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ON A CONJECTURE OF HARMONIC INDEX AND DIAMETER OF GRAPHS

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Abstract. The Harmonic index $H(G)$ of a graph G is defined as the sum of the weights $\dfrac{2}{d(u)+d(v)}$ of all edges uv of G, where $d(u)$ denotes the degree of the vertex u in G. In this work, we prove the conjecture $H(G)-D(G) \geq \dfrac{5}{6}-\dfrac{n}{2}$ given by Liu in 2013 when G is a unicyclic graph by giving a better bound, namely, $H(G)-D(G) \geq \dfrac{5}{3}-\dfrac{n}{2}$.

1. Introduction

Let $G=(V,E)$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For a graph G, the harmonic index $H(G)$ is defined as $H(G)=\sum_{uv \in E(G)} \dfrac{2}{d(u)+d(v)}$, where $d(u)$ is the degree of the vertex u in G. As far as we know, this index first appeared in [3]. Zhong found the minimum and maximum values of the harmonic index for simple connected graphs, trees and unicyclic graphs and characterized the corresponding extremal graphs [6, 7]. Wu et al. gave a best possible lower bound for the harmonic index of triangle-free graph with minimum degree at least two and characterized the extremal graphs [5]. Deng et al. considered the relation connecting the harmonic index $H(G)$ and the chromatic number $\chi(G)$ and proved that $\chi(G) \leq 2H(G)$ by using the effect of removal of a minimum degree vertex on the harmonic index [2]. Liu proved that $H(T)-D(T) \geq \frac{5}{6}-\frac{n}{2}$ and $\frac{H(T)}{D(T)} \geq \frac{1}{2}+\frac{1}{3(n-1)}$ for trees with equality for path and propose it as a conjecture for any connected graph of order n

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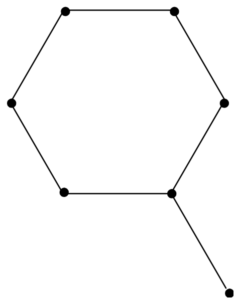
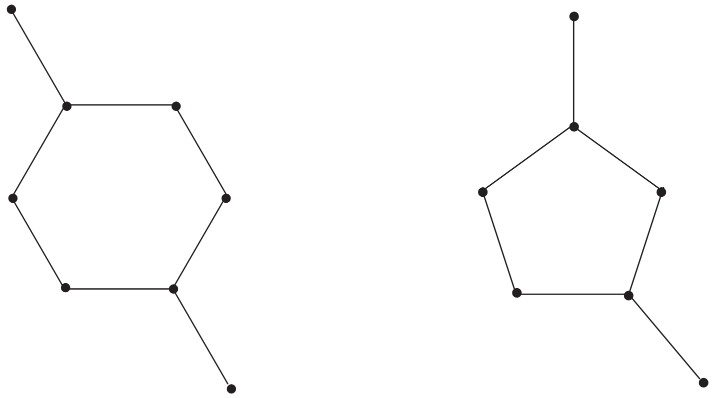


FIGURE 1. U_7^1



(A) U_8^2 (B) U_7^2

FIGURE 2.

[4]. In this work, we prove the conjecture when G is an unicyclic graph by giving a better bound, namely, $H(G) - D(G) \geq \frac{5}{3} - \frac{n}{2}$ for $n \geq 7$.

We conclude this section with some notations and terminology. Let $G = (V, E)$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex v of G is denoted by $d(v)$. If $d(v) = 1$, then v is said to be a pendant vertex of G. The edge incident with the pendant vertex v is referred to as pendant edge and the vertex adjacent to v is referred as the support vertex of v. The set of neighbours of v is denoted by $N(v)$. The eccentricity of a vertex v is the greatest distance from v to any other vertex of G. The diameter of a graph is the maximum over eccentricities of all vertices of the graph and it is denoted by $D(G)$. A diametrical path of a graph is a shortest path whose length is equal to the diameter of the graph. As usual, C_n denotes the cycle on n vertices. Let U_n^1 be a graph obtained from C_{n-1} by attaching one pendant edge to any one of the vertices of C_{n-1} and U_n^2 be a graph obtained from C_{n-2} by attaching two pendant edges to two vertices of C_{n-2}, say u and v, such that $d(u, v) = \frac{n-2}{2}$, if n is even and $d(u, v) = \frac{n-3}{2}$, if n is odd (see Figures 1 and 2). A unicyclic graph is odd or even, if the number of vertices in its cycle is odd or even. For other notations in graph theory, may be consulted [1].

2. Basic Results

In this section we have some observations and lemmas which are used to prove our main result. Analysing diametrical paths of the unicyclic graphs, we have the following observations.

1. Let $G \ncong C_n$ be a unicyclic graph on n vertices. Then at least one of the end vertices of a diametrical path of G must be a pendant vertex.
2. Let $G \ncong C_n$ be a unicyclic graph on n vertices and v be an end vertex in a diametrical path of G. If $D(G-v) = D(G) - 1$, then v must lie on every diametrical path of G.
3. Let $G \ncong C_n$ be a unicyclic graph on n vertices. If v is a pendant vertex of a diametrical path and u is the support vertex of v, then $N(u)$ has at most two vertices of degree at least 2.

Lemma 2.1. $f(x) = \dfrac{2}{x+2} - \dfrac{2}{x+1}$ *is an increasing function for* $x > 0$.

Lemma 2.2. *Let v be a pendant vertex of a connected graph G. Then $H(G) \geq H(G-v)$.*

Proof. Let u be the support vertex of v. Then

$$\begin{aligned} H(G) - H(G-v) &= \frac{2}{d(u)+1} + 2 \sum_{w \in N(u)-\{v\}} \left(\frac{1}{d(u)+d(w)} - \frac{1}{d(u)+d(w)-1} \right) \\ &\geq \frac{2}{d(u)+1} + 2(d(u)-1) \left(\frac{1}{d(u)+1} - \frac{1}{d(u)} \right) \text{ \{by Lemma 2.1\}} \\ &\geq 0. \end{aligned}$$

Hence $H(G) \geq H(G-v)$. ☐

Lemma 2.3. *Let G be a unicyclic graph of order $n \geq 7$ with $1 \leq n_1 \leq n-3$ pendant vertices, v be an end vertex of a diametrical path of G such that v is a pendant vertex of G and u be the support vertex of v. If $N(u)$ has exactly one vertex of degree ≥ 2 and $D(G-v) = D(G) - 1$, then $H(G) - H(G-v) \geq \frac{1}{2}$.*

Proof. In this case $d(u) = 2$, otherwise u has at least one more pendant neighbour contradicting $D(G-v) = D(G) - 1$. If the other neighbour of u is w, then

$$H(G) - H(G-v) = \frac{2}{2+1} + \left\{ \frac{2}{2+d(w)} - \frac{2}{1+d(w)} \right\}.$$

By Lemma 2.1 and $d(w) \geq 2$,

$$H(G) - H(G-v) \geq \frac{2}{3} + \left\{ \frac{2}{2+2} - \frac{2}{1+2} \right\} = \frac{1}{2}.$$ ☐

Lemma 2.4. *Let G be a unicyclic graph of order $n \geq 7$ with $2 \leq n_1 \leq n-4$ pendant vertices and u and v be the end vertices of a diametrical path of G such that u and v*

are the pendant vertices. Let u' and v' be their respective support vertices. If $N(u')$ and $N(v')$ each has exactly two vertices of degree ≥ 2 and $D(G-v) = D(G)-1$, then $H(G) - D(G) \geq \frac{5}{3} - \frac{n}{2}$.

Proof. We have the following cases.

Case 1: If there is no other pendant vertices in G other than u and v, then G will be of the form U_n^2. Also $H(U_n^2) = \frac{n}{2} - \frac{2}{5}$. If n is even, then $D(U_n^2) = \frac{n}{2} + 1$. Hence $H(U_n^2) - D(U_n^2) \geq \frac{-7}{5} \geq \frac{5}{3} - \frac{n}{2}$. If n is odd, then $D(U_n^2) = \frac{n}{2} + \frac{1}{2}$. Hence $H(U_n^2) - D(U_n^2) \geq \frac{-9}{10} \geq \frac{5}{3} - \frac{n}{2}$.

Case 2: If there is at least one pendant vertex in G other than u and v, then we delete those pendant vertices one by one until the resulting graph is $U^2_{n-n_1+2}$. Assume that $v_1, v_2, \ldots, v_{n_1-2}$ are the pendant vertices in the order they are deleted. Therefore by Lemma 2.2 we have $H(G) \geq H(G - v_1) \geq \cdots \geq H(G - \bigcup_{i=1}^{n_1-2} v_i) = H(U^2_{n-n_1+2})$ and $D(G) = D(G-v_1) = \cdots = D(G - \bigcup_{i=1}^{n_1-2} v_i) = D(U^2_{n-n_1+2})$. Also $H(U^2_{n-n_1+2}) = \frac{n-n_1+2}{2} - \frac{2}{5}$. If $U^2_{n-n_1+2}$ is even, then $D(U^2_{n-n_1+2}) = \frac{n-n_1+2}{2} + 1$. Hence $H(G) - D(G) \geq H(U^2_{n-n_1+2}) - D(U^2_{n-n_1+2}) = \frac{-7}{5} \geq \frac{5}{3} - \frac{n}{2}$. If $U^2_{n-n_1+2}$ is odd, then $D(U^2_{n-n_1+2}) = \frac{n-n_1+2}{2} + \frac{1}{2}$. Hence $H(G) - D(G) \geq H(U^2_{n-n_1+2}) - D(U^2_{n-n_1+2}) = \frac{-9}{10} \geq \frac{5}{3} - \frac{n}{2}$. □

3. Main Result

In this section, we give a sharp lower bound of the relationship involving the harmonic index and diameter of connected unicyclic graphs. Let $\mathscr{C}$ be a set of graphs obtained from C_4 by attaching one pendant edge and a path of length $n-5$ to two diametrically nonadjacent vertices of C_4. For each $n \geq 7$, we have exactly one graph in this set.

Theorem 3.1. *Let G be a unicyclic graph of order $n \geq 7$ and diameter D(G). Then* $H(G) - D(G) \geq \frac{5}{3} - \frac{n}{2}$, *where equality holds if $G \in \mathscr{C}$.*

Proof. We have the following cases.

Case 1: Let $G \cong C_n$. For C_n, $H(C_n) = \frac{n}{2}$. If n is odd, then $D(C_n) = \frac{n-1}{2}$. Therefore $H(C_n) - D(C_n) = \frac{n}{2} - \frac{n-1}{2} = \frac{1}{2} \geq \frac{5}{3} - \frac{n}{2}$. If n is even, then $D(C_n) = \frac{n}{2}$. Therefore $H(C_n) - D(C_n) = \frac{n}{2} - \frac{n}{2} = 0 \geq \frac{5}{3} - \frac{n}{2}$

Case 2: Let $G \ncong C_n$. Then G has at least one pendant vertex. By observation 1, at least one of the end vertices of the diametrical path of G is a pendant vertex, say v. Also $D(G-v) = D(G)$ or $D(G-v) = D(G) - 1$.

Subcase 2.1: Let $D(G-v) = D(G)$. In this case we can prove the result by induction on n. We can easily check for unicyclic graph G of order 7 such that $D(G-v) = D(G)$. Assume the result is true for all unicyclic graphs of order less than or equal to $n-1$. Let G be a unicyclic graph of order n and v be the pendant vertex of the diametrical path of G. By Lemma 2.2, $H(G) \geq H(G-v)$. Therefore

$$\begin{aligned} H(G) - D(G) &\geq H(G-v) - D(G-v) \\ &\geq \frac{5}{3} - \frac{n-1}{2} \quad \{\text{by induction}\} \\ &\geq \frac{5}{3} - \frac{n}{2}. \end{aligned}$$

Subcase 2.2: Let $D(G-v) = D(G) - 1$. Let u be the support vertex of v. If $N(u)$ has one vertex of degree at least two, we can prove by induction on n. We can easily check for unicyclic graph G of order 7 such that $D(G-v) = D(G) - 1$. Assume the result is true for all unicyclic graphs of order less than or equal to $n-1$. Let G be a unicyclic graph of order n and v be the pendant vertex of the diametrical path of G. By Lemma 2.3,

$$H(G) \geq H(G-v) + \frac{1}{2}.$$

Therefore

$$\begin{aligned} H(G) - D(G) &\geq H(G-v) + \frac{1}{2} - (D(G-v) + 1) \\ &= H(G-v) - D(G-v) - \frac{1}{2} \\ &\geq \frac{5}{3} - \frac{n-1}{2} - \frac{1}{2} \quad \{\text{by induction}\} \\ &= \frac{5}{3} - \frac{n}{2}. \end{aligned}$$

If $N(u)$ has two neighbours of degree at least two and G has exactly one pendant vertex, then $G \cong U_n^1$. Now $H(U_n^1) = \frac{n}{2} - \frac{1}{5}$. If n is odd, then $D(U_n^1) = \frac{n}{2} - \frac{1}{2}$. Hence $H(U_n^1) - D(U_n^1) = \frac{-7}{10} \geq \frac{5}{3} - \frac{n}{2}$. If n is even, then $D(U_n^1) = \frac{n}{2}$. Hence $H(U_n^1) - D(U_n^1) = \frac{-1}{5} \geq \frac{5}{3} - \frac{n}{2}$.

If $N(u)$ has two neighbours of degree at least two and G has more than one pendant vertex, then by Lemma 2.4 the result is true.

For proving the equality, assume that $H(G) - D(G) = \frac{5}{3} - \frac{n}{2}$. Since $D(G) \leq n-2$, $\quad H(G) - (n-2) \leq H(G) - D(G)$, for all G. So our search is to find that G,

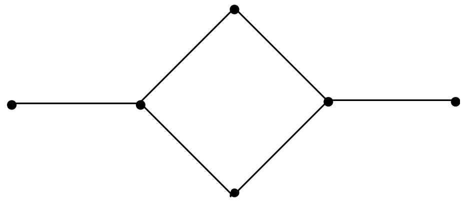


FIGURE 3. G_1

for which $D(G) = n-2$ and $H(G) - D(G) = \frac{5}{3} - \frac{n}{2}$. Among all unicyclic graphs with $D(G) = n-2$, the graph belongs to $\mathscr{C}$ is the only graph that satisfies the equality. Hence $G \in \mathscr{C}$. The converse is obviously true. $\square$

Remark 3.1. If $n \leq 6$ this lower bound is not true. One such graph is shown in Figure 3. For this graph $H(G_1) - D(G_1) = \frac{-7}{5} < \frac{-4}{3} = \frac{5}{3} - \frac{n}{2}$.

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