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On Some Irregularity Measures of Graphs

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Abstract: Let $\Gamma(G)$ be a set of all simple graphs of order n and size m, without isolated vertices, with vertex degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n > 0$. A graph G is regular if and only if $d_1 = d_2 = \cdots = d_n$. Each mapping $Irr: \quad \Gamma(G) \mapsto [0, +\infty)$ with the property $Irr(G) = 0$ if and only if G is regular, is referred to as irregularity measure of graph. In this paper we introduce some new irregularity measures and inequalities that establish relations between them.

Keywords: Zagreb indices; irregularity measures; inequalities.

1 Introduction

We consider simple graphs $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, without isolated vertices. Denote by $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(i)$, $i = 1, 2, \ldots, n$, a sequence of vertex degrees and by $d(e_1) \geq d(e_2) \geq \cdots \geq d(e_m)$, a sequence of edge degrees, whereby for each edge $e = \{i, j\} \in E$ holds $d(e) = d_i + d_j - 2$. By $i \sim j$ we denote that vertices i and j are adjacent, while by $e_i \sim e_j$ that edges e_i and e_j are adjacent in a given graph G.

Let $\mathbf{A}$ be the adjacency matrix of G. Eigenvalues of $\mathbf{A}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ represent ordinary eigenvalues of graph G. Well known properties of these are (see for example [6, 30])

$$\sum_{i=1}^{n} \lambda_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} d_i = 2m.$$

Denote by $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ a diagonal matrix of vertex degrees in G. Then $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the Laplacian matrix of G. The eigenvalues of $\mathbf{L}$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$, are Laplacian eigenvalues of G. Some well known properties of Laplacian eigenvalues are

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[6, 10]

$$\sum_{i=1}^{n-1} \mu_i = \sum_{i=1}^{n} d_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n-1} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m,$$

where

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 = \sum_{i\sim j} (d_i + d_j)$$

is the first Zagreb index [24]. In the same paper, the second Zagreb index, M_2, is defined as

$$M_2 = M_2(G) = \sum_{i\sim j} d_i d_j.$$

The first and the second Zagreb indices of a given graph satisfy so called Zagreb indices inequality (see for example [8, 18])

$$\frac{M_1}{n} \le \frac{M_2}{m}. \tag{1}$$

The first and second reformulated Zagreb indices, EM_1 and EM_2, are defined as [35, 44]

$$EM_1 = EM_1(G) = \sum_{i=1}^{m} d(e_i)^2 \qquad \text{and} \qquad EM_2 = EM_2(G) = \sum_{e_i \sim e_j} d(e_i)d(e_j).$$

A relation between Zagreb indices and reformulated Zagreb indices is established over the line-graph, $\mathscr{L} = \mathscr{L}(G)$, of the underlying graph G (see [6, 10, 30]). Namely,

$$EM_1(G) = M_1(\mathscr{L}(G)) \qquad \text{and} \qquad EM_2(G) = M_2(\mathscr{L}(G)).$$

It is not difficult to conclude that line-graph $\mathscr{L}(G)$ has m vertices and $\frac{1}{2}M_1 - m$ edges. If graph G satisfies (1), then the following is valid

$$\frac{EM_1}{2m} \le \frac{EM_2}{M_1 - 2m} \qquad \text{and} \qquad \frac{EM_1}{n} \le \frac{EM_2}{2m-n}, \quad (2m \ne n). \tag{2}$$

The forgotten topological index of G is defined as [17, 22, 24, 45]

$$F_1 = \sum_{i=1}^{n} d_i^3 = \sum_{i\sim j} (d_i^2 + d_j^2).$$

The equality that establishes a connection between graph invariants M_1, M_2, EM_1 and F_1 is given by [45]

$$EM_1 = F_1 + 2M_2 - 4M_1 + 4m. \tag{3}$$

More on these and other degree-based topological indices and their applications can be found in [12, 13, 16, 18, 20, 25, 26, 27, 28, 37, 40, 41, 42, 43].

A graph G is regular if and only if $d_1 = d_2 = \cdots = d_n > 0$. Otherwise it is irregular. Let $\Gamma(G)$ be a set of all simple graphs of order n, $n \geq 2$, and size m, without isolated vertices. Each mapping $Irr: \quad \Gamma(G) \mapsto [0, +\infty)$ with the property $Irr(G) = 0$ if and only if G is regular, is referred to as irregularity measure of graph. There have been defined a lot of irregularity measures in the literature (see for example [1, 2, 3, 5, 11, 14, 15, 19, 21, 24, 29, 34]). In the next section we mention some irregularity measures that are of interest for our work and introduce some new ones.

2 Some irregularity measures of the graph

Collatz and Sinogowitz [11] have proved that

$$\lambda_1 \geq \frac{2m}{n},$$

with equality holding if and only if G is a regular graph. Using this inequality in [21] the Collatz–Sinogowitz irregularity measure was defined via

$$Irr_{CS}(G) = \frac{n\lambda_1}{2m} - 1.$$

Nikiforov [34] introduced irregularity measure referred to as degree deviation

$$S(G) = \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right|.$$

We will call this measure Nikiforov irregularity measure and denote it by $Irr_N(G)$.

Bell [5] considered the variance of vertex degrees as irregularity measure. It is defined as

$$VAR(G) = \frac{1}{n} \sum_{i=1}^{n} \left(d_i - \frac{2m}{n} \right)^2.$$

It is not difficult to see that both $Irr_N(G)$ and $VAR(G)$ are special case of the following graph invariant

$$R_\alpha(G) = \left(\frac{1}{n^{\alpha-1}} \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right|^\alpha \right)^{1/\alpha}, \qquad \alpha \geq 1.$$

Hamzeh and Reti [24] defined the following irregularity measure

$$IRM_1(G) = M_1 - \frac{4m^2}{n}.$$

Edwards [14] considered the value C_γ as irregularity measure, defined as

$$1 + C_\gamma^2 = \frac{nM_1}{4m^2}, \tag{4}$$

whereby C_γ^2 is a real number. He proved that $C_\gamma = 0$ if and only if G is regular. It is not difficult to see that the following relation between $VAR(G)$, $IRM_1(G)$ and C_γ^2 holds

$$C_\gamma^2 = \frac{n^2}{4m^2} VAR(G) = \frac{n}{4m^2} IRM_1(G).$$

Therefore these measures can be considered as equivalent. Since $C_\gamma^2 \geq 0$, according to (4) it follows that

$$M_1 \geq \frac{4m^2}{n}. \tag{5}$$

Equality holds if and only if G is regular. More on the inequality (5) one can find in [12, 20, 25, 44]. Based on (5), in [21] Edwards' irregularity measure was defined as

$$Irr_E(G) = \sqrt{\frac{nM_1}{4m^2} - 1}.$$

Ilić and Stevanović [25] (see also [45]) proved the following inequality

$$M_2 \geq \frac{4m^3}{n^2}, \tag{6}$$

with equality holding if and only if G is a regular graph. Based on this, a new irregularity measure, Ilić–Stevanović measure, can be defined as

$$Irr_{IS}(G) = \sqrt[3]{\frac{n^2 M_2}{4m^3} - 1}.$$

In [24] the following irregularity measure was proposed

$$IRM_2(G) = M_2 - \frac{4m^3}{n^2}.$$

Since

$$IRM_2(G) = \frac{4m^3}{n^2} Irr_{IS}(G),$$

it follows that $IRM_2(G)$ is not different from the $Irr_{IS}(G)$.

From the inequality (5) the following inequality can be directly derived [13]

$$EM_1 \geq \frac{(M_1 - 2m)^2}{m} \geq \frac{4m(2m-n)^2}{n^2},$$

with equality holding if and only if G is regular. Based on this inequality, another irregularity measure can be defined as

$$Irr_1(G) = \sqrt[3]{\frac{n^2 EM_1}{4m(2m-n)^2} - 1}, \qquad 2m \neq n.$$

In [26] (see also [13]) based on inequality (6) the following one was proved

$$EM_2 \geq \frac{(M_1 - 2m)^3}{2m^2},$$

with equality holding if and only if G is regular. According to this and the inequality (5), it follows that

$$EM_2 \geq \frac{4m(2m-n)^3}{n^3}.$$

Now we can define Ilić–Zhou irregularity measure

$$Irr_{IZ}(G) = \sqrt[4]{\frac{n^3 EM_2}{4m(2m-n)^3} - 1}, \qquad 2m \neq n.$$

Denote by $F_1^{(\alpha)} = \sum_{i=1}^{n} d_i^{\alpha}$, where α ($\alpha \geq 1$) is a real number. It is not difficult to see that $F_1^{(1)} = 2m$, $F_1^{(2)} = M_1$ and $F_1^{(3)} = F_1$. According to the Chebyshev's inequality (see for example [31])

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i \geq \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i, \tag{7}$$

for $p_i = b_i = d_i$ and $a_i = d_i^{\alpha-2}$, $\alpha \geq 2$, inequality (7) becomes

$$\sum_{i=1}^{n} d_i^{\alpha} \geq \frac{M_1}{2m} \sum_{i=1}^{n} d_i^{\alpha-1}. \tag{8}$$

After iterating the above inequality, we obtain

$$F_1^{(\alpha)} = \sum_{i=1}^{n} d_i^{\alpha} \geq \frac{(M_1)^{\alpha-1}}{(2m)^{\alpha-2}}, \qquad \alpha \geq 2.$$

Based on this and inequality (5), we have that

$$F_1^{(\alpha)} = \sum_{i=1}^{n} d_i^{\alpha} \geq \frac{(2m)^{\alpha}}{(n)^{\alpha-1}}, \qquad \alpha \geq 2,$$

whereby the equality holds if and only if G is regular. This inequality enables us to define a bunch of irregularity measures for various α

$$Irr^{(\alpha)}(G) = \left(\frac{n^{\alpha-1} F_1^{(\alpha)}}{(2m)^{\alpha}} - 1 \right)^{1/\alpha}. \tag{9}$$

It is obvious that $Irr^{(2)}(G) = Irr_E(G)$. Also, the following might be an interesting irregularity measure when $\alpha = 3$

$$Irr^{(3)} = \sqrt[3]{\frac{n^2 F_1}{8m^3}} - 1.$$

Remark 1 *Let us note that equivalent irregularity measure to the above was defined in [38].*

In [19] Goldberg noticed that the simplest irregularity measure is

$$d(G) = d_1 - d_n.$$

We will call this measure Goldberg's irregularity measure, and consider it in the form

$$Irr_g(G) = \frac{d_1}{d_n} - 1.$$

Remark 2 *It is obvious that a lot of irregularity measures can be defined. The question is do we really need all of them? From each of them we can conclude whether a given graph is regular or not, but if we are interested to see how much the considered graph deviates from the regular one, the differences arise. It is desirable that irregularity measure is sensitive to the changes of all basic graph parameters: n, m, d_1 and d_n. Thus, for example Golberg's irregularity measure is really simple, but it is (completely) insensitive to the changes of parameter m when parameters d_1 and d_n remain unchanged. Thus, for example, if graph G_1 is obtained by adding edges to the graph G, while d_1 and d_n remain unchanged , then $Irr_g(G_1) = Irr_g(G)$. However, it is obvious that G deviates from regularity more than G_1. Also, the measure $Irr(G) = \frac{2m}{nd_n} - 1$, ($d_n \neq 0$), is not sensitive to the changes of parameter d_1 when n, m and d_n are unchanged. Nevertheless, these "simple" irregularity measures are useful for determining bounds of other irregularity measures or when comparing different measurs.*

3 Inequalities for irregularity measures

In this section we prove some inequalities between irregularity measures defined in the previous section. Also we establish bounds for some of them. But, first we recall some results from the literature needed for our work.

In [19] Goldberg proved the following inequality

$$Irr_{CS}(G) \geq \frac{4m^2}{n^2}\sqrt{\frac{2m}{nd_1}} Irr_E(G)^2. \tag{10}$$

Nikiforov [34] proved the following inequalities

$$\frac{\sqrt{2m}}{2n}Irr_E(G)^2 \le Irr_{CS}(G) \le \frac{n\sqrt{Irr_N(G)}}{2m}, \tag{11}$$

and

$$\frac{Irr_N(G)^2}{4nm\sqrt{2m}} \le Irr_{CS}(G) \le \sqrt{\frac{n^2}{2m}Irr_E(G)}. \tag{12}$$

Based on the Popovicu [36] and Nagy [33] inequalities (see also [39]) the following inequalities can be obtained

$$\frac{\sqrt{n}d_nIrr_g(G)}{2\sqrt{2m}} \le Irr_E(G) \le \frac{nd_nIrr_g(G)}{4m}. \tag{13}$$

In the following theorem we improve the right-hand side of inequality (13).

Theorem 1 *For each graph G, $G \in \Gamma(G)$, the following is valid*

$$Irr_E(G) \le \frac{\sqrt{\alpha(n)}nd_nIrr_g(G)}{2m}, \tag{14}$$

where

$$\alpha(n) = \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\left(1-\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right). \tag{15}$$

Equality holds if and only if G is regular.

Proof. For real numbers $p_1,p_2,\dots,p_n$, $a_1,a_2,\dots,a_n$ and $b_1,b_2,\dots,b_n$ with the property

$$p_i \ge 0, \quad 0 < r_1 \le a_i \le R_1 < +\infty, \quad 0 < r_2 \le b_i \le R_2 < +\infty, \quad i = 1,2,\dots,n$$

Andrica and Badea [4] proved the following inequality

$$\left|\sum_{i=1}^n p_i\sum_{i=1}^n p_ia_ib_i - \sum_{i=1}^n p_ia_i\sum_{i=1}^n p_ib_i\right| \le (R_1-r_1)(R_2-r_2)\sum_{i\in S}p_i\left(\sum_{i=1}^n p_i - \sum_{i\in S}p_i\right), \tag{16}$$

where S is a subset of the set $I_n = \{1,2,\dots,n\}$ for which the value

$$\left|\sum_{i\in S}p_i - \frac{1}{2}\sum_{i=1}^n p_i\right| \tag{17}$$

reaches a minimum. Let $S = \{1,2,\dots,k\}$, $1 \le k \le n$, and $p_i = 1$, for $i = 1,2,\dots,n$. Then, according to (17) $k = \lfloor\frac{n}{2}\rfloor$, i.e. $S = \{1,2,\dots,\lfloor\frac{n}{2}\rfloor\}$. Now, for $S = \{1,2,\dots,\lfloor\frac{n}{2}\rfloor\}$, $p_i = 1$, $a_i = b_i = d_i$, $i = 1,2,\dots,n$, $R_1 = R_2 = d_1$ and $r_1 = r_2 = d_n$ according to (16) we have that

$$n\sum_{i=1}^n d_i^2 - \left(\sum_{i=1}^n d_i\right)^2 \le (d_1-d_n)^2\left\lfloor\frac{n}{2}\right\rfloor\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right) \tag{18}$$

i.e.

$$nM_1 - 4m^2 \le (d_1 - d_n)^2 \left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor \right).$$

The above inequality can be rewritten in the form

$$4m^2 \left(\frac{nM_1}{4m^2} - 1 \right) \le n^2 d_n^2 \left(\frac{d_1}{d_n} - 1 \right)^2 \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor \right)$$

wherefrom we obtain the result of the theorem.

Equality in (18) holds if and only if $d_1 = d_2 = \cdots = d_n$, so the equality in (14) holds if and only if G is a regular graph.

$\square$

Remark 3 *Since*

$$\alpha(n) = \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right) = \frac{1}{4} \left(1 - \frac{(-1)^{n+1} + 1}{2n^2} \right) = \begin{cases} \frac{1}{4}, & \text{if } n \text{ is even} \\ \frac{(n-1)(n+1)}{4n^2}, & \text{if } n \text{ is odd} \end{cases},$$

the inequality (14) *is stronger than the right-hand side of inequality* (13) *for odd n.*

Corollary 1 *For each graph G, $G \in \Gamma(G)$, the following inequality is valid*

$$Irr_{CS} \le \frac{n}{2m} \sqrt{\sqrt{\alpha(n)} n d_n Irr_g(G)}.$$

Equality holds if and only if G is a regular graph.

In inequalities (11) and (12) indirect relationships between irregularity measures $Irr_N(G)$ and $Irr_E(G)$ are given. In the following theorem we prove the inequality that establishes direct connection between these irregularity measures.

Theorem 2 *For each graph G, $G \in \Gamma(G)$, the following is valid*

$$\frac{2\sqrt{2}m}{\sqrt{n}} Irr_E(G) \le Irr_N(G) \le 2m Irr_E(G). \tag{19}$$

Equality holds if and only if G is a regular graph.

Proof. For the real positive numbers $p_1, p_2, \ldots, p_n$, $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ with the property

$$\sum_{i=1}^{n} p_i = 1, \qquad 0 < r \le a_i \le R < +\infty, \qquad i = 1, 2, \ldots, n$$

Cerone and Dragomir [9] have proved the following inequality

$$\left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right| \leq \frac{R-r}{2} \sum_{i=1}^{n} p_i \left|b_i - \sum_{i=1}^{n} p_i b_i\right|.$$

For $p_i = \frac{1}{n}$, $a_i = b_i = d_i$, $i = 1, 2, \ldots, n$, $R = d_1$ and $r = d_n$, the above inequality transforms into

$$\frac{1}{n}\sum_{i=1}^{n} d_i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{n} d_i\right)^2 \leq \frac{d_1 - d_n}{2n} \sum_{i=1}^{n} \left|d_i - \frac{2m}{n}\right|,$$

i.e.

$$\frac{1}{n}M_1 - \frac{4m^2}{n^2} \leq \frac{d_1 - d_n}{2n} \sum_{i=1}^{n} \left|d_i - \frac{2m}{n}\right|.$$

From the above inequality we obtain

$$\frac{4m^2}{n^2} Irr_E(G)^2 \leq \frac{d_1 - d_n}{2n} Irr_N(G). \tag{20}$$

Based on the Lagrange's identity (see for example [32]) we have

$$\frac{4m^2}{n^2} Irr_E(G)^2 = \frac{1}{n^2}\left(nM_1 - 4m^2\right) = \frac{1}{n^2}\left(n\sum_{i=1}^{n} d_i^2 - \left(\sum_{i=1}^{n} d_i\right)^2\right) =$$

$$= \frac{1}{n^2} \sum_{1 \leq i < j \leq n} (d_i - d_j)^2 \geq \frac{1}{n^2}\left(\sum_{i=2}^{n-1}\left((d_1 - d_i)^2 + (d_i - d_n)^2\right) + (d_1 - d_n)^2\right).$$

Now according to Jennsen's inequality (see for example [31]) from the above inequality we have that

$$\frac{4m^2}{n^2} Irr_E(G)^2 \geq \frac{1}{n^2}\left(\sum_{i=2}^{n-1} \frac{(d_1 - d_n)^2}{2} + (d_1 - d_n)^2\right) = \frac{1}{2n}(d_1 - d_n)^2.$$

From the above we obtain that the following is valid

$$d_1 - d_n \leq \frac{2\sqrt{2}m}{\sqrt{n}} Irr_E(G). \tag{21}$$

Left side of the inequality (19) is obtained according to (20) and (21).

Right-hand part of inequality (19) is obtained from the Cauchy inequality for $b_i = 1$ and $a_i = \left|d_i - \frac{2m}{n}\right|$, $i = 1, 2, \ldots, n$ (see for example [32])

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i.$$

□

Corollary 2 *For every graph G, $G \in \Gamma(G)$, the following inequality is valid*

$$d_n Irr_g(G) \leq Irr_N(G) \leq nd_n \sqrt{\alpha(n)} Irr_g(G).$$

Equalities hold if and only if G is regular.

Corollary 3 *For every graph G, $G \in \Gamma(G)$, the following inequalities are valid*

$$Irr_N(G) \leq \sqrt{2m(nd_1 - 2m)} \tag{22}$$

and

$$Irr_N(G) \leq \sqrt{\frac{m(n-2)(n(n-1)-2m)}{n-2}}, \qquad n \geq 3. \tag{23}$$

Equality in (22) holds if and only if G is regular, whereas in (23) if G is a complete graph, $G = K_n$.

Proof. The inequality (22) is obtained from the right-hand side of (19) and inequality $M_1 \leq 2md_1$. The inequality (23) is obtained from the right-hand side of (19) and the inequality

$$M_1 \leq m\left(\frac{2m}{n-1} + n - 2\right),$$

proven in [7].

□

Theorem 3 *For every graph G, $G \in \Gamma(G)$, that satisfies Zagreb indices inequality, the following inequalities are valid*

$$Irr_{IS}(G)^3 \geq Irr_E(G)^2 \qquad \textit{and} \qquad Irr_{IZ}(G)^4 \geq Irr_1(G)^3.$$

Equalities hold if and only if G is regular.

Proof. Since the graph G satisfies Zagreb indices inequality, then according to (1) and (2) we have that

$$M_2 \geq \frac{m}{n} M_1 \qquad \text{and} \qquad EM_2 \geq \frac{2m-n}{n} EM_1.$$

From the above inequalities we obtain

$$Irr_{IS}(G)^3 = \frac{n^2 M_2}{4m^3} - 1 \geq \frac{n^2 m M_1}{4m^3 n} - 1 = \frac{nM_1}{4m^2} - 1 = Irr_E(G)^2,$$

and

$$Irr_{IZ}(G)^4 = \frac{n^3 EM_2}{4m(2m-n)^3} - 1 \geq \frac{n^2 EM_1}{4m(2m-n)^2} - 1 = Irr_1(G)^3,$$

which had to be proved.

□

The following theorem gives a relationship between irregularity measures $Irr_E(G)$ and $Irr^{(3)}(G)$.

Theorem 4 *For every graph G, $G \in \Gamma(G)$, the following inequalities are valid*

$$\begin{aligned}&\sqrt[3]{Irr_E(G)^2\,(Irr_E(G)^2+2)} \le Irr^{(3)}(G) \le \\ &\le \sqrt[3]{Irr_E(G)^2\,(Irr_E(G)^2+2)+\frac{n^2 d_n^2 \beta(S)}{4m^2} Irr_g(G)^2},\end{aligned} \tag{24}$$

where

$$\beta(S) = \frac{1}{2m}\sum_{i\in S} d_i \left(1-\frac{1}{2m}\sum_{i\in S} d_i\right),$$

and S is a subset of $I_n = \{1,2,\ldots,n\}$ which minimizes the expression

$$\left|\sum_{i\in S} d_i - m\right|. \tag{25}$$

Equalities hold if and only if G is regular.

Proof. For $p_i = d_i$, $i = 1,2,\ldots,n$ the inequality (17) becomes (25). Let S be a subset of $I_n = \{1,2,\ldots,n\}$ which minimizes the expression (25). Then for $p_i = a_i = b_i = d_i$, $i = 1,2,\ldots,n$, $R_1 = R_2 = d_1$ and $r_1 = r_2 = d_n$, the inequality (16) becomes

$$\sum_{i=1}^{n} d_i \sum_{i=1}^{n} d_i^3 - \left(\sum_{i=1}^{n} d_i^2\right)^2 \le (d_1-d_n)^2 \sum_{i\in S} d_i \left(\sum_{i=1}^{n} d_i - \sum_{i\in S} d_i\right),$$

i.e.

$$2mF_1 - M_1^2 \le 4m^2\beta(S)(d_1-d_n)^2.$$

If we multiply the above inequality by $\frac{n^2}{16m^4}$ it transforms into

$$\frac{n^2}{8m^3}F_1 - \left(\frac{nM_1}{4m^2}\right)^2 \le \frac{n^2\beta(S)d_n^2}{4m^2}\left(\frac{d_1}{d_n}-1\right)^2,$$

wherefrom we obtain right-hand part of (24).

For $\alpha = 3$ the inequality (8) becomes

$$F_1 \ge \frac{M_1^2}{2m}.$$

According to the above we have that the following is valid

$$\frac{n^2F_1}{8m^3} - 1 \geq \left(\frac{nM_1}{4m}\right)^2 - 1,$$

wherefrom we obtain the result of the theorem.

$\square$

Corollary 4 *For every graph G, $G \in \Gamma(G)$, the following inequality is valid*

$$Irr^{(3)}(G) \leq \sqrt[3]{Irr_E(G)^2\,(Irr_E(G)^2+2) + \frac{n^2d_n^2}{16m^2}Irr_g(G)^2}. \tag{26}$$

Equality holds if and only if G is regular.

Proof. According to the arithmetic-geometric mean inequality for real numbers (see for example [32]) we have that for each set S, $S \subseteq I_n$, holds $\beta(S) \leq \frac{1}{4}$. Now the inequality (26) can be obtained from the right-hand part of (24).

$\square$

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