ON THE APOLAR POLYNOMIALS

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Abstract. We give generalizations of theorems of Aziz on apolar polynomials. Our proofs are based on direct methods.

1. Introduction

Let \( p(z) = \sum_{k=0}^{n} \binom{n}{k} A_k z^k \) and \( q(z) = \sum_{k=0}^{m} \binom{m}{k} B_k z^k \) be two complex polynomials, where \( m \leq n \). Let us define \( A(p, q) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (A_{m-k} B_k) \).

Classical theorem of Grace states that if the polynomials \( p \) and \( q \) are apolar, that is if \( A(p, q) = 0 \) and \( m = n \), then any circular region that contains all zeros of one of these polynomials contains at least one zero of the other one.

The following theorem is due to Aziz, see [1].

Theorem 1.1. Let \( p(z) = \sum_{k=0}^{n} \binom{n}{k} A_k z^k \) and \( q(z) = \sum_{k=0}^{m} \binom{m}{k} B_k z^k \) where \( m \leq n \). Assume \( A(p, q) = 0 \). Then

(i) If \( q(z) \) has all zeros in the disc \( |z| \leq r \), then \( p(z) \) has at least one zero in the same disc.

(ii) If \( p(z) \) has all zeros in the region \( |z| \geq r \), then \( q(z) \) has at least one zero in the same region.

A different proof of this theorem is given in [5]. The following theorem is proved in [2].

Theorem 1.2. Let \( p(z) = \sum_{k=0}^{n} \binom{n}{k} A_k z^k \) and \( q(z) = \sum_{k=0}^{m} \binom{m}{k} B_k z^k \) where \( m \leq n \). Let us assume

\[
(1.1) \quad \binom{m}{0} B_0 A_n - \binom{m}{1} B_1 A_{n-1} + \binom{m}{2} B_2 A_{n-2} + \cdots + (-1)^m \binom{m}{m} B_m A_{n-m} = 0.
\]

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Then

(i) If \( q(z) \) has all zeros in the region \( |z - a| \geq r \), then \( p(z) \) has at least one zero in the same region.

(ii) If \( p(z) \) has all zeros in the disc \( |z - a| \leq r \), then \( q(z) \) has at least one zero in the same disc.

We also note that the both above theorems are proved in [3] also in the case of certain half-planes. Our goal is to give generalizations of these results.

2. Main results

Our first result is a generalization of Theorem 1.1.

**Theorem 2.1.** Let \( p(z) = \sum_{k=0}^{m} \binom{n}{k} A_{k} z^{k} \) and \( q(z) = \sum_{k=0}^{m} \binom{n}{k} B_{k} z^{k} \), where \( m \leq n \) and \( A(p, q) = 0 \). Then any circular region \( K \) that contains 0 and all zeros of \( q(z) \) contains at least one zero of \( p(z) \).

**Proof.** Let us set \( Q(z) = z^{n-m} q(z) \). Then \( A(p, q) = 0 \) implies that \( p(z) \) and \( Q(z) \) are apolar. All the zeros of \( Q(z) \) are contained in \( K \). By theorem of Grace, at least one zero of \( p(z) \) is in \( K \). \( \square \)

Also, if all the zeros of \( p(z) \) are outside \( K \), then at least one zero of \( q(z) \) is also outside \( K \). This is an equivalent form of stating the above theorem.

We are going to use the following result.

**Theorem 2.2.** [4] For polynomials \( f(z) = a_{0} + a_{1} z + \cdots + z^{n} \) and \( g(z) = b_{0} + b_{1} z + \cdots + z^{n} \) the following holds: if all the zeros of \( f(z) \) are contained in some disc of radius \( r \), then a zero of \( g(z) \) is contained in concentric disc of radius \( r + |A(f, g)|^{1/n} \).

If \( K \) is a disc, we can generalize Theorem 2.1 to include the case \( A(p, q) \neq 0 \). Namely, we have the following theorem, which can be deduced from Theorem 2.2.

**Theorem 2.3.** Let \( p(z) = \sum_{k=0}^{m} \binom{n}{k} A_{k} z^{k} \) and \( q(z) = \sum_{k=0}^{m} \binom{n}{k} B_{k} z^{k} \), where \( m \leq n \) and \( A_{n} B_{m} \neq 0 \). Assume a disc \( K \) of radius \( r \) contains 0 and all zeros of \( q(z) \). Then the concentric disc of radius \( r + \left| \frac{A(f, g)}{A_{n} B_{m}} \right|^{1/n} \) contains at least one zero of \( p(z) \).

**Proof.** We note that polynomials \( p(z)/A_{n} \) and \( Q(z)/B_{m} \) are monic, where \( Q(z) = z^{n-m} q(z) \), and that all the zeros of \( Q(z)/B_{m} \) are contained in \( K \). Since \( \deg Q = \deg p = n \), Theorem 2.2 implies that at least one zero of \( p(z) \) is contained in the concentric disc of radius \( r + |A(p/A_{n}, Q/B_{m})|^{1/n} \). However, by a simple calculation, \( |A(p/A_{n}, Q/B_{m})| = |A(p, q)/A_{n} B_{m}| \) and this completes the proof. \( \square \)

Our last theorem generalizes Theorem 1.2.

**Theorem 2.4.** Assume polynomials \( p(z) \) and \( q(z) \) are as in Theorem 1.2. Assume condition (1.1) is satisfied, and let \( K \) be a closed disc or a closed half-plane. Then
(i) If all the zeros of \( p(z) \) are in \( K \), then at least one zero of \( q(z) \) is in \( K \).

(ii) If all the zeros of \( q(z) \) are outside \( K \), then at least one of the zeros of \( p(z) \) is outside \( K \).

Proof. At first let us note that the two assertions are logically equivalent, so it suffices to prove the first one. We assume all the zeros of \( p(z) \) are in \( K \). Then, by Gauss-Lucas theorem, the same is true for the polynomial \( p^{(n-m)}(z) \). Condition (1.1) tells us that in fact \( p^{(n-m)}(z) \) is apolar to \( q(z) \), and the proof is finished by invoking the theorem of Grace.

\[ \square \]

References


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