EXTREMAL PROPERTIES OF THE CHROMATIC POLYNOMIALS
OF CONNECTED 3-CHROMATIC GRAPHS

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Abstract. In this paper the greatest \([n/2]\) values of \(P(G;3)\) in the class of connected 3-chromatic graphs \(G\) of order \(n\) are found, where \(P(G;\lambda)\) denotes the chromatic polynomial of \(G\).

1. Preliminary definitions and results

Let \(G\) be a graph of order \(n\) and let \(P(G;\lambda)\) be its chromatic polynomial [1]. A \(k\)-color partition of \(G\) is a partition of the vertex set \(V(G)\) into \(k\) classes where each class is an independent set of vertices. The number of \(k\)-color partitions of \(G\) and the chromatic number of \(G\) will be denoted by \(\text{Col}_k(G)\) and by \(\chi(G)\), respectively. It is well known that \(P(G;\lambda)\) can be expressed in terms of the number of \(k\)-color partitions as follows

\[ P(G;\lambda) = \sum_{k=1}^{n} (\lambda)_k \text{Col}_k(G), \]

where \((\lambda)_k = \lambda(\lambda - 1) \cdots (\lambda - k + 1)\).

It follows that if \(\chi(G) = k\), then \(\text{Col}_k(G) = P(G;\lambda)/k!\). Let \(xy\) be an edge of \(G\). By \(G - xy\) we mean the graph obtained from \(G\) by deleting edge \(xy\). Also \(G/xy\) denotes the graph obtained from \(G\) by identifying vertices \(x\) and \(y\), i.e., (i) by deleting both \(x\) and \(y\) and all the edges incident to them, and (ii) by introducing a new vertex \(z\) and joining \(z\) to both all the neighbors of \(x\) different from \(y\) and all the neighbors of \(y\) different from \(x\) in \(G\).

The following lemma describes some properties of \(P(G;\lambda)\), which we will use later [2].

Lemma 1.1. The following properties hold:

(i) Reduction Formula. Let \(a\) and \(b\) be two adjacent vertices of \(G\). Then \(P(G;\lambda) = P(G - ab;\lambda) - P(G/ab;\lambda)\).

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111
(ii) Let $G$ and $H$ be two graphs that overlap in a complete graph $K_r$ on $r$ vertices. Then the chromatic polynomial of this overlap graph is

$$P(G; \lambda)P(H; \lambda)/P(K_r; \lambda).$$

Let $G$ be a graph and $H$ an induced subgraph of $G$. The graph obtained from $G$ by the contraction of $H$ is the graph $G_1$ derived from $G$ by the following operations: suppress all vertices of $H$ and the edges incident with them, and replace them with a new vertex $w \notin V(G)$ and edges $wx$ such that $wx \in E(G_1)$ if and only if there exists $y \in V(G)$ such that $xy \in E(G)$ and $x \in V(G) - V(H)$.

The cycle with $n$ vertices will be denoted by $C_n$ and $C_{n}^{k}$ will denote the graph consisting of $C_n$ and one more vertex adjacent to only one vertex of $C_n$. The following theorem was proved in [4].

**Theorem 1.2.** The maximum number of 3-color partitions of a connected graph $G$ having $n$ vertices and chromatic number $\chi(G) = 3$ is $(2^{n-1} - 1)/3$ for odd $n$, and $(2^{n-1} - 2)/3$ for even $n$. Moreover, if $n$ is odd, the unique connected graph that achieves the maximum number of 3-color partitions is $C_n$, while if $n$ is even, the unique graph is $C_{n-1}^{1}$.

By $H(n, 2r + 1)$ we denote the class of connected graphs $G$ of order $n$ containing $n$ edges and a unique cycle $C_{2r+1}$, where $3 \leq 2r + 1 \leq n$. It is clear that the graph $G \in H(n, 2r + 1)$ by contracting $C_{2r+1}$ is a tree on $n - 2r$ vertices. By Rényi’s formula [3], the number of labeled graphs in $H(n, 2r + 1)$ is equal to $(n - 1)_{2r}n^{-2r-1}/2$.

Let $D_{n}$ ($n \geq 5$) be the graph consisting of a 4-cycle in which two nonadjacent vertices are connected by a newly added path of length $n - 3$. Note that $\chi(D_{n}) = 3$ for even $n$ and $\chi(D_{n}) = 2$ for odd $n$. If “nonadjacent” is replaced by “adjacent”, the resulting graph is denoted by $F_{n}$. Hence, $F_{n}$ consists of two cycles $C_{4}$ and $C_{n-2}$ having a common edge. Also, $\chi(F_{n}) = 3$ for odd $n$ and $\chi(F_{n}) = 2$ for even $n$.

The following two properties were deduced in [5].

**Lemma 1.3.** For every $n \geq 5$, the following equalities hold: $P(D_{n}; 3) = 2^{n} - 2^{n-2} - (-1)^{n-6}$ and $P(F_{n}; 3) = 2^{n} - 2^{n-2} + (-1)^{n-6}$.

**Theorem 1.4.** (a) If $G$ is a 2-connected graph of order $n$, $n \geq 5$, such that $P(G; 3)$ is maximum in the class $F_{n} \setminus \{C_{n}, K_{2,n-2}, D_{n}\}$, where $F_{n}$ denotes the class of all 2-connected graphs of order $n$, then $G \cong F_{n}$ for odd $n$.

(b) If $G$ is a 2-connected graph of order 6 such that $P(G; 3)$ is maximum in the class $F_{6} \setminus \{C_{6}, K_{2,4}, F_{6}, K_{3,3} - e\}$, then $G \cong K_{3,3}$ or $D_{6}$.

(c) If $G$ is a 2-connected graph of order $n$, $n \geq 8$, such that $P(G; 3)$ is maximum in the class $F_{n} \setminus \{C_{n}, K_{2,n-2}, F_{n}\}$, then $G \cong D_{n}$ for even $n$; for $n = 8$ there exists another extremal graph, $E_{8,3}$.

Note that the graph $E_{8,3}$, described in [5], has $\chi(E_{8,3}) = 2$; also $\chi(K_{2,n-2}) = \chi(K_{3,3} - e) = \chi(K_{3,3}) = 2$. 


Lemma 1.5. Let $G$ be a graph of order $n \geqslant 5$ consisting of two cycles $C_{2r+1}$ and $C_{n-2r}$, having exactly one vertex in common. Then $P(G; 3) < 2^n - 2^{n-2} - 6$.

Proof. By Lemma 1.1(ii) we get

$$P(G; \lambda) = ((\lambda - 1)^{2r+1} - (\lambda - 1)((\lambda - 1)^{n-2r} + (-1)^{n-2r} (\lambda - 2))/\lambda$$

since $P(C_n; \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$. It follows that

$$P(G; 3) = (2^{2r+1} - 2)(2^n - 2^{n-2r} + (-1)^{n-2r} 2)/3$$

$$\leq (2^{2r+1} - 2)(2^n - 2^{n-2r} + 2)/3 = 2(2^n + 2^{2r+1} - 2^{n-2r} - 2)/3.$$ 

Since $n - 2r \geqslant 3$, we shall consider two subcases: Case I. $2r \leq n - 4$, and Case II. $2r = n - 3$.

Case I. If $2r \leq n - 4$ we deduce $2(2^n + 2^{2r+1} - 2^{n-2r} - 2)/3 \leq 2(2^n + 2^{n-3} - 2^4 - 2)/3 = 2^n - 2^{n-2} - 12 < 2^n - 2^{n-2} - 6$.

Case II. In this case $n - 2r = 3$ and $P(G; 3) = (2^n - 2)(2^3 - 2)/3 < 2^n - 2^{n-2} - 6$. 

We define the skeleton $S(G)$ of a connected graph $G$ as follows:

(\alpha) If $G$ has no vertex of degree one, then $S(G) = G$.

(\beta) Otherwise, let $x$ be a vertex of degree one of $G$; then $G$ is replaced by $G - x$. Repeat (\alpha).

For example, $S(T)$ consists of a unique vertex if $T$ is a tree, and $S(G) = C_{2r+1}$ for any graph $G \in H(n, 2r + 1)$.

Lemma 1.6. Let $G$ be a graph of order $n$ such that its skeleton $S(G)$ has order $r$. Then $P(G; \lambda) = P(S(G); \lambda)(\lambda - 1)^{n-r}$.

Proof. One applies Lemma 1.1(ii) since $P(K_2; \lambda) = \lambda(\lambda - 1)$. 

Corollary 1.7. For every $G \in H(n, 2r + 1)$, where $3 \leq 2r + 1 \leq n$, we have

$$P(G; \lambda) = (\lambda - 1)^n - (\lambda - 1)^{n-2r}$$

Lemma 1.8. Let $G$ be a connected graph of order $n$ consisting of two vertex disjoint cycles $C_r$ and $C_s$, joined by a path of length $t$ ($r + s + t = n + 1$). Then

$$P(G; \lambda) = P(H; \lambda)(\lambda - 1)^t$$

where $H$ is the graph of order $r + s - 1$ consisting of cycles $C_r$ and $C_s$ having a unique common vertex.

Proof. This equality is a consequence of Lemma 1.1(ii).

Lemma 1.9. Let $G$ be a graph of order $2r + s + p$ consisting of two cycles—one cycle with $s \geq 3$ vertices and another odd cycle with $2r + 1 \geq 3$ vertices, having in common a path of length $p \geq 1$. Then

$$P(G; 3) < P(H; 3) = 2^{2r+s-p} - 2^{2r+s-p-2},$$

where $H \in H(2r + s - p, 3)$. 

Proof. Suppose that the common path with $p + 1$ vertices of the two cycles of $G$ has extremities $a$ and $b$. It follows that $1 \leq p \leq 2r - 1$ and $p \leq s - 2$. If $p \geq 2$ then vertices $a$ and $b$ are not adjacent and by Lemma 1.1 we deduce

$$P(G; \lambda) = P(G_1; \lambda) + P(G_2; \lambda) =$$

$$= ((\lambda - 1)^{t - p} + (-1)^{t - p}(\lambda - 1))(\lambda - 1)^p + (-1)^p(\lambda - 1)) \times$$

$$\times ((\lambda - 1)^{2r-p+1} + (-1)^{2r-p+1}(\lambda - 1))/\lambda^2 +$$

$$+ ((\lambda - 1)^{t - p+1} + (-1)^{t - p+1}(\lambda - 1))(\lambda - 1)^{p+1} + (-1)^{p+1}(\lambda - 1)) \times$$

$$\times ((\lambda - 1)^{2r-p+2} + (-1)^{2r-p}(\lambda - 1))/(\lambda^2(\lambda - 1)^2),$$

where $G_1$ consists of three cycles with $p$, $s - p$ and $2r - p + 1$ vertices having a common vertex and $G_2$ of three cycles with $p + 1$, $s - p + 1$ and $2r - p + 2$ vertices having a common edge. Hence (1) is equivalent to

$$2^{2r+s-p} > (-1)^t2^{2s-p+4} - 2^{t-p+3} + (-1)^{t+1}2^{p+3} + (-1)^{t-p+1}8. \quad (2)$$

For $s = 3$ we deduce $p = 1$ which contradicts our hypothesis. If $s \geq 4$ we can write $2^{2r+s} - (-1)^t2^{2s-p+4} \geq 2^{2r+s-p} - 2^{s-p+4} = 2^{2r-p+4}(2^{s-p} - 1) \geq 2^{5(2^{s-p} - 1)} = 2^{5s - 5} \geq 2^{5s - 5}$ since $p \leq 2r - 1$. Since $p \leq s - 2$, $2^{t-p+3} + (1)^t2^{p+3} \geq 2^{s-p+3} - 2^{p+3} \geq 2^s - 2^{p+1} \geq 2^s - 2^t$ for $p \leq s - 2$. But this inequality can be deduced from (2) for $p = 1$ and it is also true for $s = 3$. ■

2. Main result

We shall denote by $C_{n,3}$ the class of connected 3-chromatic graphs of order $n$. The following theorem is an extension of Theorem 1.2.

Theorem 2.1. Let $n \geq 5$. Then:

(a) For every $r = \lceil n/2 \rceil - 1$, $r = \lceil n/2 \rceil - 2$, \ldots, 1, if $G$ is a connected 3-chromatic graph of order $n$, such that $P(G; 3)$ is maximum in the class of graphs

$$C_{n,3} \cup \bigcup_{t \geq r+1} H(n, 2s + 1),$$

then $G \in H(n, 2r + 1)$ and $P(G; 3) = 2^n - 2^{n-2r}$.

(b) If $P(G; 3)$ is maximum in the class of graphs

$$C_{n,3} \cup \bigcup_{t \geq 1} H(n, 2s + 1),$$

then $G \cong F_n$, for odd $n$, $G \cong D_n$, for even $n$ and in this case $P(G; 3) = 2^n - 2^{n-2} - 6.$
Proof. (a) Let $G \in \mathcal{C}_{n,3}$. It follows that $G$ contains an odd cycle $C_{2r+1}$. If for every edge $e \in E(G) \setminus E(C_{2r+1})$ the graph $G - e$ is not connected then $G \in H(n, 2r + 1)$. Otherwise, by Lemma 1.1(ii) we have

$$P(G - e; 3) = P(G; 3) + P(G/e; 3).$$

(3)

But $\chi(G/e) = 3$ since $G/e$ contains an odd cycle even if $e$ is a chord of $C_{2r+1}$. It follows that $P(G/e; 3) > 0$ and (3) implies that $P(G - e; 3) > P(G; 3)$. By applying several times this operation of deleting edges not belonging to $C_{2r+1}$ without disconnecting the resulting graph, one obtains a graph $H \in H(n, 2r + 1)$ such that $P(H; 3) > P(G; 3)$. By Corollary 1.7 if $3 \leq 2j + 1 < 2i + 1 \leq n$ then $G_1 \in H(n, 2i + 1)$ and $G_2 \in H(n, 2j + 1)$ imply

$$P(G_1; 3) = 2^n - 2^{n-2i} > 2^n - 2^{n-2j} = P(G_2; 3)$$

and (a) is proved for $r = [n/2] - 1$ (this is the property expressed by Theorem 1.2).

Let $G \in \bigcup_{r \geq 2} H(n, 2s + 1)$ and $a, b$ be two nonadjacent vertices of $G$. We shall prove that if $e = ab$ then

$$P(G + e; 3) < 2^n - 2^{n-2} = P(H; 3),$$

(4)

where $H \in H(n, 3)$.

It is clear that the skeleton $S(G + e)$ consists of: I. Two vertex disjoint cycles joined by a path of length $t \geq 1$; II. Two cycles having exactly one common vertex; III. Two cycles having in common a path of length $p \geq 1$. In all cases at least one cycle is odd. Suppose that $|S(G + e)| = m$.

Case I. In this case by Lemmas 1.6 and 1.8 one deduces

$$P(G + e; \lambda) = P(S(G + e); \lambda)(\lambda - 1)^{n - m} = P(H; \lambda)(\lambda - 1)^{n - m + t},$$

where $H$ has order $m - t$ and consists of two cycles (one is odd) having one vertex in common. By Lemma 1.5 we get

$$P(G + e; 3) = P(H; 3) 2^{m - m + t} < (2m - 2^{m-2}) 2^{n - m} = 2^n - 2^{n-2}$$

by Lemmas 1.5, 1.6 and 1.9. Let now $r$ be such that $1 \leq r \leq [n/2] - 2$ and $G$ be such that $P(G; 3)$ is maximum in the class $\mathcal{C}_{n,3} \setminus \bigcup_{r \geq 1} H(n, 2s + 1)$. If $G \in \bigcup_{r \leq n-1} H(n, 2s + 1)$ it follows that $G \in H(n, 2r + 1)$ and the property is proved. Otherwise, there exists an edge $e \in E(G)$ such that $G - e \in \mathcal{C}_{n,3}$. Since $P(G; 3)$ is maximum in the class $\mathcal{C}_{n,3} \setminus \bigcup_{r \geq 1} H(n, 2s + 1)$, it follows that $G - e \in \bigcup_{r \geq 1} H(n, 2s + 1)$, i.e., there exists a graph $H$ in $\bigcup_{r \geq 1} H(n, 2s + 1)$ such that $G \geq H + e$. By (4) this leads to a contradiction.

(b) Let $G \in \mathcal{C}_{n,3} \setminus \bigcup_{r \geq 1} H(n, 2s + 1)$ be such that $P(g; 3)$ is maximum. We have seen that the greatest values of $P(G; 3)$ in the class $\mathcal{C}_{n,3}$ are obtained for graphs in $\bigcup_{r \geq 1} H(n, 2s + 1)$, and for graphs not belonging to this class the greatest values of $P(G; 3)$ are obtained for graphs of the form $H + e$, where $H \in \bigcup_{r \geq 1} H(n, 2s + 1)$ and
$e \notin E(H)$. It follows that $G \cong H + e$, where $H \in \bigcup_{t \geq 1} H(n, 2s + 1)$ and $e \notin E(H)$. Suppose that $|S(H + e)| = m$. As for the case (a) we may distinguish cases I–III concerning the structure of $S(H + e)$. Using the same notation, in the case I one obtains $P(H + e; 3) < (2^{m-t} - 2^{m-2} - 6)2^{n-m+t} < 2^n - 2^{n-2} - 6$ since $n - m + t \geq 1$. In the case II by Lemma 1.5, $P(H + e; 3) < (2^m - 2^{m-2} - 6)2^{n-m} \leq 2^n - 2^{n-2} - 6$.

In the case III the skeleton $S(H + e)$ is 2-connected and by Lemmas 1.3, 1.6 and Theorem 1.4 one deduces

$$P(H + e; 3) \leq (2^m - 2^{m-2} - 6)2^{n-m} \leq 2^n - 2^{n-2} - 6$$

and equality holds if and only if $m = n$ and $G \cong F_n$ for odd $n$ and $G \cong D_n$ for even $n$. ■

Note that $\text{Col}_3(F_n)$ for odd $n$, resp. $\text{Col}_3(D_n)$ for even $n$ is equal to $\text{Col}_3(H) - 1 = 2^{n-3} - 1$ for any $H \in H(n, 3)$.

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