HESSIAN DETERMINANTS OF COMPOSITE FUNCTIONS WITH APPLICATIONS FOR PRODUCTION FUNCTIONS IN ECONOMICS

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ABSTRACT. B.-Y. Chen [7] derived an explicit formula for the Hessian determinants of composite functions defined by

\[ f = F(h_1(x_1) + \cdots + h_n(x_n)) \]

In this paper, we introduce a new formula for the Hessian determinants of composite functions of the form

\[ f = F(h_1(x_1) \times \cdots \times h_n(x_n)) \]

Several applications of the new formula to the well-known Cobb-Douglas production functions in economics are also given.

1. INTRODUCTION

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}, \ f = f(x_1, \ldots, x_n) \), be a twice differentiable function. Then the Hessian matrix \( \mathcal{H}(f) \) is the square matrix \( (f_{x_ix_j}) \) of second-order partial derivatives of the function \( f \). If the second-order partial derivatives of \( f \) are all continuous in a neighborhood \( D \), then the Hessian of \( f \) is a symmetric matrix throughout \( D \) (cf. [7]).

For applications of Hessian matrices to production models in economics, we refer the reader to B.-Y. Chen’s papers [7, 8]. In addition, the Hessian matrices have an important geometric interpretation as following.

Let \( f = f(x_1, \ldots, x_n) \) be a twice differentiable real valued function. Then the Hessian matrix \( \mathcal{H}(f) \) of \( f \) is singular if and only if the graph of \( f \) in \( \mathbb{R}^{n+1} \) has null Gauss-Kronecker curvature [7].

Key words and phrases. Hessian matrix, Hessian determinant, Production function, Generalized Cobb-Douglas production function, Composite function.

2010 Mathematics Subject Classification. Primary: 91B38. Secondary: 15A15.

Received: July 28, 2013
Accepted: September 28, 2014.

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On the other hand, the bordered Hessian matrix of the function $f$ is given by

$$
\mathcal{H}^B (f) = \begin{pmatrix}
0 & f_{x_1} & \cdots & f_{x_n} \\
f_{x_1} & f_{x_1x_1} & \cdots & f_{x_1x_n} \\
\vdots & \vdots & \ddots & \vdots \\
f_{x_n} & f_{x_nx_1} & \cdots & f_{x_nx_n}
\end{pmatrix},
$$

where $f_{x_i} = \frac{\partial f}{\partial x_i}$ and $f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ for all $i, j \in \{1, \ldots, n\}$.

The bordered Hessian matrices of functions have important applications in many areas of mathematics. For instance, the bordered Hessian matrices are used to analyze quasi-convexity and quasi-concavity of the functions. If the signs of the bordered principal diagonal determinants of the bordered Hessian matrix of a function are alternate (resp. negative), then the function is quasi-concave (resp. quasi-convex). For more detailed properties see [4, 12, 13, 14].

Another example is the application of the bordered Hessian matrices to elasticity of substitutions of production functions in economics. Explicitly, let $f = f (x_1, \ldots, x_n)$ be a production function. Then the Allen’s elasticity of substitution of the $i$–th production variable with respect to the $j$–th production variable is defined by

$$
A_{ij} (x) = - \frac{(x_1 f_{x_1} + x_2 f_{x_2} + \cdots + x_n f_{x_n})}{x_i x_j} \frac{\mathcal{H}^B (f)_{ij}}{\det \mathcal{H}^B (f)}
$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$, $i, j \in \{1, \ldots, n\}$, $i \neq j$, where $\mathcal{H}^B (f)_{ij}$ is the co-factor of the element $f_{x_i x_j}$ in the determinant of $\mathcal{H}^B (f)$ [15, 17]. The authors [2, 3] called the bordered Hessian matrix $\mathcal{H}^B (f)$ by Allen’s matrix and $\det \mathcal{H}^B (f)$ by Allen determinant.

Let $f$ be a composite function of the form

$$
f (x) = F (h_1 (x_1) \times \cdots \times h_n (x_n)).
$$

In [3] for the composite functions of the form (1.1), an Allen Determinant Formula was obtained as follows

$$
\det (\mathcal{H}^B (f)) = -u^{n+1} \left( \tilde{F} \right)^{n+1} \sum_{j=1}^{n} \left( \frac{h_j'}{h_1} \right)^2 \left( \frac{h_{j-1}'}{h_j} \right)^2 \cdots \left( \frac{h_{j+1}'}{h_{j+1}} \right)^2 \left( \frac{h_{n+1}'}{h_n} \right)^2,
$$

where $h_j' = \frac{dh_j}{dx_j}$ and $\tilde{F} = \tilde{F} (u)$ for $u = h_1 (x_1) \times \cdots \times h_n (x_n)$.

In this paper, we obtain a new formula for Hessian determinants $\mathcal{H} (f)$ of composite functions of the form (1.1). Several applications of the new formula to production functions in economics are also given.

2. Production models in economics

In economics, a production function is a mathematical expression which denotes the physical relations between the output generated of a firm, an industry or an economy...
and inputs that have been used. Explicitly, a production function is a map which has non-vanishing first derivatives defined by

$$ f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+, \ f = f(x_1, \ldots, x_n), $$

where $f$ is the quantity of output, $n$ are the number of inputs and $x_1, \ldots, x_n$ are the inputs.

A production function $f(x_1, \ldots, x_n)$ is said to be homogeneous of degree $p$ or $p$-homogeneous if

$$ f(tx_1, \ldots, tx_n) = t^p f(x_1, \ldots, x_n) $$

holds for each $t \in \mathbb{R}_+$ for which (2.1) is defined. A homogeneous function of degree one is called linearly homogeneous. If $p > 1$, the function exhibits increasing return to scale, and it exhibits decreasing return to scale if $p < 1$. If it is homogeneous of degree 1, it exhibits constant return to scale [5].

Many important properties of homogeneous production functions in economics were interpreted in terms of the geometry of their graphs by [5, 9, 10, 18, 19].


$$ Y = bL^kC^{1-k}, $$

where $b$ presents the total factor productivity, $Y$ the total production, $L$ the labor input and $C$ the capital input. This function is nowadays called Cobb-Douglas production function.

The Cobb–Douglas production function with $n$–factor, also called generalized Cobb–Douglas production function, is given by

$$ f(x) = \gamma x_1^{\alpha_1} \cdots x_n^{\alpha_n}, $$

where $\gamma$ is a positive constant and $\alpha_1, \ldots, \alpha_n$ are nonzero constants [6].

3. HESSIAN DETERMINANT FORMULA

Let us denote the first derivative of $h_i(x_i)$ with respect to $x_i$ by a prime ($'$) and that of $F(u)$ with respect to $u$ by a dot ($\cdot$).

Throughout this article, we assume that $h_1, \ldots, h_n : \mathbb{R} \rightarrow \mathbb{R}$ are thrice differentiable functions with $h'_i(x_i) \neq 0$ and $F : I \subset \mathbb{R} \rightarrow \mathbb{R}$ a twice differentiable function with $F(u) \neq 0$ such that $I \subset \mathbb{R}$ is an interval of positive length.

The following provides an explicit formula for the Hessian determinant of the composite function given by (1.1).
The determinant of the Hessian matrix $\mathcal{H}(f)$ of the composite function $f = F(h_1(x_1) \times \cdots \times h_n(x_n))$ is given by

$$\det(\mathcal{H}(f)) = (uF)^n \left\{ \left( \frac{h'_1}{h_1} \right)' \cdots \left( \frac{h'_n}{h_n} \right)' \right\} \left(1 + u\frac{\ddot{F}}{F} \right) \sum_{j=1}^{n} \left( \frac{h'_{j-1}}{h_{j-1}} \right)' \left( \frac{h'_{j}}{h_{j}} \right) \left( \frac{h'_{j+1}}{h_{j+1}} \right)' \cdots \left( \frac{h'_{n}}{h_{n}} \right)' \right\},$$

(3.1)

where $h'_j = \frac{dh_j}{dx_j}$, $h''_j = \frac{d^2h_j}{dx_j^2}$, $\dot{F} = \frac{dF}{du}$, and $\ddot{F} = \frac{d^2F}{du^2}$ for $u = h_1(x_1) \times \cdots \times h_n(x_n)$.

**Proof.** Let $f$ be a twice differentiable composite function given by

$$f(x) = F(h_1(x_1) \times \cdots \times h_n(x_n))$$

(3.2)

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. It follows from (3.2) that

$$f_{x_i} = \frac{\partial f}{\partial x_i} = \frac{h'_i}{h_i} uF,$$

$$f_{x_ix_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{h'_ih'_j}{h_ih_j} u \left[ \dot{F} + u\ddot{F} \right], \quad 1 \leq i \neq j \leq n,$$

and

$$f_{x_i x_i} = \frac{\partial^2 f}{\partial x_i^2} = \frac{h''_i}{h_i} uF + \left( \frac{h'_i}{h_i} \right)^2 u^2 \ddot{F}.$$  

(3.4)

By using (3.3) and (3.4) the determinant of Hessian matrix $\mathcal{H}(f)$ of the composite function given by (3.2) is

$$\det(\mathcal{H}(f)) =$$

$$\begin{vmatrix}
\frac{h''_1}{h_1} uF + \left( \frac{h'_1}{h_1} \right)^2 u^2 \ddot{F} & \frac{h'_1 h'_2}{h'_1 h_2} \frac{h''_1}{h_1} u \left[ \dot{F} + u\ddot{F} \right] & \frac{h'_1 h'_3}{h'_1 h_3} \frac{h''_1}{h_1} u \left[ \dot{F} + u\ddot{F} \right] & \cdots & \frac{h'_1 h'_{n}}{h'_1 h_n} \frac{h''_1}{h_1} u \left[ \dot{F} + u\ddot{F} \right] \\
\frac{h'_1 h'_2}{h'_1 h_2} \frac{h''_2}{h_2} u \left[ \dot{F} + u\ddot{F} \right] & \frac{h''_2}{h_2} u \left[ \dot{F} + u\ddot{F} \right] + \left( \frac{h'_2}{h_2} \right)^2 u^2 \ddot{F} & \frac{h'_2 h'_3}{h'_2 h_3} \frac{h''_2}{h_2} u \left[ \dot{F} + u\ddot{F} \right] & \cdots & \frac{h'_2 h'_{n}}{h'_2 h_n} \frac{h''_2}{h_2} u \left[ \dot{F} + u\ddot{F} \right] \\
\frac{h'_1 h'_3}{h'_1 h_3} \frac{h''_3}{h_3} u \left[ \dot{F} + u\ddot{F} \right] & \frac{h'_2 h'_3}{h'_2 h_3} \frac{h''_3}{h_3} u \left[ \dot{F} + u\ddot{F} \right] & \frac{h''_3}{h_3} u \left[ \dot{F} + u\ddot{F} \right] + \left( \frac{h'_3}{h_3} \right)^2 u^2 \ddot{F} & \cdots & \frac{h'_3 h'_{n}}{h'_3 h_n} \frac{h''_3}{h_3} u \left[ \dot{F} + u\ddot{F} \right] \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{h'_1 h'_{n}}{h'_1 h_n} u \left[ \dot{F} + u\ddot{F} \right] & \frac{h'_2 h'_{n}}{h'_2 h_n} u \left[ \dot{F} + u\ddot{F} \right] & \frac{h'_3 h'_{n}}{h'_3 h_n} u \left[ \dot{F} + u\ddot{F} \right] & \cdots & \frac{h''_{n}}{h_n} u \left[ \dot{F} + u\ddot{F} \right] + \left( \frac{h'_n}{h_n} \right)^2 u^2 \ddot{F}
\end{vmatrix}.$$
After calculating the determinant in the previous formula, we obtain
\[
\operatorname{det}(\mathcal{H}(f)) = \left| \begin{array}{cccccc}
\frac{h_1''}{h_1} u \dot{F} + \left( \frac{h_1'}{h_1} \right)^2 u^2 \ddot{F} - \frac{h_1' h_2'}{h_1 h_2} \left( \frac{h_1'}{h_1} \right)' u \ddot{F} & \cdots & \frac{h_1' h_{n-1}'}{h_1 h_{n-1}} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_1' h_n'}{h_1 h_n} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_1' h_n'}{h_1 h_n} u \left[ \dddot{F} + u \dddot{F} \right] \\
\frac{h_1' h_2'}{h_1 h_2} u \left[ \dddot{F} + u \dddot{F} \right] & \left( \frac{h_2'}{h_2} \right)' u \ddot{F} & \cdots & \frac{h_2' h_{n-1}'}{h_2 h_{n-1}} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_2' h_n'}{h_2 h_n} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_2' h_n'}{h_2 h_n} u \left[ \dddot{F} + u \dddot{F} \right] \\
\frac{h_1' h_3'}{h_1 h_3} u \left[ \dddot{F} + u \dddot{F} \right] & 0 & \cdots & \frac{h_3' h_{n-1}'}{h_3 h_{n-1}} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_3' h_n'}{h_3 h_n} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_3' h_n'}{h_3 h_n} u \left[ \dddot{F} + u \dddot{F} \right] \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{h_1' h_n'}{h_1 h_n} u \left[ \dddot{F} + u \dddot{F} \right] & 0 & \cdots & \frac{h_n' h_{n-1}'}{h_n h_{n-1}} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_n' h_n'}{h_n h_n} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_n' h_n'}{h_n h_n} u \left[ \dddot{F} + u \dddot{F} \right] \\
\end{array} \right|
\]

By similar elementary transformations, we get
\[
\operatorname{det}(\mathcal{H}(f)) = \left| \begin{array}{cccccc}
\frac{h_1''}{h_1} u \dot{F} + \left( \frac{h_1'}{h_1} \right)^2 u^2 \ddot{F} - \frac{h_1' h_2'}{h_1 h_2} \left( \frac{h_1'}{h_1} \right)' u \ddot{F} & \cdots & \frac{h_1' h_{n-1}'}{h_1 h_{n-1}} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_1' h_n'}{h_1 h_n} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_1' h_n'}{h_1 h_n} u \left[ \dddot{F} + u \dddot{F} \right] \\
\frac{h_1' h_2'}{h_1 h_2} u \left[ \dddot{F} + u \dddot{F} \right] & \left( \frac{h_2'}{h_2} \right)' u \ddot{F} & \cdots & \frac{h_2' h_{n-1}'}{h_2 h_{n-1}} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_2' h_n'}{h_2 h_n} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_2' h_n'}{h_2 h_n} u \left[ \dddot{F} + u \dddot{F} \right] \\
\frac{h_1' h_3'}{h_1 h_3} u \left[ \dddot{F} + u \dddot{F} \right] & 0 & \cdots & \frac{h_3' h_{n-1}'}{h_3 h_{n-1}} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_3' h_n'}{h_3 h_n} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_3' h_n'}{h_3 h_n} u \left[ \dddot{F} + u \dddot{F} \right] \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{h_1' h_n'}{h_1 h_n} u \left[ \dddot{F} + u \dddot{F} \right] & 0 & \cdots & \frac{h_n' h_{n-1}'}{h_n h_{n-1}} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_n' h_n'}{h_n h_n} u \left[ \dddot{F} + u \dddot{F} \right] & \cdots & \frac{h_n' h_n'}{h_n h_n} u \left[ \dddot{F} + u \dddot{F} \right] \\
\end{array} \right|
\]

After calculating the determinant in the previous formula, we obtain
\[
\operatorname{det}(\mathcal{H}(f)) = (u \dot{F})^n \frac{h_1''}{h_1} \left( \frac{h_2'}{h_2} \right)' \left( \frac{h_3'}{h_3} \right)' \cdots \left( \frac{h_n'}{h_n} \right)' + (u \dot{F})^n \times
\]
\[
\times \left\{ \left( \frac{h_1'}{h_1} \right)' \left( \frac{h_2'}{h_2} \right)^2 \left( \frac{h_3'}{h_3} \right)' \cdots \left( \frac{h_n'}{h_n} \right)' + \left( \frac{h_1'}{h_1} \right)' \left( \frac{h_2'}{h_2} \right)' \left( \frac{h_3'}{h_3} \right)' \cdots \left( \frac{h_n'}{h_n} \right)' \cdots \right. \\
\left. \cdots + \left( \frac{h_1'}{h_1} \right)' \left( \frac{h_2'}{h_2} \right)' \left( \frac{h_3'}{h_3} \right)' \cdots \left( \frac{h_n'}{h_n} \right)^2 \right\}
\]
\[
+ (u)^{n+1} \left( \frac{h_1'}{h_1} \right)^{n-1} \left( \dddot{F} + u \dddot{F} \right) \left\{ \left( \frac{h_1'}{h_1} \right)^2 \left( \frac{h_2'}{h_2} \right)' \left( \frac{h_3'}{h_3} \right)' \cdots \left( \frac{h_n'}{h_n} \right)' + \left( \frac{h_1'}{h_1} \right)' \left( \frac{h_2'}{h_2} \right)^2 \left( \frac{h_3'}{h_3} \right)' \cdots \left( \frac{h_n'}{h_n} \right)' \right. \\
\left. \cdots + \left( \frac{h_1'}{h_1} \right)' \left( \frac{h_2'}{h_2} \right)' \left( \frac{h_3'}{h_3} \right)' \cdots \left( \frac{h_n'}{h_n} \right)^2 \right\}.
\]
After adding and substracting \( (u \dot{F})^n \left( \frac{h'_1}{h_1} \right)^2 \left( \frac{h'_2}{h_2} \right) \cdots \left( \frac{h'_n}{h_n} \right) \) we deduce
\[
\det (\mathcal{H}(f)) = (u \dot{F})^n \left\{ \prod_{j=1}^n \left( \frac{h'_j}{h_j} \right) \right. + \left. \left( 1 + \frac{\ddot{F}}{F} \right) \sum_{j=1}^n \left( \frac{h'_1}{h_1} \right) \cdots \left( \frac{h'_{j-1}}{h_{j-1}} \right) \left( \frac{h'_j}{h_j} \right)^2 \left( \frac{h'_{j+1}}{h_{j+1}} \right) \cdots \left( \frac{h'_n}{h_n} \right) \right\}.
\]
This completes the proof of the formula (3.1).

4. CHARACTERIZATIONS OF CD PRODUCTION FUNCTIONS

Next, we provide the following characterization of the generalized Cobb-Douglas production function with constant return to scale via the Theorem 3.1.

**Theorem 4.1.** Let \( F(u) \) be a twice differentiable function with \( \dot{F}(u) \neq 0 \) and let \( f \) be a composite function given by
\[
(4.1) \quad f = F\left((x_1 + \zeta_1)^{\alpha_1} \times \cdots \times (x_n + \zeta_n)^{\alpha_n}\right)
\]
for some constants \( \alpha, \zeta \). The Hessian matrix \( \mathcal{H}(f) \) of \( f \) is singular if and only if either

(i) at least one of the \( \alpha_1, \ldots, \alpha_n \) vanishes, or

(ii) up to suitable translations of \( x_1, \ldots, x_n \), \( f \) is a generalized Cobb-Douglas production function with constant return to scale.

**Proof.** Let us assume that the Hessian matrix of \( f \) is singular. By the hypothesis of the theorem, we have \( h_j(x_j) = (x_j + \zeta_j)^{\alpha_j} \). Thus we get
\[
h_j'(x_j) = \alpha_j (x_j + \zeta_j)^{\alpha_j-1}, \quad h_j''(x_j) = \alpha_j (\alpha_j - 1) (x_j + \zeta_j)^{\alpha_j-2}
\]
for all \( j \in \{1, \ldots, n\} \). After applying the formula (3.1), we write
\[
(4.2) \quad 0 = (-1)^{n-1} \left( u \tilde{F} \right)^n \prod_{j=1}^n \frac{\alpha_j}{(x_j + \zeta_j)^2} \left\{ -1 + \sum_{j=1}^n \alpha_j \right\} + u \tilde{F} \left( \sum_{j=1}^n \alpha_j \right),
\]
where \( u = (x_1 + \zeta_1)^{\alpha_1} \times \cdots \times (x_n + \zeta_n)^{\alpha_n} \). Since \( u \neq 0 \) and \( \ddot{F} \neq 0 \), the equation (4.2) reduces to
\[
(4.3) \quad 0 = \prod_{j=1}^n \frac{\alpha_j}{(x_j + \zeta_j)^2} \left[ -1 + \sum_{j=1}^n \alpha_j + u \frac{\tilde{F}}{F} \sum_{j=1}^n \alpha_j \right].
\]
From the equation (4.3), it is easily seen that either at least one of the \( \alpha_1, \ldots, \alpha_n \) vanishes or
\[
(4.4) \quad 1 - \sum_{j=1}^n \alpha_j = u \frac{\tilde{F}}{F} \sum_{j=1}^n \alpha_j.
\]
For (4.4), if $F$ is a linear function, then $\sum_{j=1}^{n} \alpha_j = 1$, which implies that, up to suitable translations of $x_1, \ldots, x_n$, $f$ is a generalized Cobb-Douglas production function with constant return to scale. If $F$ is a non-linear function, then by (4.4) we derive
\[
\dot{F} \frac{1 - \sum_{j=1}^{n} \alpha_j}{\sum_{j=1}^{n} \alpha_j} \frac{1}{u} = 0,
\]
which implies that
\[
(4.5) \quad F = \frac{\delta}{\gamma + 1} (u)^{\gamma + 1} + \varepsilon,
\]
where $\gamma, \delta$ are nonzero constants and $\varepsilon$ some constant such that
\[
(4.6) \quad \gamma = \frac{1 - \sum_{j=1}^{n} \alpha_j}{\sum_{j=1}^{n} \alpha_j}.
\]
Combining (4.1), (4.5) and (4.6) gives that, up to suitable translations of $x_1, \ldots, x_n$, $f$ is a generalized Cobb-Douglas production function with constant return to scale.

Conversely, it is straightforward to verify that cases (i) and (ii) imply that $f$ has vanishing Hessian determinant. □

**Theorem 4.2.** Let $F = u^r$ be a power function such that $r \neq 0, 1$ and let $f$ be a composite function given by
\[
(4.7) \quad f = F (h_1 (x_1) \times \cdots \times h_n (x_n)).
\]
The Hessian matrix $\mathcal{H} (f)$ of $f$ is singular if and only if either
(i) $f = F (\gamma e^{\alpha_1 x_1 + \alpha_2 x_2} x_3 (x_3) \times \cdots \times h_n (x_n))$ for nonzero constants $\gamma, \alpha_1, \alpha_2$, or
(ii) up to suitable translations of $x_1, \ldots, x_n$, $f$ is a generalized Cobb-Douglas production function with constant return to scale.

**Proof.** Let us assume that the Hessian matrix of $f$ is singular. Then we have $\det (\mathcal{H} (f)) = 0$. From the hypothesis of theorem, we get
\[
(4.8) \quad \dot{F} = ru^{r-1} \quad \text{and} \quad \ddot{F} = r(r-1)u^{r-2}.
\]
After substituting (4.8) into the formula (3.1), we derive
\[
(4.9) \quad 0 = \prod_{j=1}^{n} \left( \frac{h_j'}{h_j} \right) + r \sum_{j=1}^{n} \left( \frac{h_j'}{h_1} \right) \cdots \left( \frac{h_j'}{h_j-1} \right) \cdots \left( \frac{h_j'}{h_j+1} \right) \cdots \left( \frac{h_j'}{h_n} \right).
\]
For (4.9) we have two cases:
**Case (a):** At least one of \( \left( \frac{h'_1}{h_1} \right)' , \ldots , \left( \frac{h'_n}{h_n} \right)' \) vanishes. Without loss of generality, we may assume that

\[
\left( \frac{h'_1}{h_1} \right)' = 0.
\]

Then from (4.9), we find

\[
0 = \left( \frac{h'_1}{h_1} \right)^2 \left( \frac{h'_2}{h_2} \right)' \left( \frac{h'_3}{h_3} \right)' \cdots \left( \frac{h'_n}{h_n} \right)' .
\]

Without loss of generality, we may assume from (4.11) that

\[
\left( \frac{h'_2}{h_2} \right)' = 0.
\]

After solving (4.10) and (4.12), we obtain \( h_j(x_j) = \gamma_j e^{\alpha_j x_j}, \) \((j = 1, 2)\) for nonzero constants \( \gamma_j, \alpha_j. \) This gives the statement (i).

**Case (b):** \( \left( \frac{h'_1}{h_1} \right)' , \ldots , \left( \frac{h'_n}{h_n} \right)' \) are nonzero. Then from (4.9), by dividing with the product \( \left( \frac{h'_1}{h_1} \right)' \cdots \left( \frac{h'_n}{h_n} \right)' \), we write

\[
0 = 1 + r \left( \left( \frac{h'_1}{h_1} \right)^2 + \cdots + \left( \frac{h'_n}{h_n} \right)^2 \right).
\]

Taking partial derivative of (4.13) with respect to \( x_i \), we find

\[
2 \left( \frac{h'_i}{h_i} \right)^2 = \left( \frac{h'_i}{h_i} \right) \left( \frac{h'_i}{h_i} \right)'' .
\]

By solving (4.14), we get

\[
h_j(x_j) = \gamma_j (x_j + \zeta_j)^{\alpha_j},
\]

where \( \gamma_j, \alpha_j \) are nonzero constants with \( \sum_{j=1}^n \alpha_j = \frac{1}{r} \), and \( \zeta_j \) some constants. Combining (4.7) and (4.15) gives the statement (ii).

Converse is straightforward to verify that cases (i) and (ii) imply that \( f \) has vanishing Hessian determinant. \( \square \)

5. **Further applications**

We provide the following as further applications of Theorem 3.1.

**Theorem 5.1.** Let \( f \) be a twice differentiable composite function given by

\[
f = \ln (h_1(x_1) \times \cdots \times h_n(x_n)).
\]

The Hessian matrix \( \mathcal{H}(f) \) of \( f \) is singular if and only if at least one of the \( h_1(x_1) , \ldots , h_n(x_n) \) is of the form \( \gamma_j e^{\alpha_j x_j} \) for nonzero constants \( \gamma_j, \alpha_j. \)
Proof. Let assume that the Hessian matrix $H(f)$ of $f$ is singular. Then under the hypothesis of the theorem, we get

$$F(u) = \ln u, \quad \dot{F}(u) = \frac{1}{u}, \quad \ddot{F} = -\frac{1}{u^2}. \tag{3.1}$$

After applying the formula (3.1), we derive $0 = \prod_{j=1}^{n} \left( \frac{h'_j}{h_j} \right)'$. Because of $h'_j (x_j) \neq 0$, at least one of $\left( \frac{h'_j}{h_j} \right)'$ vanishes which implies that at least one of $h_j$ is of the form $\gamma_j e^{\alpha_j x_j}$ for nonzero constants $\gamma_j, \alpha_j$.

Converse is easy to verify. $\square$

Corollary 5.1. Let $f = F(h_1(x_1) \times \cdots \times h_n(x_n))$ be a twice differentiable composite function. If at least two of $h_1(x_1), \ldots, h_n(x_n)$ is of the form $\gamma_j e^{\alpha_j x_j}$ for nonzero constants $\gamma_j, \alpha_j$, then the Hessian matrix $H(f)$ of $f$ is singular.

Proof. Let $f = F(h_1(x_1) \times \cdots \times h_n(x_n))$ be a twice differentiable composite function such that at least two of $h_1(x_1), \ldots, h_n(x_n)$ is of the form $\gamma_j e^{\alpha_j x_j}$ for nonzero constants $\gamma_j, \alpha_j$. Without lose of generality, we may assume that $h_1 = \gamma_1 e^{\alpha_1 x_1}$ and $h_2 = \gamma_2 e^{\alpha_2 x_2}$.

Thus we get

$$\left( \frac{h'_1}{h_1} \right)' = 0 \quad \text{and} \quad \left( \frac{h'_2}{h_2} \right)' = 0. \tag{5.1}$$

Substituting (5.1) into (3.1) gives the proof. $\square$

References


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