Trees Whose Second Largest Eigenvalue Does Not Exceed $\frac{\sqrt{5}+1}{2}$

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Abstract: The second largest eigenvalue ($\lambda_2$) provides significant information on characteristics and structure of graphs. Therefore, finding bounds for $\lambda_2$ is a topic of interest in many fields. In this paper we prove one general theorem about values of $\lambda_2$ of graphs with a cut-vertex and after that we determine all trees with the property $\lambda_2 \leq \frac{1+\sqrt{5}}{2}$.

Keywords: spectral graph theory, tree, second largest eigenvalue

1 Introduction

In this paper we consider connected simple graphs, i.e. undirected, with no loops or multiple edges. When we use the term subgraph (subtree), it means the induced subgraph (subtree). Naturally, $H$ is a supergraph of $G$ if $G$ is a subgraph (induced!) of $H$. If $A$ is (0,1)-adjacency matrix of a graph $G$, then its characteristic polynomial is defined by $P_G(\lambda) = \det(\lambda I - A)$. The roots of the characteristic polynomial are called the eigenvalues of $G$. The family of eigenvalues is the spectrum of $G$. $A$ is a real and symmetric matrix, and, therefore, its eigenvalues are real. We assume their non-increasing order: $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$. The largest eigenvalue $\lambda_1(G)$ is called the index of $G$. For connected graphs $\lambda_1(G) > \lambda_2(G)$ holds. If graph $G$ is not connected, then its spectrum is the union of the spectra of its components.

The second largest eigenvalue of a graph is a subject of investigations in spectral graph theory, but also in computer science and various fields across the science in which networks as mathematical models are widely used.

In spectral graph theory, graphs with the second largest eigenvalue bounded by a constant $a \in \mathbb{R}$ have been investigated by many authors. Some of the bounds considered so far are: $a = \frac{1}{3}$ [1], $a = \sqrt{2} - 1$ [6], $a = \frac{\sqrt{5}-1}{2}$ [3], $a = 1$ [4], $a = \sqrt{2}$ [5], $a = \sqrt{3}$ [5]. Reflexive graphs ($a = 2$) have been investigated in many articles, for example [8, 9, 10, 11, 12] where

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Theorem 1 of [10] (further, RS-theorem) was often used to prove whether a connected graph with a cut-vertex is reflexive or not.

In this paper we shall prove a generalization of this theorem and then use it in determining all trees whose second largest eigenvalue does not exceed $\sqrt{5+1}/2$. This bound for the second largest eigenvalue has not been considered before.

The paper is structured in the following way. In Section 2, we present the main tools used in our investigations including the RS-theorem. Section 3 brings the Generalized RS-theorem, along with two auxiliary lemmas. In section 4, applying this theorem and by further analysis of remaining cases, we determine all trees whose second largest eigenvalue does not exceed $\sqrt{5+1}/2$.

2 Auxiliary results and the RS-theorem

The following theorem shows the interrelation between the spectra of a graph and its induced subgraph.

**The Interlacing Theorem** (e.g.[2]) Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of a simple graph $G$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ the eigenvalues of its induced subgraph $H$. Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ $(i = 1, 2, \ldots, m)$ hold.

If $G$ is a connected graph and $m = n - 1$, then $\lambda_1 > \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots$.

**Schwenk’s Lemma** (e.g.[6]) Given a graph $G$, let $C(v)$ and $C(uv)$ denote the set of all cycles containing a vertex $v$ and an edge $uv$ of $G$, respectively. Then

1. $P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{u \in Adj(v)} P_{G-v-u}(\lambda) - 2 \sum_{C \in C(v)} P_{G-V(C)}(\lambda)$,
2. $P_G(\lambda) = P_{G-w}(\lambda) - P_{G-v-u}(\lambda) - 2 \sum_{C \in C(uv)} P_{G-V(C)}(\lambda)$,

where $Adj(v)$ denotes the set of neighbours of $v$, while $G-V(C)$ is the graph obtained from $G$ by removing the vertices belonging to the cycle $C$.

The only connected graphs for which $\lambda_1 = 2$ holds are called Smith graphs [13]. For every connected graph exactly one of the following statements hold: 1) a graph is a Smith graph, 2) a graph is a proper subgraph of some Smith graphs and its index is less than 2, 3) a graph is a proper supergraph of some of Smith graphs and its index is greater than 2.

It can be established effortlessly in many cases whether a graph with a cut-vertex is reflexive or not using a theorem of [10] (Z. Radosavljevic, S. Simic) which for convenience we will call RS-theorem.

**RS-theorem** [10] Let $G$ be a graph with a cut-vertex $u$.

1. If at least two components of $G-u$ are supergraphs of Smith graphs, and if at least one of them is a proper supergraph, then $\lambda_2(G) > 2$.
2. If at least two components of $G-u$ are Smith graphs, and the rest are subgraphs of Smith graphs, then $\lambda_2(G) = 2$. 
3. If at most one component of $G - u$ is a Smith graph, and the rest are proper subgraphs of Smith graphs, then $\lambda_2(G) < 2$.

After removing a cut-vertex of the graph $G$, we get several new connected graphs. They are comparable to the Smith graphs, in the sense that they are either their subgraphs or supergraphs.

In many cases the theorem gives the answer about the reflexivity of the graph, but there is one case when it does not, namely when after the removal of the cut-vertex $u$ we get one proper supergraph, and the rest are proper subgraphs of Smith graphs. In these cases other techniques are used, but often, at the subgraph level analysis, the RS-theorem proves useful again.

In former work, many classes of reflexive graphs have been described. Since reflexivity is a hereditary property, i.e. every subgraph preserves it, it is natural to present classes of resulting graphs through the sets of maximal reflexive graphs. In this case maximal means that its supergraphs are not reflexive. It turned out that Smith graphs play an essential role also in the construction of maximal reflexive graphs [7, 8, 9, 11, 12] By generalizing the RS-theorem we get a useful instrument for comparing $\lambda_2$ with arbitrary $a > 0$ for many graphs with a cut-vertex.

3 The Generalized RS-theorem

In this section we will prove the Generalized RS-theorem. Before that, we present two lemmas that will be used in the proof of the theorem.

**Lemma 3.1** Consider graph $G$ in Fig. 1, where $G_1$ is a connected graph with the index $a$, $a > 0$, and $u$ is an extra vertex connected to some of the vertices of the graph $G_1$ ($v_1, v_2, \ldots, v_m$). Then, $P_G(a) < 0$, and, consequently, $\lambda_2(G) < a < \lambda_1(G)$.

![Fig. 1.](image)

**Proof:** By the Interlacing theorem $\lambda_1(G) > a$ and $\lambda_2(G) \leq a$. Applying the Schwenk’s lemma at vertex $u$, we get the characteristic polynomial of the graph $G$:

$$P_G(\lambda) = \lambda P_{G_1}(\lambda) - \sum_{i=1}^{m} P_{G_1 - v_i}(\lambda) - 2 \sum_{C \in C(u)} P_{G - C}(\lambda).$$
Now, \( \lambda_1(G_1 - v_i) < a \) holds, implying \( P_{G_1 - v_i}(a) > 0 \) for \( i = 1, \ldots, m \), and also \( P_{G - C}(a) > 0 \) holds, since graph \( G - C (C \in C(u)) \) is a subgraph of the graph \( G_1 - v_i \) for some \( i = 1, \ldots, m \). Therefore, \( P_{G_1}(a) = 0 \) implies \( P_{G}(a) < 0 \) and \( \hat{\lambda}_2(G) < a \). 

**Lemma 3.2** Let \( G \) be a graph with \( n \) vertices and the eigenvalues \( \lambda_1(G) \geq \lambda_2(G) \geq \lambda_3(G) \geq \ldots \geq \lambda_{m-1}(G) \geq \lambda_m(G) \). Let \( a = \lambda_{m+1}(G) = \lambda_{m+2}(G) = \ldots = \lambda_{m+k}(G) \) be the eigenvalue of multiplicity \( k \). If the polynomial \( Q_G(\lambda) \) is defined by the relation \( P_G(\lambda) = (\lambda - \alpha)^k Q_G(\lambda) \), then \( \text{sgn}(Q_G(\alpha)) = (-1)^m \).

**Proof:** The characteristic polynomial of the graph \( G \) is factorized in the following way: 

\[
P_G(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i) \cdot (\lambda - \alpha)^k \prod_{i=m+k+1}^{n}(\lambda - \lambda_i).
\]

Thus we have proved \( \text{sgn}(Q_G(\alpha)) = (-1)^m \).

Here are some simple and useful consequences of the Lemma 3.2. In a connected graph \( G \), if \( \lambda_2 = \alpha \), then \( Q_G(\alpha) < 0 \); or, if \( \lambda_3 = \alpha \) and \( \lambda_2 > \lambda_3 \), then \( Q_G(\alpha) > 0 \).

**Theorem 3.3** (GT) Let \( G \) be the graph in Fig. 2, with a cut-vertex \( u \). Let components of the graph \( G - u \), the graphs \( G_1, \ldots, G_n \), be connected graphs for which \( \lambda_2(G_i) \leq a, i = 1, \ldots, n \), holds. For \( a > 0 \) it holds:

1. If at most one of the graphs \( G_1, \ldots, G_n \) has index \( a \), and for the rest of them the indices are less than \( a \), then \( \lambda_2(G) < a \).

2. If at least two of the graphs \( G_1, \ldots, G_n \) have indices \( a \), and for the rest of them the indices are not greater than \( a \), then \( \lambda_2(G) = a \).

3. If only one of the graphs \( G_1, \ldots, G_n \) has index greater than \( a \), and at least one of the remaining graphs has index \( a \), then \( \lambda_2(G) > a \).

**Proof:** 1. If \( \lambda_1(G_1), \lambda_1(G_2), \ldots, \lambda_1(G_n) < a \), then \( \lambda_1(\bigcup_{i=1}^{n} G_i) < a \) and, therefore, by the Interlacing theorem, \( \lambda_2(G) < a \). Now consider the case when exactly one of the graphs \( G_i \) has index \( a \), say \( \lambda_1(G_1) = a \), and \( \lambda_1(G_2), \lambda_1(G_3), \ldots, \lambda_1(G_n) < a \). \( \lambda_1(G_1) = a \) implies \( \lambda_2(G_1) < a \). Then, \( \lambda_1(\bigcup_{i=1}^{n} G_i) = a \) and \( \lambda_2(\bigcup_{i=1}^{n} G_i) < a \), and, therefore, by the Interlacing theorem, \( \lambda_2(G) \leq a \). \( \lambda_1(G_1) = a \) implies \( P_{G_1}(a) = 0 \), and \( \lambda_1(G_2), \lambda_1(G_3), \ldots, \lambda_1(G_n) < a \)
implies \( P_{G_2}(a), P_{G_1}(a), \ldots, P_{G_n}(a) < a \). Applying the Schwenk’s lemma at the vertex \( u \) of the graph \( G \), we compute the characteristic polynomial of \( G \).

\[
P_G(a) = aP_{G_1}(a) \cdots P_{G_n}(a) \\
- \left( \sum_{v \in \text{Adj.} \cap G_1} P_{G_1-v}(a) + 2 \sum_{C \in (u) \cap G_1} P_{G_1-V(C)}(a) \right) P_{G_2}(a) \cdots P_{G_n}(a) \\
- P_{G_1}(a) \left( \sum_{v \in \text{Adj.} \cap (G-G_1)} P_{G-V_1-u-v}(a) + 2 \sum_{C \in (u) \cap (G-G_1)} P_{G-V(C)}(a) \right) P_{G_2}(a) \cdots P_{G_n}(a) \\
= - \left( \sum_{v \in \text{Adj.} \cap G_1} P_{G_1-v}(a) + 2 \sum_{C \in (u) \cap G_1} P_{G_1-V(C)}(a) \right) P_{G_2}(a) \cdots P_{G_n}(a).
\]

Let us introduce the graph \( K = G - (G_2 \cup \ldots \cup G_n) \). Then,

\[
P_K(a) = aP_{G_1}(a) - \sum_{v \in \text{Adj.} \cap G_1} P_{G_1-v}(a) - 2 \sum_{C \in (u) \cap G_1} P_{G_1-V(C)}(a)
\]

Then, \( P_{G_2}(a) = P_K(a) \cdot P_{G_2}(a) \cdots P_{G_n}(a) \). \( P_K(a) < 0 \) by the Lemma 3.1, and therefore, \( P_G(a) < 0 \), implying \( \lambda_2(G) < a \).

2. A clear consequence of the Interlacing Theorem.

3. If the index of one of the graphs \( G_1, \ldots, G_n \) is greater than \( a \), say \( \lambda_1(G_1) > a \), and one is equal to \( a \), say \( \lambda_1(G_2) = a \), the Interlacing theorem implies \( \lambda_2(G) \geq a \). To prove that a strong inequality \( \lambda_2(G) > a \) holds, let us notice that it is sufficient to prove it in the case when graph \( G - u \) consists only of the two mentioned components \( G_1 \) and \( G_2 \). If \( \lambda_2(G_1) > a \), then \( \lambda_2(G_1 \cup G_2) > a \), and, therefore, \( \lambda_2(G) > a \). Now, let us consider the case \( \lambda_2(G_1) = a \). Let us introduce the graphs \( H = G - G_1 \) and \( K = G - G_2 \). Applying the Schwenk’s lemma at the vertex \( u \) of the graphs \( H \) and \( K \) we get:

\[
P_K(\lambda) = \lambda P_{G_1}(\lambda) - \sum_{v \in \text{Adj.} \cap G_1} P_{G_1-v}(\lambda) - 2 \sum_{C \in (u) \cap K} P_{K-V(C)}(\lambda)
\]

\[
P_H(\lambda) = \lambda P_{G_2}(\lambda) - \sum_{v \in \text{Adj.} \cap G_2} P_{G_2-v}(\lambda) - 2 \sum_{C \in (u) \cap H} P_{H-V(C)}(\lambda).
\]

For the graph \( G \) we get:

\[
P_G(\lambda) = \lambda P_{G_1}(\lambda) P_{G_2}(\lambda)
\]

\[
- \left( \sum_{v \in \text{Adj.} \cap G_1} P_{G_1-v}(\lambda) + 2 \sum_{C \in (u) \cap K} P_{K-V(C)}(\lambda) \right) P_{G_2}(\lambda)
\]

\[
- \left( \sum_{v \in \text{Adj.} \cap G_2} P_{G_2-v}(\lambda) + 2 \sum_{C \in (u) \cap H} P_{H-V(C)}(\lambda) \right) P_{G_1}(\lambda).
\]
Finally, \( P_G(\lambda) = P_K(\lambda)P_G(\lambda) + P_{G_1}(\lambda)P_H(\lambda) - \lambda P_{G_1}(\lambda)P_G(\lambda) \). Since \( P_G(a) = 0 \), we have \( P_G(a) = P_{G_1}(a)P_H(\lambda) \). Since \( \lambda_1(G_2) = a \), Lemma 3.1 implies \( P_H(a) < 0 \).

a) If \( \lambda_2(G_1) < a < \lambda_1(G_1) \), then \( P_{G_1}(a) < 0 \), implying \( P_G(a) > 0 \), and, therefore, \( \lambda_2(G) > a \).

b) Let us consider the case \( \lambda_2(G_1) = \lambda_3(G_1) = \ldots = \lambda_k(G_1) = a \) and \( \lambda_{k+1}(G_1) < a \), \( k \geq 2 \). By the Interlacing theorem we have \( \lambda_2(K) \geq a \). Similarly, if \( \lambda_2(K) > a \), then \( \lambda_2(G) > a \).

Now, let us consider the case \( \lambda_2(K) = a \). By the Interlacing theorem \( \lambda_3(K) = \lambda_4(K) = \ldots = \lambda_k(K) = a \) holds. Let us introduce the polynomial \( Q_K(\lambda) \) by \( P_K(\lambda) = (\lambda - 1)^k - 1 \). \( Q_K(\lambda) \). Similarly, we have \( P_{G_1}(\lambda) = (\lambda - 1)^k - 1 \cdot Q_{G_1}(\lambda) \) and \( P_{G_2}(\lambda) = (\lambda - 1) \cdot Q_{G_2}(\lambda) \). Notice that, by Lemma 3.2, \( Q_{G_1}(a) < 0 \). Now,

\[
P_G(\lambda) = (\lambda - 1)^k - 1 \cdot Q_K(\lambda) \cdot (\lambda - 1) \cdot Q_{G_2}(\lambda) + (\lambda - 1)^k - 1 \cdot Q_{G_1}(\lambda) \cdot P_H(\lambda) - \lambda \cdot (\lambda - 1)^k - 1 \cdot Q_{G_1}(\lambda) \cdot (\lambda - 1) \cdot Q_{G_2}(\lambda).
\]

Let us introduce the polynomial \( Q_G(\lambda) \) by \( P_G(\lambda) = (\lambda - 1)^k - 1 \cdot Q_G(\lambda) \). Then, \( Q_G(a) = Q_{G_1}(a) \cdot P_H(a) \). Since \( Q_{G_1}(a) < 0 \) and \( P_H(a) < 0 \), we get \( Q_G(a) > 0 \), implying \( \lambda_2(G) > a \). 

The Generalized RS-theorem brings improvement of the Interlacing theorem for the cases 1. and 3.

4 Trees whose second largest eigenvalue does not exceed \( \frac{\sqrt{5} + 1}{2} \)

Let us determine all trees \( T \) with the property \( \lambda_2(T) \leq \frac{\sqrt{5} + 1}{2} \), by describing all maximal trees for this property.

The bound \( \frac{\sqrt{5} + 1}{2} \) has not been considered before. We shall denote it by \( \phi \). This number is the greater root of the polynomial \( \phi^2 - \phi - 1 \), hence, it is an index of the path \( P_4 \). For every tree one and only one of the following statements holds: 1) a tree is the path \( P_4 \), 2) a tree is a proper subgraph of the path \( P_4 \), 3) a tree is a proper supergraph of the path \( P_4 \), 4) a tree is the star \( K_{1,3} \). The only trees for which \( \lambda_1 < \phi \) holds are paths \( P_1, P_2 \) and \( P_3 \). On the other hand, minimal forbidden trees for the property \( \lambda_1 \leq \phi \) are path \( P_5 \) and star \( K_{1,3} \) (for both of these trees \( \lambda_1 = \sqrt{3} \) holds).

By \( T_\infty \) we shall denote the family of trees \( T \) with a cut-point \( u \), for which all components of \( T - u \) are paths \( P_1, P_2, P_3 \) and \( P_4 \). For these graphs, by GT, \( \lambda_2 \leq \phi \) holds. Also, we shall say that a tree is GT-decidable for \( \lambda_2 = \phi \) if we can find whether its second largest eigenvalue is less than \( \phi \), equal to \( \phi \), or greater than \( \phi \), only by applying GT to one of its cut-points. All graphs of the family \( T_\infty \) are GT-decidable for \( \lambda_2 = \phi \). Graphs whose at least one component is path \( P_4 \) are GT-decidable, for \( \lambda_2 = \phi \), too.
In certain cases, the sign of the \( P_T(\phi) \) will be used to compare values \( \lambda_2(T) \) and \( \phi \). This is explained by the next Lemma, which is a simple consequence of the Interlacing theorem.

**Lemma 4.1** For a tree \( T \), let \( \lambda_2(T) < \phi < \lambda_1(T) \). Let \( \tau \) be the tree \( T \) extended with a pendant edge. Then the following statements hold:

1. If \( P_\tau(\phi) < 0 \), then \( \lambda_2(\tau) < \phi \).
2. If \( P_\tau(\phi) = 0 \), then \( \lambda_2(\tau) = \phi \).
3. If \( P_\tau(\phi) > 0 \), then \( \lambda_2(\tau) > \phi \).

In the next Lemma we also give values of the \( P_T(\phi) \) for some simple trees.

**Lemma 4.2** 1) \( P_1(\phi) = P_2(\phi) = \phi \), 2) \( P_3(\phi) = 1 \), 3) \( P_5(\phi) = 0 \), 4) \( P_{5,\phi}(\phi) = -P_5(\phi) \), \( n \in \mathbb{N} \), 5) \( P_{5,\phi}(\phi) = P_5(\phi), n,k \in \mathbb{N} \), 6) \( P_{5,\phi}(\phi) = \phi^{\cos^2(\frac{k\pi}{n+1})} - \phi^{\cos^2(\frac{k\pi}{n+1})} \).

**Proof**: These are simple consequences of \( P_T(\phi) = \prod_{k=1}^n(\phi - 2\cos\frac{k\pi}{n+1}) \), Schwenk’s lemma and the fact that \( \phi^2 = \phi + 1 \).

Now, we shall describe all trees \( T \) with the property \( \lambda_2(T) \leq \sqrt{\frac{125}{2}} \) that are not GT-decidable for \( \phi \).

**Theorem 4.3** Let \( T \) be a tree which is not GT-decidable for \( \phi \). Then, \( \lambda_2(T) \leq \phi \) if and only if \( T \) is a subgraph of some of the trees \( T_1-T_{39} \) (Figure 3).

**Proof**: Using GT we easily get that \( \lambda_2(P_n) > \phi \), for \( n \geq 10 \) (\( P_n \) is shown in Figure 3, with appropriate labels). In this case, the middle vertex (or one of two such vertices) is considered as a cut-point. Therefore, a diameter of a tree with the property \( \lambda_2 \leq \phi \) must be less than 9. If the diameter of a tree is 8, the tree is GT-decidable for \( \phi \). If it is a path \( P_8 \), then \( \lambda_2 = \phi \) (cut-vertex is again the middle vertex and two components are paths \( P_8 \)). If it is a supergraph of \( P_8 \), then \( \lambda_2 > \phi \). Let us notice that a tree is GT-decidable for \( \phi \) (and belongs to the family \( T_n \)) if the diameter of a tree is 2, too. We shall continue by discussing the length of the diameter.

Let \( \text{diam}(T) = 7 \). Then \( T \) contains a path \( P_8 \), which we observe as a basic tree, whose vertices (different from end-vertices) may be additionally loaded with some subtrees. At least one of the vertices \( v_2, v_3 \) and \( v_4 \) (or \( v_5, v_6 \) and \( v_7 \)) must be additionally loaded, otherwise a tree will be GT-decidable for \( \phi \). If there is a pendant edge added to any of the vertices \( v_2, v_3, v_5 \) (or \( v_7 \)), the tree is GT-decidable, too. The vertex \( v_4 \) is considered as a cut-vertex and we get \( \lambda_2(T) > \phi \) (components are \( K_{1,3} \) and \( P_3 \)) or \( \lambda_2(T) = \phi \) (components are two paths \( P_3 \)). Therefore, there must be at least a pendant edge added to each of the vertices \( v_4 \) and \( v_5 \). For such tree \( T \) \( \lambda_2(T) < \sqrt{\frac{125}{2}} \) holds. By making further extensions, we can get three different maximal trees, \( T_1, T_2 \) and \( T_3 \) (Figure 4). For all of them, \( \lambda_2 = \phi \) holds. This can
be easily proved by using Schwenk’s lemma (and also checked by using NEWGRAPH). For example, for the tree $T_1$, by applying Schwenk’s lemma to the vertex $v_4$ and by using $\phi^2 - \phi - 1 = 0$, we get $P_{T_1}(\phi) = -\phi^3 + \phi^3 + \phi^2 + \phi - \phi^3 = 0$. For any subtree of the tree $T_1$, we can check in the same way that its characteristic polynomial at the point $\phi$ is less than zero. Hence, if we add a pendant edge to any vertex of the tree $T_1$, the characteristic polynomial of this new tree at the point $\phi$ shall exceed 0. Therefore, $T_1$ is a maximal tree for the property $\lambda_2 = \sqrt{5} + 1$.

Let $\text{diam}(T) = 6$. The basic tree is the path $P_7$. At least one of the vertices $v_2$, $v_3$ and $v_4$ (or $v_4$, $v_5$ and $v_6$) must be additionally loaded, otherwise the tree will be GT-decidable for $\phi$. If there is a pendant edge added to a vertex $v_2$, then the vertices $v_5$ and $v_6$ may not be
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additionally loaded (otherwise, the tree is GT-decidable for $\varphi$ and $\lambda_2 > \varphi$). Then the vertex $v_4$ must be additionally loaded. By further extensions and using Lemma 4.1 we can get two maximal trees, $T_6$ and $T_7$, for which $\lambda_2 = \varphi$ holds. Let the vertex $v_2$ be of the degree 2 and let the vertex $v_3$ be additionally loaded. Then the degrees of the vertices $v_5$ and $v_6$ have to be 2, too. Otherwise, if we observe $v_4$ as a cut-vertex, we get GT-decidable graph. Therefore, the vertex $v_4$ must be additionally loaded. Further on, we can discuss possible degrees of the vertices $v_3$ and $v_4$.

Let $d(v_3) = 3$ and $d(v_4) = r + 2$. If there is a pendant edge added to the vertex $v_3$, the tree becomes GT-decidable for $\varphi$. If there is a path $P_n (n > 3)$ added to a vertex $v_3$, the diameter of the graph becomes greater than 6. Therefore, there must be the path $P_3$ added to the vertex $v_3$. Let nothing but pendant edges be added to $v_4$. Then we get $P(\varphi) = \varphi'^{r+1} - \varphi'^{r+2} + r\varphi'^{r-1} + \varphi'^{r+1}$. Using $\varphi^2 - \varphi - 1 = 0$, $P(\varphi) = \varphi'^{-1}(r - 2\varphi - 1)$ holds. Therefore, $r \leq 4$. For $r = 4$, $P(\varphi) < 0$. There is no possible extension $\tau$ such that $P_7(\varphi) < 0$, so we get the maximal tree $T_6$. For $r = 3$, $r = 2$ and $r = 1$ the only maximal trees that we get by further extending are the trees $T_7$, $T_8$ and $T_9$, respectively.

Let pendant edges be added to the vertices $v_3$ and $v_4$ and let $d(v_4) = 3$ and $d(v_3) = r + 2$. Applying Schwenk’s lemma to $v_3$ we get $P(\varphi) = -\varphi\varphi'^{r+1} + \varphi'^{r+1} + r\varphi' - \varphi'^{r+2}$. Using $\varphi^2 - \varphi - 1 = 0$, $P(\varphi) = \varphi'(r - \varphi - 2)$ holds. Hence, $r \leq 3$. For $r = 3$, the only possible extension is the tree $T_{10}$. For $r = 2$, there are four possible extensions: $T_{11}$, $T_{12}$, $T_{13}$ and $T_{14}$.

If the only loaded vertex is $v_4$, then this vertex must be an end-vertex of a bridge which connects the basic tree $P_7$ and another tree $\tau$ whose index is greater than $\varphi$. If the tree $\tau$ is the path $P_4$, then it must be leaning on its middle vertex, otherwise the diameter becomes greater than 6. For the whole tree $\lambda_2 < \varphi$ holds, but the only possible extension leads to the tree $T_8$. If the tree $\tau$ is the star $K_{1,3}$, it must be leaning on its middle vertex (otherwise, $\lambda_2 > \varphi$). Hence, $\lambda_2$ of the whole tree is less than $\varphi$, and this case allows 4 new extensions: the trees $T_{15}$–$T_{18}$.

Let $\text{diam}(T) = 5$. The basic tree is the path $P_6$. At least one of the vertices $v_2$, $v_3$ and $v_4$ (or $v_3$, $v_4$ and $v_5$) must be additionally loaded, otherwise the tree will be GT-decidable for $\varphi$. If there is a pendant edge added to the vertex $v_2$, then at least one of the vertices $v_3$, $v_4$ or $v_5$ has to be additionally loaded. If it is $v_5$, we get maximal tree $T_{10}$. If it is $v_4$ and if it is loaded with a pendant edge, the tree becomes GT-decidable. Let $d(v_4) = 4$ (for $d(v_4) = 5$, $P(\varphi)$ becomes greater than 0). There are two possible maximal extensions – $T_{20}$ and $T_{21}$. In the case of $d(v_4) = 3$, there has to be a path $P_3$ leaned on the vertex $v_4$ and the only maximal extension is $T_{22}$. Now let only $v_2$ and $v_3$ be loaded. Let $d(v_2) = i$ and $d(v_3) = j$. Using Schwenk’s lemma we get $P(\varphi) = \varphi^{2+i}(j \varphi^{i-1}) - (i + 1)\varphi'^{(i+1)}(j - (j + 1)(\varphi + 1)\varphi^{i+j+1} = \varphi^{i+j+1}((i + 1)j - (j + 1)(\varphi + 1))$. Hence, if there is only a pendant edge leaned on $v_3$, maximum degree of $v_2$ is 6 and the tree $T_{23}$ is maximal. For $d(v_2) = 5$, the only maximal extension is $T_{24}$. For $d(v_2) = 4$, we get 4 maximal extensions: $T_{25}$–$T_{28}$. For $d(v_2) = 3$, $v_3$ must be an end-vertex of a bridge and the other end-vertex must be the middle vertex of the star $K_{1,3}$ (otherwise, the tree remains GT-decidable, or its diameter exceeds 5). But this case produces only maximal tree $T_{28}$. Let $d(v_2) = d(v_5) = 2$, $d(v_3) = i + 2$ and $d(v_4) = j + 2$. As before, we get $P(\varphi) = \varphi^{i+j}(i - j - \varphi)$. Hence, $\min(i, j) \leq 2$. Let $i \leq j$. If $i = 1$ there
A tree $T$ has the property $D. C$ for $d_T(v)$ end-vertex is trees $T$. Further extending we get maximal trees $T$ has to be the path $P$ whose index is greater than $\phi$. Because of $\text{diam}(T) = 5$, the tree $\tau$ is the star $K_{1,3}$. For the whole tree $P(\phi) < 0$ holds, but further extending brings only trees that are already described - $T_{23} - T_{28}$.

Let $\text{diam}(T) = 4$. The basic tree is the path $P_3$. Vertices $v_2$ and $v_4$ are to be loaded only by pendant edges. Let $d(v_2) = i$, $d(v_3) = 2$ and $d(v_4) = j$. As before, using Schwenk’s lemma we get $P(\phi) = \phi^{i+j} = (i-j)(\phi + 1 + 3\phi + 1)$, and hence, $\min(i, j) \leq 3$. Otherwise the tree becomes GT-decidable for $\phi$. Let $i \leq j$. If $i = 3$, then $j \leq 5$. Let $i = 3$: for $j = 5$, we get maximal tree $T_{36}$; for $j = 4$, the vertex $v_3$ can be additionally loaded and by further extending we get maximal trees $T_{37} - T_{40}$. For $j = 3$, there must be a bridge whose end-vertex is $v_3$ and the star $K_{1,3}$ is leaned on the other end. This tree is not maximal, but it can be extended in only one way, which gives maximal tree $T_{41}$. If $j = 2$, then $d(v_3) \geq 3$. For $d(v_3) = 3$, the path $P_3$ must be leaned on the vertex $v_3$ and $P(\phi) = \phi^{j-2}(j-3\phi - 3)$ holds, and hence, $j \leq 7$. For $j = 7$ we get maximal tree $T_{42}$. Let $d(v_3) = k \geq 4$. Then from $P(\phi) = \phi^{i+k} = (k - 3 - 2k + 5 + 2\phi - k\phi)$ we get $j \leq 6$. For $j = 6$ there is only one maximal tree $T_{43}$, for $j = 5$ only one maximal tree $T_{44}$, while for $j = 4$ we get 4 maximal trees $T_{35} - T_{38}$. For $j = 3$ and $j = 2$, there must be a bridge to the star $K_{1,3}$ leaned on the vertex $v_3$ which leads to maximal trees $T_{49} - T_{57}$.

Let $\text{diam}(T) = 3$. Then there are only pendant edges leaned on the vertices $v_2$ and $v_3$. Let $d(v_2) = i - 2$ and $d(v_3) = j - 2$, hence $\min(i, j) \geq 4$ (otherwise such a tree belongs to the family $T_{\infty}$. As before, we get $P(\phi) = \phi^{i+j-4}((i-2)(j-2) - \phi(i+j-4))$. Therefore, $\min(i, j) \leq 5$. Let $i \leq j$: for $i = 4$, we get $j_{\max} = 10$ and for $i = 5$, $j_{\max} = 5$, which brings maximal trees $T_{58}$ and $T_{59}$.

From the previous results it follows the main theorem.

**Theorem 4.4** A tree $T$ has the property $\lambda_3(T) \leq \sqrt{\frac{3+1}{2}}$ if and only if it belongs to the family of trees $T_{\infty}$ or it is a subgraph of some of the trees $T_1 - T_{59}$.

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**References**

Trees Whose Second Largest Eigenvalue Does Not Exceed $\frac{\sqrt{5}+1}{2}$


