POTENTIALLY GRAPHIC SEQUENCES OF SPLIT GRAPHS

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Abstract. A sequence $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative integers is said to be graphic if it is the degree sequence of a simple graph $G$ on $n$ vertices, and such a graph $G$ is referred to as a realization of $\pi$. The set of all non-increasing non-negative integer sequences $\pi = (d_1, d_2, \ldots, d_n)$ is denoted by $NS_n$. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph $G$ on $n$ vertices, and such a graph $G$ is called a realization of $\pi$. The set of all graphic sequences in $NS_n$ is denoted by $GS_n$. A graphic sequence $\pi$ is potentially $H$-graphic if there is a realization of $\pi$ containing $H$ as a subgraph. In this paper, we determine the graphic sequences of subgraphs $H$, where $H$ is $S_{r_1,s_1} + S_{r_2,s_2} + S_{r_3,s_3} + \ldots + S_{r_m,s_m}$, $S_{r_1,s_1} \vee S_{r_2,s_2} \vee \ldots \vee S_{r_m,s_m}$, and $S_{r_1,s_1} \times S_{r_2,s_2} \times \ldots \times S_{r_m,s_m}$ and $+$, $\vee$ and $\times$ denotes the standard join operation, the normal join operation and the cartesian product in these graphs respectively.

1. Introduction

Let $G$ be an undirected simple graph (graph without multiple edges and loops) with $n$ vertices and $m$ edges having vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Any undefined notations follows that of Bondy and Murty [1]. Throughout the paper, we denote such a graph by $G(n,m)$. The set of all non-increasing non-negative integer sequences $\pi = (d_1, d_2, \ldots, d_n)$ is denoted by $NS_n$. There are several famous results, Havel and Hakimi [5, 6] and Erdős and Gallai [3] which give necessary and sufficient conditions for a sequence $\pi = (d_1, d_2, \ldots, d_n)$ to be the degree sequence of a simple graph $G$. Unfortunately, knowing that a sequence has a realization gives no information about the properties that such a graph might have. In this paper, we explore this question of properties of a graph which is related to work originally introduced by A. R. Rao [9]. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph $G$ on $n$ vertices, and such a graph $G$ is called a realization of $\pi$. The sequence

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\[ \pi = (d_1, d_2, \ldots, d_n) \] is graphic if and only if the sequence \( \pi' \) obtained from \( \pi \) by laying off an element is graphic \([7]\). Also \( d^{r_1 \times r_2} \) means \( d \) occurs \( r_1 \times r_2 \) times in \( \pi \). The set of all graphic sequences in \( NS_n \) is denoted by \( GS_n \). A graphic sequence \( \pi \) is potentially \( H \)-graphic if there is a realization of \( \pi \) containing \( H \) as a subgraph, while \( \pi \) is forcibly \( H \)-graphic if every realization of \( \pi \) contains \( H \) as a subgraph. If \( \pi \) has a realization in which the \( r + 1 \) vertices of largest degree induce a clique, then \( \pi \) is said to be potentially \( A_{r+1} \)-graphic. The graphic sequence \( \pi \) is potentially \( K_{k+1} \)-graphic if and only if \( \pi \) is potentially \( A_{k+1} \)-graphic \([10]\). Let \( \sigma(\pi) = d_1 + d_2 + \ldots + d_n \). If \( G \) and \( G_1 \) are graphs, then \( G \cup G_1 \) is the disjoint union of \( G \) and \( G_1 \). If \( G = G_1 \), we abbreviate \( G \cup G_1 \) as \( 2G \). We denote \( G + H \) as the graph with \( V(G + H) = V(G) \cup V(H) \) and \( E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\} \). Let \( K_k, C_k, T_k \) and \( P_k \) respectively denote a complete graph on \( k \) vertices, a cycle on \( k \) vertices, a tree on \( k + 1 \) vertices and a path on \( k + 1 \) vertices. Let \( F_k \) denote the friendship graph on \( 2k + 1 \) vertices, that is, the graph of \( k \) triangles intersecting in a single vertex. For \( 0 \leq r \leq t \), denote the generalized friendship graph on \( kt - kr + r \) vertices by \( F_{t,r,k} \), where \( F_{t,r,k} \) is the graph of \( k \) copies of \( K_t \) meeting in a common \( r \) set.

Given a graph \( H \), what is the maximum number of edges of a graph with \( n \) vertices not containing \( H \) as subgraph? This number is denoted by \( ex(n, H) \), and is known as the Turan number. In terms of graphic sequences, the number \( 2ex(n, H) + 2 \) is the minimum even integer \( l \) such that every \( n \)-term graphic sequence \( \pi \) with \( \sigma(\pi) \geq l \) is forcibly \( H \)-graphic. Erdős, Jacobson and Lehel \([2]\) first considered the following variant: determine the minimum even integer \( l \) such that every \( n \)-term graphic sequence \( \pi \) with \( \sigma(\pi) \geq l \) is potentially \( H \)-graphic. We denote this minimum \( l \) by \( \sigma(H, n) \). A sequence \( \pi = (d_1, d_2, \ldots, d_n) \) is said to be potentially \( K_{r+1} \)-graphic if there is a realization \( G \) of \( \pi \) containing \( K_{r+1} \) as a subgraph. If \( \pi \) is a graphic sequence with a realization \( G \) containing \( H \) as a subgraph, then there is a realization \( G' \) of \( \pi \) containing \( H \) with the vertices of \( H \) having \( |V(H)| \) largest degree of \( \pi \) \([4]\). Let \( S_{r,s} = K_r + \overline{K}_s \) be split graph on \( r + s \) vertices, where \( \overline{K}_s \) is the complement of \( K_s \) and \( + \) denotes the standard join operation. As seen in \([11]\), \( S_{r,1} = K_{r+1} \) and so the graph \( S_{r,s} \) is an extension of the graph \( K_{r+1} \). A sequence \( \pi = (d_1, d_2, \ldots, d_n) \) is said to be potentially \( S_{r,s} \)-graphic if there is a realization \( G \) of \( \pi \) containing \( S_{r,s} \) as a subgraph. Yin Jain Hua and Haikou \([11]\) obtained a Havel-Hakimi type procedure and a simple sufficient condition for \( \pi \) to be potentially \( S_{r,s} \)-graphic. We have the following definitions.

**Definition 1.1.** \([3]\) For the graphs \( G_1, G_2 \) with disjoint vertex set \( V(G_1), V(G_2) \) the cartesian product is a graph \( G = G_1 \times G_2 \) with vertex set \( V(G_1) \times V(G_2) \) and an edge \( (u_1, v_1), (u_2, v_2) \) iff \( u_1 = u_2 \) and \( v_1 v_2 \) is an edge of \( G_2 \).

**Definition 1.2.** \([11]\) The standard join of \( S_{r_1,s_1}, S_{r_2,s_2} \) is a graph \( S = S_{r_1,s_1} \cup S_{r_2,s_2} \) with vertex set \( V(S_{r_1,s_1}) \cup V(S_{r_2,s_2}) \) and an edge set consisting of all edges of \( S_{r_1,s_1} \) and \( S_{r_2,s_2} \) together with the edges joining each vertex of \( K_{r_1} \) of \( S_{r_1,s_1} \) with every vertex of \( S_{r_2,s_2} \) and \( s_1 \) vertices of \( S_{r_1,s_1} \) are joined with only vertices of \( K_{r_2} \) in \( S_{r_2,s_2} \).
Definition 1.3. [3] The join (complete product) of $G_1$ and $G_2$ is a graph $G = G_1 \cup G_2$ with vertex set $V(G_1) \cup V(G_2)$ and an edge set consisting of all edges of $G_1$ and $G_2$ together with the edges joining each vertex of $G_1$ with every vertex of $G_2$.

Definition 1.4. [9] The split graph $K_r + \overline{K_s}$ on $r+s$ vertices is denoted by $S_{r,s}$ where $+$ denotes the standard join operation and $\overline{K_s}$ is the complement of $K_s$. A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative integers is said to be potentially $S_{r,s}$-graphic if there exists a realization $G$ of $\pi$ containing $S_{r,s}$ as a subgraph.

Definition 1.5. If $\pi$ has a realization $G$ containing $K_{r+1}$ on those vertices having degree $d_1, d_2, \ldots, d_{r+1}$, then $\pi$ is potentially $A_{r+1}$-graphic.

Definition 1.6. [10] The tensor product (conjunction), denoted by $G = G_1 \wedge G_2$, is the graph with vertex set $V = V_1 \times V_2$ and for any two vertices $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ in $V$; $u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$, there is an edge $w_1w_2 \in E(G)$ if and only if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

2. Main Results

We start with the following result.

Theorem 2.1. If $\pi_1 = (d_1^1, d_2^1, \ldots, d_m^1)$ is potentially $K_{p_1}$-graphic and $\pi_2 = (d_1^2, d_2^2, \ldots, d_n^2)$ is potentially $K_{p_2}$-graphic, $p_1 \leq m$ and $p_2 \leq n$, then the graphic sequence $\pi$ of $G = G_1 \times G_2$ is potentially $p_1 + p_2 - 2$ regular graphic.

Proof. Let $\pi_1 = (d_1^1, d_2^1, \ldots, d_m^1)$ and $\pi_2 = (d_1^2, d_2^2, \ldots, d_n^2)$ be respectively $K_{p_1}$-graphic and $K_{p_2}$-graphic. Then there exists graphs $G_1$ and $G_2$ respectively realizing $\pi_1$ and $\pi_2$ and respectively containing $K_{p_1}$ and $K_{p_2}$ as subgraphs. Let $G = G_1 \times G_2$ be the cartesian product of $G_1$ and $G_2$ and let $\pi_3 = (d_1, d_2, \ldots, d_m, d_{m+1}, d_{m+2}, \ldots, d_{2m}, \ldots, d_{m^1}, d_{m^2}, \ldots, d_{mn})$ be the graphic sequence of $G_1 \times G_2$. Then $d_{ij} = d_{ij}^1 + d_{ij}^2$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ where $d_{ij}$ is the degree of $ij$th vertex in $G$. We have to show that the realization $G$ of $\pi$ contains $p_1 + p_2 - 2$ as a regular subgraph. To prove this, it is enough to show that sum of degrees of this subgraph is equal to $p_1p_2(p_1 + p_2 - 2)$.

Clearly,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^1 + d_{ij}^2 = (d_{1}^1 + d_{1}^2) + \ldots + (d_{m}^1 + d_{m}^2) + \ldots + (d_{m}^1 + d_{n}^2).$$

This is true for all $m$ and $n$. In particular, it holds for $m = p_1$ and $n = p_2$. Therefore

$$\sum_{i=1}^{p_1} \sum_{j=1}^{p_2} d_{ij} = (d_{1}^1 + d_{1}^2) + (d_{1}^1 + d_{2}^2) + \ldots + (d_{1}^1 + d_{p_2}^2) + \ldots + (d_{p_1}^1 + d_{p_2}^2)$$

$$= (p_1 - 1 + p_2 - 1) + (p_1 - 1 + p_2 - 1) + \ldots + (p_1 - 1 + p_2 - 1)$$

$$+ \ldots + (p_1 - 1 + p_2 - 1)$$

$$= p_1p_2(p_1 + p_2 - 2).$$

□
Theorem 2.1 can be generalized as follows.

**Theorem 2.2.** If \( \pi_i = (d_{i1}, d_{i2}, \ldots, d_{in_j}) \) is potentially \( K_{p_i} \)-graphic for \( i, j = 1, 2, \ldots, r \) with \( p_i \leq n_j \), then the graphic sequence \( \pi \) of \( G = G_1 \times G_2 \times \ldots \times G_r \) is a potentially \( \sum_{i=1}^{r} p_i - r \) regular graphic.

*Proof.* The proof follows by induction on \( r \). \( \square \)

**Theorem 2.3.** If \( \pi_1 = (d_{11}, d_{12}, \ldots, d_{1m}) \) is potentially \( K_{p_1} \)-graphic and \( \pi_2 = (d_{21}, d_{22}, \ldots, d_{2n}) \) is potentially \( K_{p_2} \)-graphic, \( p_1 \leq m \) and \( p_2 \leq n \), then the graphic sequence \( \pi \) of \( G = G_1 + G_2 \) is potentially \( K_{p_1+p_2} \)-graphic.

*Proof.* Let \( \pi_1 = (d_{11}, d_{12}, \ldots, d_{1m}) \) be potentially \( K_{p_1} \)-graphic. Then there exists a graph \( G_1 \) which realizes \( \pi_1 \) and will contain \( K_{p_1} \) as a subgraph. Let \( \pi_2 = (d_{21}, d_{22}, \ldots, d_{2n}) \) be potentially \( K_{p_2} \)-graphic, so there exists a graph \( G_2 \) which realizes \( \pi_2 \) and will contain \( K_{p_2} \) as a subgraph. Let \( G = G_1 + G_2 \) be the join of \( G_1 \) and \( G_2 \) and let \( \pi = (d_1, d_2, \ldots, d_{m+n}) \) be the graphic sequence of \( G = G_1 + G_2 \). Then we have

\[
d_i = d_{i1} + n \quad \text{for} \quad i = 1, 2, \ldots, m
\]
\[
d_{m+j} = d_{2j} + m \quad \text{for} \quad j = 1, 2, \ldots, n.
\]

(2.1)

We have to show that the realization of \( \pi \) contains \( K_{p_1+p_2} \) as a subgraph. To prove this it is enough to show that

\[
\sum_{i=1}^{p_1} d_i + \sum_{j=1}^{p_2} d_{m+j} = (p_1 + p_2)(p_1 + p_2 - 1).
\]

We take the summation to the equations in (2.1) respectively from \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \) and get \( \sum_{i=1}^{m} d_i = \sum_{i=1}^{m} d_{i1} + \sum_{i=1}^{m} n \) and \( \sum_{j=1}^{n} d_{m+j} = \sum_{j=1}^{n} d_{2j} + \sum_{j=1}^{n} m \). These two equations imply

(2.2) \[
\sum_{i=1}^{m} d_i = \sum_{i=1}^{m} d_{i1} + mn
\]

and

(2.3) \[
\sum_{j=1}^{n} d_{m+j} = \sum_{j=1}^{n} d_{2j} + nm.
\]

As (2.2) and (2.3) is true for all \( m \) and \( n \), therefore, in particular it is true for \( m = p_1 \) and \( n = p_2 \). So,
Proof. This can be proved by induction on \( r \).

\[ p = 1 \]

Theorem 2.3 can be generalized as follows.

**Theorem 2.4.** If \( \pi_i = (d_{i1}, d_{i2}, \ldots, d_{in_i}) \) is potentially \( K_{p_i} \)-graphic for \( i = 1, 2, \ldots, r \) with \( p_i \leq n_j \). Then the graphic sequence \( \pi \) of \( G = G_1 + G_2 + \ldots + G_r \) is potentially \( K_{\sum_{i=1}^r p_i} \)-graphic.

**Proof.** This can be proved by induction on \( r \).

\[ \sum_{i=1}^{p_1} d_i + \sum_{j=1}^{p_2} d_{m+j} = \sum_{i=1}^{p_1} d_i^1 + \sum_{i=1}^{p_2} d_{m+j}^2 + 2p_1p_2 \]

\[ = d_1^1 + d_2^1 + \ldots + d_{p_1}^1 + d_1^2 + d_2^2 + \ldots + d_{p_2}^2 + 2p_1p_2 \]

\[ = (p_1 - 1) + \ldots + (p_1 - 1) + (p_2 - 1) + \ldots + (p_2 - 1) + 2p_1p_2 \]

\[ = p_1(p_1 - 1) + p_2(p_2 - 1) + p_1p_2 \]

\[ = p_1(p_1 + p_2 - 1) + p_2(p_2 + p_1 - 1) \]

\[ = (p_1 + p_2)(p_1 + p_2 - 1). \]

\[ \square \]

**Theorem 2.5.** If \( \pi_i \) is potentially \( S_{r_i,s_i} \)-graphic for \( i = 1, 2, \ldots, m \), then

1. The graphic sequence \( \pi \) of \( G = G_1 + G_2 + \ldots + G_m \) is potentially \( \sum_{i=1}^m r_i \sum_{i=1}^m s_i \)-graphic, where \( + \) denotes the standard join operation in \( S_{r_i,s_i} \).

2. The graphic sequence of \( \sum_{i=1}^m r_i \sum_{i=1}^m s_i \) for \( j = 1, 2, \ldots, m \) is

\[ \pi' = \left( \left( \sum_{i=1}^m (r_i + s_i - 1) \right)^{r_j}, \left( \sum_{i=1}^m r_i \right)^{s_j} \right), \]

3. Also, \( \sigma(\pi') = \left( \sum_{i=1}^m r_i \right)^2 + 2 \left( \sum_{i=1}^m r_i \right) \left( \sum_{i=1}^m s_i \right) - \left( \sum_{i=1}^m r_i \right). \]

**Proof.** Let \( \pi \) be potentially \( S_{r_i,s_i} \)-graphic for \( i = 1, 2, \ldots, m \). Then there exists a graph \( G_i \) which realizes \( \pi_i \) and will contain \( S_{r_i,s_i} \) as a subgraph. Let \( G = G_1 + G_2 + \ldots + G_m \) be the graph obtained from \( G_1, G_2, \ldots, G_m \) by using join operation. Therefore, clearly the graphic sequence \( \pi \) of \( G \) is potentially \( \sum_{i=1}^m r_i \sum_{i=1}^m s_i \)-graphic follows from Theorem 2.4. To prove part (2), we use induction on \( m \). For \( k = 1 \), the result is obvious. For \( k = 2 \), we have \( G = G_1 + G_2 \). Therefore, in particular \( S_{r_1+r_2,s_1+s_2} = S_{r_1,s_1} + S_{r_2,s_2} \). Now by Theorem 2.4 we have for every \( i = 1, 2, \ldots, r_1 \) and \( i = 1, 2, 3, \ldots, r_2 \) and \( j = 1, 2, 3, \ldots, s_1 \) and \( j = 1, 2, 3, \ldots, s_2 \)

\[ \overline{d}_i = d_i + r_2 + s_1 + s_2 \]
π is the degree of $i$ th vertex in $K_{r_1}$. Equations (2.4) and (2.5) hold for every i, j. Thus

$$\pi^2 = \left( \left( r_1 + r_2 + s_1 + s_2 - 1 \right)^{r_1}, \left( r_1 + r_2 + s_1 + s_2 - 1 \right)^{r_2}, \left( r_1 + r_2 \right)^{s_1}, \left( r_1 + r_2 \right)^{s_2} \right)$$

This shows that the result is true for $k = 2$. Assume that the result holds for $k = m - 1$, therefore $\pi^{m-1} = \left( \left( \sum_{i=1}^{m-1} (r_i + s_i) \right)^{r_j}, \left( \sum_{i=1}^{m-1} r_i \right)^{s_j} \right)$, for $j = 1, 2, \ldots, m - 1$. Now for $k = m$ we have that $G = S_{r_1,s_1} + S_{r_2,s_2} + \ldots + S_{r_{m-1},s_{m-1}} + S_{r_m,s_m} = A + S_{r_m,s_m}$, where $A = S_{r_1,s_1} + S_{r_2,s_2} + \ldots + S_{r_{m-1},s_{m-1}}$.

Since the result is proved for every $k = m - 1$ and using the fact that the result is proved for each pair and since the result is already proved for $k = 2$, it follows by induction hypothesis that the result holds for $k = m$ also. That is,

$$\pi = \pi^m = \left( \left( \sum_{i=1}^{m} (r_i + s_i - 1) \right)^{r_j}, \left( \sum_{i=1}^{m} r_i \right)^{s_j} \right).$$

This proves part (2). To prove part (3), we have for $j = 1, 2, \ldots, m$ that

$$\sigma(\pi') = r_j \left( \sum_{i=1}^{m} (r_i + s_i - 1) \right) + s_j \left( \sum_{i=1}^{m} r_i \right)$$

$$= r_j \left( \sum_{i=1}^{m} (r_i + s_i) \right) - r_j + s_j \left( \sum_{i=1}^{m} r_i \right)$$

$$= \sum_{j=1}^{m} r_j \left( \sum_{i=1}^{m} (r_i + s_i) \right) - \sum_{j=1}^{m} r_j + \sum_{j=1}^{m} s_j \sum_{i=1}^{m} r_i$$

$$= \left( \sum_{i=1}^{m} r_i \right)^2 + 2 \left( \sum_{i=1}^{m} r_i \right) \left( \sum_{i=1}^{m} s_i \right) - \left( \sum_{i=1}^{m} r_i \right).$$

\[ \square \]

**Theorem 2.6.** If $\pi_1 = (d_1, d_2, \ldots, d_m)$ is potentially $S_{r_1,s_1}$-graphic and $\pi_2 = (d_1^2, d_2^2, \ldots, d_m^2)$ is potentially $S_{r_2,s_2}$-graphic. Then

1. $\pi_{1} \times_{s_2} S_1 \times S_2$ is graphic,
2. the graphic sequence of $S_1 \times S_2$ is $\pi_{1} \times_{s_2} S_1 \times S_2 = (d_1^{r_1 \times r_2}, d_2^{r_1 \times s_2}, d_1^{s_1 \times r_2}, d_2^{s_1 \times s_2})$, where $d_{ij}$ is the degree of $ij$th vertex in $S_1 \times S_2$. 


Proof. Let \( \pi_1 = (d_1^1, d_2^1, \ldots, d_n^1) \) be potentially \( S_{r_1,s_1} \)-graphic. Then there exists a graph \( G_1 \) which realizes \( \pi_1 \) and will contain \( S_{r_1,s_1} \) as a subgraph. Let \( \pi_2 = (d_1^2, d_2^2, \ldots, d_n^2) \) be potentially \( S_{r_2,s_2} \)-graphic so that there exists a graph \( G_2 \) which realizes \( \pi_2 \) and will contain \( S_{r_2,s_2} \) as a subgraph. Let \( G = G_1 \times G_2 \) be the cartesian product of \( G_1 \) and \( G_2 \). Then we have \( d_{ij} = d_i + d_j \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). This relation is true for every vertex of the graph \( G \), therefore it also holds for the graph \( S = S_1 \times S_2 \). Thus we can write \( S = S_{r_1,s_1} \times S_{r_2,s_2} \). We have

\[
d_{ij} = d_i + d_j \quad \text{for} \quad 1 \leq i \leq r_1 + s_1 \quad \text{and} \quad 1 \leq j \leq r_2 + s_2.
\]

for \( 1 \leq i \leq r_1 + s_1 \) and \( 1 \leq j \leq r_2 + s_2 \).

If \( d_i \) is the degree of \( i \)th vertex of \( r_1 \) in \( S_{r_1,s_1} \) and \( d_j \) is the degree of \( j \)th vertex of \( r_2 \) in \( S_{r_2,s_2} \), it can be seen by construction that degree of \( i \)th vertex of \( r_1 \times r_2 \) in \( S \) is \( d_{ij} \), where \( d_{ij} \) is defined above and this term occurs in \( r_1 \times r_2 \) in \( S_{r_1,s_1} \times S_{r_2,s_2} \). Similarly other degree terms of the sequence occurs in \( r_1 \times s_2, s_1 \times r_2, s_1 \times s_2 \) by using definition of cartesian product of graphs. Thus \( \pi_{s_1 \times s_2} = (d_1^{r_1 \times r_2}, d_1^{r_1 \times s_2}, d_1^{r_1 \times r_2}, d_1^{r_1 \times s_2}) \). This completes the proof of the theorem. \( \square \)

The following result is a generalization of Theorem 2.6 whose proof follows simply by induction.

**Theorem 2.7.** If \( \pi_i = (d_i^1, d_i^2, \ldots, d_i^n) \) is potentially \( S_{r_i,s_i} \)-graphic, then

1. the sequence \( \pi \) of \( G = S_{r_1,s_1} \times S_{r_2,s_2} \times \cdots \times S_{r_m,s_m} \) is graphic,
2. the graphic sequence of \( \pi \) is \( \pi = \pi_{r_1,s_1} \times \pi_{r_2,s_2} \times \cdots \times \pi_{r_m,s_m} = (d_{ij}^{r_1 \times r_2}, d_{ij}^{r_1 \times s_2}, \ldots, d_{ij}^{r_1 \times r_2}, d_{ij}^{r_1 \times s_2}, \ldots), \) where \( d_{ij}^{r_1 \times s_2} = d_i + d_j + d_k + \cdots + d_m \).

**Proof.** This can be proved by induction on \( r \). \( \square \)

**Theorem 2.8.** If \( \pi_i \) is potentially \( S_{r_i,s_i} \)-graphic for \( i = 1, 2, \ldots, m \), then

1. the graphic sequence \( \pi \) of \( G = G_1 \lor G_2 \lor \cdots \lor G_m \) is potentially \( S_{\sum_{i=1}^m r_i, \sum_{i=1}^m s_i} \)-graphic, where \( \lor \) denotes the join operation in \( G_1, G_2, \ldots, G_n \),
2. the graphic sequence of \( S_{\sum_{i=1}^m r_i, \sum_{i=1}^m s_i} \) is

\[
\pi' = \left( \sum_{i=1}^m (r_i + s_i - 1) \right)^{1/r_j}, \left( \sum_{i=1}^m r_i + \sum_{i=1, i \neq j}^m s_i \right)^{1/s_j}, \text{ for } j = 1, 2, \ldots, m,
\]

3. and \( \sigma(\pi') = \left( \sum_{i=1}^m r_i \right)^2 + 2 \sum_{i=1}^m r_i \sum_{j=1}^m s_j + \left( \sum_{i=1}^m s_i \right)^2 + \sum_{j=1}^m s_j \left( \sum_{i=1, i \neq j} s_i \right) - \sum_{i=1}^m r_i \).

**Proof.** Let \( \pi \) be potentially \( S_{r_1,s_1} \)-graphic for \( i = 1, 2, \ldots, m \). Then there exists a graph \( G_i \) which realizes \( \pi_i \) and will contain \( S_{r_i,s_i} \) as a subgraph. Let \( G = G_1 \lor G_2 \lor \cdots \lor G_m \) be the graph obtained from \( G_1, G_2, \ldots, G_m \) by using join operation. Therefore, clearly the graphic sequence \( \pi \) of \( G \) is potentially \( S_{\sum_{i=1}^m r_i, \sum_{i=1}^m s_i} \)-graphic.
To prove part (2), we use induction on \( m \). For \( k = 1 \), the result is obvious. For \( k = 2 \), we have \( G = G_1 \lor G_2 \), therefore, in particular we take the normal join operation between graphs \( S_{r_1,s_1} \) and \( S_{r_2,s_2} \). Thus we have \( S_{1,2} = S_{r_1,s_1} \lor S_{r_2,s_2} \). Now by Theorem 2.6 we have for every \( i = 1, 2, \ldots, r_1 \) and \( j = 1, 2, 3, \ldots, s_1 \) and \( j = 1, 2, 3, \ldots, s_2 \)

\[
\overline{d}_i = d_i + r_2 + s_1 + s_2
\]

and

\[
\overline{d}_j = r_1 + r_2 + s_2,
\]

where \( \overline{d}_i \) and \( \overline{d}_j \) are respectively the degree of \( v^i_i \) and \( v^j_j \) vertex in \( S_{r_1+r_2,s_1+s_2} \) and \( d_i \) is the degree of \( i^{th} \) vertex in \( K_{r_1} \). Equations (2.7) and (2.8) hold for every \( i, j \). Thus for \( j = 1, 2 \)

\[
\pi^2 = \left( \left( r_1 + r_2 + s_1 + s_2 - 1 \right)^{r_1}, \left( r_1 + r_2 + s_1 + s_2 - 1 \right)^{r_2} \right),
\]

\[
\left( r_1 + r_2 + s_2 \right)^{s_1}, \left( r_1 + r_2 + s_1 \right)^{s_2} \right) = \left( \left( \sum_{i=1}^{r_j} (r_i + s_i - 1) \right)^{r_j}, \left( \sum_{i=1}^{r_j} r_i + \sum_{i=1,i \neq j}^{m-1} s_i \right)^{r_j} \right).
\]

This shows that the result is true for \( k = 2 \). Assume that the result holds for \( k = m - 1 \), therefore \( \pi^{m-1} = \left( \left( \sum_{i=1}^{m-1} (r_i + s_i - 1) \right)^{r_j}, \left( \sum_{i=1}^{m-1} r_i + \sum_{i=1,i \neq j}^{m-1} s_i \right)^{r_j} \right) \), for all \( j = 1, 2, \ldots, m - 1 \). Now for \( k = m \) we have that \( G = S_{r_1,s_1} \lor S_{r_2,s_2} \lor \ldots \lor S_{r_{m-1},s_{m-1}} \lor S_{r_m,s_m} = A \lor S_{r_m,s_m} \), where \( A = S_{r_1,s_1} \lor S_{r_2,s_2} \lor \ldots \lor S_{r_{m-1},s_{m-1}} \).

Since the result is proved for all \( k = m - 1 \) and using the fact that the result is proved for each pair and since the result is already proved for \( k = 2 \), it follows by induction hypothesis that result holds for \( k = m \) also. That is,

\[
\pi = \pi^m = \left( \left( \sum_{i=1}^{m} (r_i + s_i - 1) \right)^{r_j}, \left( \sum_{i=1}^{m} r_i + \sum_{i=1,i \neq j}^{m} s_i \right)^{s_j} \right).
\]

This proves part (2). To prove part (3), we have for all \( j = 1, 2, \ldots, m \)

\[
\sigma(\pi^j) = r_j \left( \sum_{i=1}^{m} (r_i + s_i - 1) \right) + s_j \left( \sum_{i=1}^{m} r_i + \sum_{i=1,i \neq j}^{m} s_i \right)
\]

\[
= r_j \left( \sum_{i=1}^{m} (r_i + s_i) \right) - r_j + s_j \left( \sum_{i=1}^{m} r_i + \sum_{i=1,i \neq j}^{m} s_j \right).
\]
\[
\left( \sum_{j=1}^{m} r_i \right)^2 + 2 \sum_{i=1}^{m} r_i \sum_{i=1}^{m} s_i + \sum_{i=1}^{m} s_j \left( \sum_{i=1, i \neq j}^{m} s_i \right) - \sum_{i=1}^{m} r_i. 
\]

\[\square\]

**Remark 2.1.** Let \( \pi_1 = (d_1^1, d_2^1, \ldots, d_m^1) \) be potentially \( K_{p_1} \)-graphic \( \pi_2 = (d_1^2, d_2^2, \ldots, d_m^2) \) be potentially \( K_{p_2} \)-graphic. Then the graphic sequence \( \pi \) of \( G = G_1 \land G_2 \) is potentially \( H_p \)-graphic, where \( H_p \) is a \( p \)-regular graph and \( p \) depends upon \( p_1 \) and \( p_2 \). If \( p_1 = 3 \) and \( p_2 = 2 \), then \( \pi \) of \( G = G_1 \land G_2 \) is potentially \( H_2 \)-graphic. If \( p_1 = 3 \) and \( p_2 = 3 \), then \( \pi \) of \( G = G_1 \land G_2 \) is potentially \( H_3 \)-graphic. If \( p_1 = 4 \) and \( p_2 = 4 \), then \( \pi \) of \( G = G_1 \land G_2 \) is potentially \( H_4 \)-graphic. If \( p_1 = 3 \) and \( p_2 = 4 \), then \( \pi \) of \( G = G_1 \land G_2 \) is potentially \( H_6 \)-graphic. From this we conclude that \( p \) depends upon \( p_1 \) and \( p_2 \).

**References**


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