassen [3], Seoudy [14], Wanas and Majeed [25] and Srivastava and Wanas [21] have obtained sandwich results for certain classes of analytic functions. Motivated by aforementioned works to investigate sufficient condition for \( f \) based on Wanas differential operator we define a new subclasses of normalized analytic functions satisfying the following:

\[ q_1(z) \prec \left( \frac{W_{\alpha,\beta}^k f(z)}{z} \right)^\gamma \prec q_2(z) \]

and

\[ q_1(z) \prec \left( \frac{W_{\alpha,\beta}^{k+1} f(z)}{W_{\alpha,\beta}^k f(z)} \right)^\gamma \prec q_2(z), \]

where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = q_2(0) = 1 \).

To establish our main results, we need the following definition and lemmas.

**Definition 1** ([8]). Denote by \( Q \) the set of all functions \( f \) that are analytic and injective on \( \mathbb{U} \setminus E(f) \), where

\[ E(f) = \left\{ \zeta \in \partial \mathbb{U} : \lim_{z \to \zeta} f(z) = \infty \right\} \]

and are such that \( f'(\zeta) \neq 0 \) for \( \zeta \in \partial \mathbb{U} \setminus E(f) \).

**Lemma 1** ([8]). Let \( q \) be univalent in the unit disk \( \mathbb{U} \) and let \( \theta \) and \( \phi \) be analytic in a domain \( D \) containing \( q(\mathbb{U}) \) with \( \phi(w) \neq 0 \) when \( w \in q(\mathbb{U}) \). Set \( Q(z) = zq'(z)\phi(q(z)) \) and \( h(z) = \theta(q(z)) + Q(z) \). Suppose that

1. \( Q(z) \) is starlike univalent in \( \mathbb{U} \),
2. \( \Re \left( \frac{zh'(z)}{Q(z)} \right) > 0 \) for \( z \in \mathbb{U} \).

If \( \xi \) is analytic in \( \mathbb{U} \), with \( \xi(0) = q(0) \), \( \xi(\mathbb{U}) \subset D \) and

\[ \theta(\xi(z)) + z\xi'(z)\phi(\xi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \]

then \( \xi \prec q \) and \( q \) is the best dominant of (5).

**Lemma 2** ([9]). Let \( q \) be a convex univalent function in \( \mathbb{U} \) and let \( \mu \in \mathbb{C} \), \( \nu \in \mathbb{C} \setminus \{0\} \) with

\[ \Re \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\Re \left( \frac{\mu}{\nu} \right) \right\}. \]
If $\xi$ is analytic in $U$ and
\[(6)\quad \mu \xi (z) + \nu z \xi' (z) < \mu q (z) + \nu z q' (z),\]
then $\xi \prec q$ and $q$ is the best dominant of (6).

Lemma 3 ([9]). Let $q$ be convex univalent in $U$ and let $\nu \in \mathbb{C}$. Further assume that $\Re (\nu) > 0$. If $\xi \in H [q (0), 1] \cap Q$ and $\xi (z) + \nu z \xi' (z)$ is univalent in $U$, then
\[(7)\quad q (z) + \nu z q' (z) \prec \xi (z) + \nu z \xi' (z),\]
which implies that $q \prec \xi$ and $q$ is the best subordinant of (7).

Lemma 4 ([5]). Let $q$ be convex univalent in the unit disk $U$ and let $\theta$ and $\phi$ be analytic in a domain $D$ containing $q (U)$. Suppose that
\[(1)\quad \Re \left( \frac{\theta' (q (z))}{\phi' (q (z))} \right) > 0 \text{ for } z \in U,
\]
\[(2)\quad Q (z) = z q' (z \phi (q (z))) \text{ is starlike univalent in } U.
\]
If $\xi \in H [q (0), 1] \cap Q$, with $\xi (U) \subset D$, $\phi (\xi (z)) + z \xi' (z) \phi (\xi (z))$ is univalent in $U$ and
\[(8)\quad \theta (q (z)) + z q' (z) \phi (q (z)) < \theta (\xi (z)) + z \xi' (z) \phi (\xi (z)),\]
then $q \prec \xi$ and $q$ is the best subordinant of (8).

2. Main Results

Theorem 1. Let $q$ be convex univalent in $U$ with $q (0) = 1$, $\sigma \in \mathbb{C} \setminus \{0\}$, $\gamma > 0$ and suppose that $q$ satisfies
\[(9)\quad \Re \left( 1 + \frac{z q'' (z)}{q' (z)} \right) > \max \left\{ 0, -\Re \left( \frac{\gamma}{\sigma} \right) \right\}.
\]
If $f \in \mathcal{A}$ satisfies the subordination
\[(10)\quad \left[ 1 - \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \binom{m}{\alpha} + 1 \right) \left( \frac{W^{k, \delta}_{\alpha, \beta} f (z)}{z} \right)^{\gamma} \right]
\]
\[+ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \binom{m}{\alpha} + 1 \right) \left( \frac{W^{k, \delta+1}_{\alpha, \beta} f (z)}{W^{k, \delta}_{\alpha, \beta} f (z)} \right)^{\gamma} \left( \frac{W^{k, \delta}_{\alpha, \beta} f (z)}{W^{k, \delta+1}_{\alpha, \beta} f (z)} \right) \]
\[\prec q (z) + \frac{\sigma}{\gamma} z q' (z),\]
then
\[(11)\quad \left( \frac{W^{k, \delta}_{\alpha, \beta} f (z)}{z} \right)^{\gamma} \prec q (z)\]
and $q$ is the best dominant of (10).
Proof. Define the function $\xi$ by
\[
\xi(z) = \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma}, \quad (z \in U).
\]

Differentiating (12) logarithmically with respect to $z$, we get
\[
\frac{z \xi'(z)}{\xi(z)} = \gamma \left( \frac{z \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)' - 1}{W_{\alpha,\beta}^{k,\delta} f(z)} \right).
\]

Now, in view of (4), we obtain the following subordination
\[
\frac{z \xi'(z)}{\xi(z)} = \sum_{m=1}^{k} \left( \frac{k}{m} \right) (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^{m} + 1 \right) \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right) \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} - 1 \right).
\]

Therefore,
\[
\frac{z \xi'(z)}{\gamma} = \sum_{m=1}^{k} \left( \frac{k}{m} \right) (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^{m} + 1 \right) \times \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} - 1 \right).
\]

The subordination (10) from the hypothesis becomes
\[
\xi(z) + \frac{\sigma}{\gamma} z \xi'(z) < q(z) + \frac{\sigma}{\gamma} z q'(z).
\]

Hence, an application of Lemma 2 with $\mu = 1$ and $\nu = \frac{\sigma}{\gamma}$, we obtain (11). \qed

**Theorem 2.** Let $\eta, \tau \in \mathbb{C}$, $\gamma > 0$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $q$ be convex univalent in $U$ with $q(0) = 1$, $q(z) \neq 0$ $(z \in U)$ and assume that $q$ satisfies
\[
\Re \left( 1 + \frac{\tau}{\lambda} q(z) + \frac{z q''(z)}{q'(z)} - \frac{z q'(z)}{q(z)} \right) > 0.
\]

Suppose that $\frac{z q'(z)}{q(z)}$ is starlike univalent in $U$. If $f \in A$ satisfies
\[
\Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) \prec \eta + \tau q(z) + \lambda \frac{z q'(z)}{q(z)},
\]

where
\[
\Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) = \eta + \tau \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)^{\gamma} + \gamma \lambda \sum_{m=1}^{k} \left( \frac{k}{m} \right) (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^{m} + 1 \right) \left( \frac{W_{\alpha,\beta}^{k,\delta+2} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} - \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right),
\]
then

\[ \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)^{\gamma} < q(z) \]

and \( q \) is the best dominant of (14).

**Proof.** Define the function \( \xi \) by

\[ \xi(z) = \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)^{\gamma}, \quad (z \in \mathbb{U}). \]

By a straightforward computation and using (4), we have

\[ \eta + \tau \xi(z) + \lambda \frac{z \xi'(z)}{\xi(z)} = \Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z), \]

where \( \Omega(\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) \) is given by (15). From (14) and (17), we obtain

\[ \eta + \tau \xi(z) + \lambda \frac{z \xi'(z)}{\xi(z)} < \eta + \tau q(z) + \lambda \frac{z q'(z)}{q(z)}. \]

By setting

\[ \theta(w) = \eta + \tau w \quad \text{and} \quad \phi(w) = \frac{\lambda}{w}, \quad w \neq 0, \]

we see that \( \theta(w) \) is analytic in \( \mathbb{C} \), \( \phi(w) \) is analytic in \( \mathbb{C} \setminus \{0\} \) and that \( \phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\} \). Also, we get

\[ Q(z) = z q'(z) \phi(q(z)) = \lambda \frac{z q'(z)}{q(z)} \]

and

\[ h(z) = \theta(q(z)) + Q(z) = \eta + \tau q(z) + \lambda \frac{z q'(z)}{q(z)}. \]

It is clear that \( Q(z) \) is starlike univalent in \( \mathbb{U} \),

\[ \Re \left( \frac{zh'(z)}{Q(z)} \right) = \Re \left( 1 + \frac{\tau}{\lambda} q(z) + \frac{z q''(z)}{q'(z)} - \frac{z q'(z)}{q(z)} \right) > 0. \]

Thus, by Lemma 1, we get \( \xi(z) < q(z) \). By using (16), we obtain the desired result. \( \square \)

**Theorem 3.** Let \( q \) be convex univalent in \( \mathbb{U} \) with \( q(0) = 1 \), \( \gamma > 0 \) and \( \Re(\sigma) > 0 \). Let \( f \in \mathcal{A} \) satisfies

\[ \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \in H[q(0), 1] \cap Q \]

and

\[ \left[ 1 - \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^{m+1} \right] \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \]
\[ + \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \left( \frac{W_{\alpha,\beta}^k f(z)}{z} \right)^{\gamma} \left( \frac{W_{\alpha,\beta}^k f(z)}{W_{\alpha,\beta}^k f(z)} \right) \]

be univalent in \( \mathbb{U} \). If

\[ q(z) + \frac{\sigma}{\gamma} z q'(z) \]

\[ \prec \left[ 1 - \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \right] \left( \frac{W_{\alpha,\beta}^k f(z)}{z} \right)^{\gamma} \]

\[ + \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \left( \frac{W_{\alpha,\beta}^k f(z)}{z} \right)^{\gamma} \times \]

\[ \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right), \]

then

\[ q(z) \prec \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \]

and \( q \) is the best subordinant of (18).

**Proof.** Let \( \xi \) be defined by (12), then differentiating \( \xi \) with respect to \( z \), we get

\[ \frac{z \xi'(z)}{\xi(z)} = \gamma \left( \frac{z \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)'}{W_{\alpha,\beta}^{k,\delta} f(z)} - 1 \right). \]

By using (4) for \( \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)' \), in (20), we have

\[ \left[ 1 - \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \right] \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \]

\[ + \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \times \]

\[ \times \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right) = \xi(z) + \frac{\sigma}{p\gamma} z \xi'(z). \]

From (18) and (21), we get

\[ q(z) + \frac{\sigma}{\gamma} z q'(z) \prec \xi(z) + \frac{\sigma}{\gamma} z \xi'(z). \]

Hence, by using Lemma 3 with \( \mu = 1 \) and \( \nu = \frac{\sigma}{\gamma} \), we obtain (19). \( \square \)
Theorem 4. Let $\eta \in \mathbb{C}$, $\gamma > 0$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $q$ be convex univalent in $U$ with $q(0) = 1$, $q(z) \neq 0$ ($z \in U$) and assume that $q$ satisfies

$$\Re \left( \frac{\tau}{\lambda} q(z) \right) > 0. \quad (22)$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $U$. If $f \in A$ satisfies

$$\left( \frac{W^k_{\alpha,\beta} f(z)}{W^{k+1}_{\alpha,\beta} f(z)} \right) ^{\gamma} \in H[q(0), 1] \cap Q$$

and $\Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z)$ is univalent in $U$, where $\Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z)$ is given by (15). If

$$\eta + \tau q(z) + \lambda \frac{zq'(z)}{q(z)} \prec \Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z), \quad (23)$$

then

$$q(z) \prec \left( \frac{W^k_{\alpha,\beta} f(z)}{W^{k+1}_{\alpha,\beta} f(z)} \right) ^{\gamma}$$

and $q$ is the best subordinant of (23).

Proof. Assume that the function $\xi$ be defined by (16). By a straightforward computation, we have

$$\Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) = \eta + \tau \xi(z) + \lambda \frac{z \xi'(z)}{\xi(z)}, \quad (24)$$

where $\Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z)$ is given by (15). From (23) and (24), we obtain

$$\eta + \tau q(z) + \lambda \frac{zq'(z)}{q(z)} \prec \eta + \tau \xi(z) + \lambda \frac{z \xi'(z)}{\xi(z)}.$$

By setting $\theta(w) = \eta + \tau w$ and $\phi(w) = \frac{\lambda}{w}$, $w \neq 0$, we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z) \phi(q(z)) = \lambda \frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in $U$,

$$\Re \left( \frac{\theta'(q(z))}{\phi(q(z))} \right) = \Re \left( \frac{\tau}{\lambda} q(z) \right) > 0.$$

Thus, by Lemma 4, we get $q(z) \prec \xi(z)$. By using (16), we obtain the desired result. \qed

Concluding the results of differential subordination and superordination, we state the following “sandwich results”.


Theorem 5. Let \( q_1 \) and \( q_2 \) be convex univalent in \( U \) with \( q_1(0) = q_2(0) = 1 \). Suppose \( q_2 \) satisfies (9), \( \gamma > 0 \) and \( \Re(\sigma) > 0 \). Let \( f \in A \) satisfies
\[
\left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \in H[1,1] \cap Q
\]
and
\[
\left[ 1 - \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \right] + \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)
\]
be univalent in \( U \). If
\[
q_1(z) + \frac{\sigma}{\gamma} z q_1'(z)
\]
\[
\prec \left[ 1 - \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \right] + \sigma \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \left( \frac{\alpha}{\beta} \right)^m + 1 \right) \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)
\]
\[
\prec q_2(z) + \frac{\sigma}{\gamma} z q_2'(z),
\]
then
\[
q_1(z) \prec \left( \frac{W_{\alpha,\beta}^{k,\delta} f(z)}{z} \right)^{\gamma} \prec q_2(z)
\]
and \( q_1 \) and \( q_2 \) are, respectively, the best subordinant and the best dominant.

Theorem 6. Let \( q_1 \) and \( q_2 \) be convex univalent in \( U \) with \( q_1(0) = q_2(0) = 1 \). Suppose \( q_1 \) satisfies (22) and \( q_2 \) satisfies (13). Let \( f \in A \) satisfies
\[
\left( \frac{W_{\alpha,\beta}^{k,\delta+1} f(z)}{W_{\alpha,\beta}^{k,\delta} f(z)} \right)^{\gamma} \in H[1,1] \cap Q
\]
and \( \Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) \) is univalent in \( U \), where \( \Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z) \) is given by (15). If
\[
\eta + \tau q_1(z) + \lambda \frac{z q_1'(z)}{q_1(z)} \prec \Omega (\eta, \tau, \gamma, \lambda, k, \delta, \alpha, \beta; z)
\]
\[
\prec \eta + \tau q_2(z) + \lambda \frac{z q_2'(z)}{q_2(z)},
\]
then
\[ q_1(z) \prec \left( \frac{W^{k,\delta+1}_{\alpha,\beta} f(z)}{W^{k,\delta}_{\alpha,\beta} f(z)} \right)^\gamma \prec q_2(z) \]
and \( q_1 \) and \( q_2 \) are, respectively, the best subordinant and the best dominant.

**Remark 2.** By selecting the particular values of \( \delta, k, \alpha \) and \( \beta \), we can derive a number of known results. Some of them are given below:

1. Taking \( \delta = 0 \) in Theorem 1, we obtain the results obtained by Murugusundaramoorthy and Magesh [10, Corollary 3.3],
2. Putting \( k = 1, \alpha = 1 - \beta \) and \( \beta \geq 0 \) in Theorems 1, 3 and 5, we get the results obtained by Răducanu and Nechita [11, Theorem 3.1, Theorem 3.6, Theorem 3.9],
3. Setting \( \alpha = 0 \) and \( k = \beta = 1 \) in Theorems 1, 3 and 5, we get the results obtained by Răducanu and Nechita [11, Corollary 3.3, Corollary 3.8, Corollary 3.11].

**References**


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