BOUNDS ON ROMAN DOMINATION NUMBERS OF GRAPHS

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Abstract. Roman dominating function of a graph $G$ is a labeling function $f: V(G) \rightarrow \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The Roman domination number $\gamma_R(G)$ of $G$ is the minimum of $\sum_{v \in V(G)} f(v)$ over such functions. In this paper, we find lower and upper bounds for Roman domination numbers in terms of the diameter and the girth of $G$.

1. Introduction

For $G$, a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly $V$ and $E$), the open neighborhood $N(v)$ of the vertex $v$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. Similarly, the open neighborhood of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and its closed neighborhood is $N[S] = N(S) \cup S$. The minimum and maximum vertex degrees in $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A subset $S$ of vertices of $G$ is a dominating set if $N[S] = V$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. A subset $S$ of vertices of $G$ is a 2-packing if for each pair of vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$.

A Roman dominating function (RDF) on a graph $G = (V, E)$ is defined in [13], [15] as a function $f: V \rightarrow \{0, 1, 2\}$ satisfying the condition that a vertex $v$ with $f(v) = 0$ is adjacent to at least one vertex $u$ with $f(u) = 2$. The weight of a RDF is defined as $w(f) = \sum_{v \in V} f(v)$. The Roman domination number of a graph $G$, denoted by $\gamma_R(G)$, equals the minimum weight of a RDF on $G$. A $\gamma_R(G)$-function is a Roman dominating function of $G$ with weight $\gamma_R(G)$. Observe that a Roman dominating function $f: V \rightarrow \{0, 1, 2\}$ can be presented by an ordered partition $(V_0, V_1, V_2)$ of $V$, where $V_i = \{v \in V \mid f(v) = i\}$.

Cockayne et. al [3] initiated the study of Roman domination, suggested originally in a Scientific American article by Ian Stewart [15]. Since $V_1 \cup V_2$ is a dominating set when $f$ is a RDF, and since placing weight 2 at the vertices of a dominating set yields a RDF, they observed that

$$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G).$$

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In a sense, $2\gamma(G) - \gamma_R(G)$ measures “inefficiency” of domination, since the vertices with weight 1 in a RDF serve only to dominate themselves. The authors [3] investigated graph theoretic properties of RDFs and characterized $\gamma_R(G)$ for specific graphs. They found out the graphs $G$ with $\gamma_R(G) = \gamma(G) + k$ when $k \leq 2$; and then for larger $k$ by Xing et al. [16]. They also characterized the graphs $G$ with property $\gamma_R(G) = 2\gamma(G)$ in terms of 2-packings, referring them to as Roman graphs. Henning [9] characterized Roman trees, while Song and Wang [14] identified the trees $T$ with $\gamma_R(T) = \gamma(T) + 3$. Computational complexity of $\gamma_R(G)$ is considered in [4]. In [12], linear-time algorithms are given for $\gamma_R(G)$ on interval graphs and on cographs, along with a polynomial-time algorithm for AT-free graphs. Chambers et al. [2] proved that $\gamma_R(G) \leq 4n^5$ when $G$ is a connected graph of order $n \geq 3$, and determined when equality holds. They have also obtained sharp upper and lower bounds for $\gamma_R(G) + \gamma_R(\overline{G})$ and $\gamma_R(G)\gamma_R(\overline{G})$, where $\overline{G}$ denotes the complement of $G$. Favaron et al. [7] proved that $\gamma_R(G) + \frac{\gamma(G)}{2} \leq n$ for any connected graph $G$ of order $n \geq 3$. Other related domination models are studied in [1, 5, 6, 10, 11].

The purpose of this paper is to establish sharp lower and upper bounds for Roman domination numbers in terms of the diameter and the girth of $G$.

Cockayne et al. in [3] proved that:

**Theorem A.** For a graph $G$ of order $n$,

$$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G),$$

with equality in lower bound if and only if $G = K_n$.

**Theorem B.** For paths $P_n$ and cycles $C_n$,

$$\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil.$$

**Theorem C.** Let $G = K_{m_1, \ldots, m_n}$ be the complete $n$-partite graph with $m_1 \leq m_2 \leq \ldots \leq m_n$. If $m_1 = 2$, then $\gamma_R(G) = 3$.

**Theorem D.** Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R$-function for a simple graph $G$, such that $|V_1^f|$ is minimum. Then $V_1^f$ is a 2-packing.

2. Bounds in terms of the diameter

In this section sharp lower and upper bounds for $\gamma_R(G)$ in terms of $\text{diam}(G)$ are presented. Recall that the eccentricity of vertex $v$ is $\text{ecc}(v) = \max\{d(v, w) : w \in V\}$ and the diameter of $G$ is $\text{diam}(G) = \max\{\text{ecc}(v) : v \in V\}$. Throughout this section we assume that $G$ is a nontrivial graph of order $n \geq 2$.

**Theorem 1.** If a graph $G$ has diameter two, then $\gamma_R(G) \leq 2\delta$. Furthermore, this bound is sharp for infinite family of graphs.
Proof. Since $G$ has diameter two, $N(u)$ dominates $V(G)$ for all vertex $u \in V(G)$. Now, let $u \in V(G)$ and $\deg(u) = \delta$. Define $f : V(G) \rightarrow \{0,1,2\}$ by $f(x) = 2$ for $x \in N(u)$ and $f(x) = 0$ otherwise. Obviously $f$ is a RDF of $G$. Thus $\gamma_R(G) \leq 2\delta$.

To prove sharpness, let $G$ be obtained from Cartesian product $P_2 \square K_m (m \geq 3)$ by adding a new vertex $x$ and jointing it to exactly one vertex at each copy of $K_m$. Obviously, $\text{diam}(G) = 2$ and $\gamma_R(G) = 4 = 2\delta$. This completes the proof. $\blacksquare$

Next theorem presents a lower bound for Roman domination numbers in terms of the diameter.

**Theorem 2.** For a connected graph $G$,

$$\gamma_R(G) \geq \left\lceil \frac{\text{diam}(G) + 2}{2} \right\rceil.$$  

Furthermore, this bound is sharp for $P_3$ and $P_4$.

**Proof.** The statement is obviously true for $K_2$. Let $G$ be a connected graph of order $n \geq 3$ and $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$-function. Suppose that $P = v_1v_2\ldots v_{\text{diam}(G)+1}$ is a diametral path in $G$. This diametral path includes at most two edges from the induced subgraph $G[N[v]]$ for each $v \in V_1^f \cup V_2^f$. Let $E' = \{v_iv_{i+1} | 1 \leq i \leq \text{diam}(G)\} \cap \bigcup_{v \in V_1^f \cup V_2^f} E(G[N[v]])$. Then the diametral path contains at most $|V_2^f| - 1$ edges not in $E'$, joining the neighborhoods of the vertices of $V_2^f$. Since $G$ is a connected graph of order at least 3, $V_2^f \neq \emptyset$. Hence,

$$\text{diam}(G) \leq 2|V_2^f| + 2|V_1^f| + (|V_2^f| - 1) \leq 2\gamma_R(G) - 2,$$

and the result follows. $\blacksquare$

In the following theorem, an upper bound is presented for Roman domination numbers.

**Theorem 3.** For any connected graph $G$ on $n$ vertices,

$$\gamma_R(G) \leq n - \left\lceil \frac{1 + \text{diam}(G)}{3} \right\rceil.$$  

Furthermore, this bound is sharp.

**Proof.** Let $P = v_1v_2\ldots v_{\text{diam}(G)+1}$ be a diametral path in $G$. Moreover, let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(P)$-function. By Theorem B, the weight of $f$ is $\lceil 2\text{diam}(G)+2 \rceil$. Define $g: V(G) \rightarrow \{0,1,2\}$ by $g(x) = f(x)$ for $x \in V(P)$ and $g(x) = 1$ for $x \in V(G) \setminus V(P)$. Obviously $g$ is a RDF for $G$. Hence,

$$\gamma_R(G) \leq w(f) + (n - \text{diam}(G) - 1) = n - \left\lceil \frac{1 + \text{diam}(G)}{3} \right\rceil.$$  

To prove sharpness, let $G$ be obtained from a path $P = v_1v_2\ldots v_{3k} (k \geq 2)$ by adding a pendant edge $v_{3k}u$. Obviously, $G$ achieves the bound and the proof is complete. $\blacksquare$
For a connected graph $G$ with $\delta \geq 3$, the bound in Theorem 3 can be improved as follows.

**Theorem 4.** For any connected graph $G$ of order $n$ with $\delta \geq 3$, 

$$\gamma_R(G) \leq n - \left\lceil \frac{1 + \text{diam} (G)}{3} \right\rceil - (\delta - 2) \left\lceil \frac{\text{diam} (G) + 2}{3} \right\rceil.$$ 

**Proof.** Let $P = v_1v_2\ldots v_{\text{diam} (G)+1}$ be a diametral path in $G$ and $f = (V^f_0, V^f_1, V^f_2)$ be a $\gamma_R (P)$-function for which $|V^f_i|$ is minimized and $V^f_2$ is a 2-packing. Obviously, $|V^f_2| = \lfloor \frac{\text{diam} (G)+2}{3} \rfloor$. Let $V^f_2 = \{u_1, \ldots, u_k\}$ where $k = \lfloor \frac{\text{diam} (G)+2}{3} \rfloor$. Since $P$ is a diametral path, each vertex of $V^f_2$ has at least $\delta - 2$ neighbors in $V(G) \setminus V(P)$ and $N(u_i) \cap N(u_j) = \emptyset$ if $u_i \neq u_j$. Define $g : V(G) \rightarrow \{0, 1, 2\}$ by $g(x) = f(x)$ for $x \in V(P)$, $g(x) = 0$ for $x \in \bigcup_{i=1}^{k} N(u_i) \cap (V(G) \setminus V(P))$ and $g(x) = 1$ when $x \in V(G) \setminus (V(P) \cup \bigcup_{i=1}^{k} N(u_i)))$. Obviously $g$ is a RDF for $G$ and so 

$$\gamma_R (G) \leq w(g) = w(f) + n - \text{diam} (G) - 1 - (\delta - 2) \left\lceil \frac{\text{diam} (G) + 2}{3} \right\rceil.$$ 

Now the result follows from $w(f) = \lfloor \frac{2\text{diam} (G)+2}{3} \rfloor$. ■

The next theorem speaks of an interesting relationship between the diameter of $G$ and the Roman domination number of $\overline{G}$, the complement of $G$.

**Theorem 5.** For a connected graph $G$ with $\text{diam} (G) \geq 3$, $\gamma_R (\overline{G}) \leq 4$.

**Proof.** Let $P = v_1v_2\ldots v_m$ be a diametral path in $G$ where $m \geq 4$. Let $S = \{v_1, v_m\}$. Since $\text{diam} (G) \geq 3$, each vertex $v \in V(G) \setminus S$ can be adjacent to at most one vertex of $S$ in $G$. Consequently, $S$ is a dominating set for $\overline{G}$. By (1), $\gamma_R (\overline{G}) \leq 2\gamma(\overline{G}) \leq 4$ and the proof is complete. ■

## 3. Bounds in terms of the girth

In this section we present bounds on Roman domination numbers of a graph $G$ containing cycles, in terms of its girth. Recall that the girth of $G$ (denoted by $g(G)$) is the length of a smallest cycle in $G$. Throughout this section, we assume that $G$ is a nontrivial graph of order $n \geq 3$ and contains a cycle.

The following result is very crucial for this section.

**Lemma 6.** For a graph $G$ of order $n$ with $g(G) \geq 3$ we have $\gamma_R (G) \geq \lfloor \frac{2g(G)}{3} \rfloor$.

**Proof.** First note that if $G$ is an $n$-cycle then $\gamma_R (G) = \lfloor \frac{2n}{3} \rfloor$ by Theorem B. Now, let $C$ be a cycle of length $g(G)$ in $G$. If $g(G) = 3$ or 4, then we need at least 1 or 2 vertices, respectively, to dominate the vertices of $C$ and the statement follows by Theorem A. Let $g(G) \geq 5$. Then a vertex not in $V(C)$, can be adjacent to at most one vertex of $C$ for otherwise we obtain a cycle of length less than $g(G)$ which is a contradiction. Now the result follows by Theorem A. ■
Theorem 7. If $g(G) = 4$, then $\gamma_R(G) \geq 3$. Equality holds if and only if $G$ is a bipartite graph with partite sets $X$ and $Y$ with $|X| = 2$, where $X$ has one vertex of degree $n - 2$ and the other of degree at least two.

Proof. Let $g(G) = 4$. Then $\gamma_R(G) \geq 3$ by Lemma 6. If $G$ is a bipartite graph satisfying the conditions, then obviously $g(G) = 4$ and $\gamma_R(G) = 3$ by Theorem C. Now let $g(G) = 4$ and $\gamma_R(G) = 3$ and $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$-function. Obviously, $|V_1^f| = |V_2^f| = 1$. Suppose that $V_1^f = \{u\}$ and $V_2^f = \{v\}$. Since $\gamma_R(G) = 3$, $\{u, v\}$ is an independent set and $v$ is adjacent to at most one vertex of $V(G) \setminus \{u, v\}$. Let $X = \{u, v\}$ and $Y = V(G) \setminus X$. Since $g(G) = 4$, $Y$ is an independent set. Henceforth, $u$ and $v$ are contained in each 4-cycle of $G$. It follows that $u$ has degree at least two. This completes the proof. ■

Theorem 8. Let $G$ be a simple connected graph of order $n$, $\delta(G) \geq 2$ and $g(G) \geq 5$. Then $\gamma_R(G) \leq n - \lceil \frac{2g(G)}{3} \rceil$. Furthermore, the bound is sharp for cycles $C_n$ with $n \geq 5$.

Proof. Let $G$ be such a graph. Assume $C$ is a cycle of $G$ with $g(G)$ edges. If $G = C$, then the statement is valid by Theorem B. Now let $G'$ be obtained from $G$ by removing the vertices of $V(C)$. Since $g(G) \geq 5$, each vertex of $G'$ can be adjacent to at most one vertex of $C$ which implies $\delta(G') \geq 1$. Thus, $\gamma_R(G') \leq n - g(G)$. Let $f$ and $g$ be a $\gamma_R(G')$-function and $\gamma_R(C)$-function, respectively. Define $h : V(G) \to \{0, 1, 2\}$ by $h(v) = f(v)$ for $v \in V(G')$ and $h(v) = g(v)$ for $v \in V(C)$. Obviously, $h$ is a RDF of $G$ and the result follows. ■

Theorem 9. For a simple connected graph $G$ of order $n$, if $g(G) \geq 5$, then $\gamma_R(G) \geq 2\delta$. The bound is sharp for $C_5$ and $C_6$.

Proof. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$-function such that $|V_1^f|$ is minimum and let $C$ be a cycle with $g(G)$ edges. If $n = 5$, then $G$ is a 5-cycle and $\gamma_R(G) = 4 = 2\delta$. For $n \geq 6$, if $\delta \leq 2$, then $\gamma_R(G) \geq \lceil \frac{2g(G)}{3} \rceil \geq 2\delta$ by Lemma 6. Now, let $\delta \geq 3$. First suppose that $V_1^f = \emptyset$. Assume $v \in V_0^f$ and $N(v) = \{v_1, \ldots, v_k\}$ for some $k \geq \delta$. Without loss of generality, one may suppose $v_1, \ldots, v_r \in V_0^f$ and $v_{r+1}, \ldots, v_k \in V_0^f$ and for $j = r + 1, \ldots, k$, $v_jv_j' \in E(G)$ where $v_j' \in V_2^f$ and $k > r$. Since $g(G) \geq 5$, the vertices of $v_1, \ldots, v_r, v_{r+1}, \ldots, v_k$ are distinct. Consequently, $|V_2^f| \geq 2k$ which implies $\gamma_R(G) \geq 2k \geq 2\delta$. For the case $V_1^f \neq \emptyset$, by definition of $f$, $V_1^f$ is an independent set. Suppose that $u \in V_1^f$ and $N(u) = \{u_1, \ldots, u_k\}$ for some $k \geq \delta$. Obviously, $N(u) \subseteq V_0^f$. For each $j = 1, \ldots, k$, one may consider $u_jv_j \in E(G)$ where $v_j \in V_2^f$. Since $g(G) \geq 5$, the vertices $v_1, \ldots, v_k$ are distinct. Hence, $\gamma_R(G) = 2|V_2^f| + |V_1^f| \geq 2\delta + 1$ and the proof is complete. ■

Theorem 10. For a simple connected graph $G$ with $\delta \geq 2$ and $g(G) \geq 6$, $\gamma_R(G) \geq 4(\delta - 1)$. This bound is sharp for $C_6$.

Proof. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$-function such that $|V_1^f|$ is minimum. Therefore, $V_1^f$ is an independent set and $N(w_1) \cap N(w_2) = \emptyset$ if $w_1 \neq w_2$ for
$w_1, w_2 \in V_1^f$. For $V_1^f \neq \emptyset$ and $u \in V_1^f$, $N(u) = \{u_1, \ldots, u_{\text{deg}(u)}\} \subseteq V_0^f$. Suppose that $N(u_1) = \{w_1, \ldots, w_r\}$ where $u = w_1$. Since $g(G) \geq 6$, $N(u) \cap N(u_1) = \emptyset$ and $N(u_i) \cap N(w_i) = \emptyset$ for each $i, j$. In this way, each vertex of $V_2^f$ can be adjacent to at most one vertex in $(N(u) \cup N(u_1)) \cap V_0^f$. This implies that $|V_2^f| \geq 2(\delta - 1)$ which follows the statement.

For $V_1^f = \emptyset$, $|V_1^f| \geq 2$ holds clearly. If $G[V_1^f]$ has an edge $uv$, analogous reasoning proves the statement. Let $V_0^f$ be an independent set in $G$ with $|V_0^f| \geq 2$ and $u, v \in V_0^f$. Since $g(G) \geq 6$ and $V_0^f$ is an independent set, $|N(u) \cap N(v)| \leq 1$ and $N(u) \cup N(v) \subseteq V_2^f$. This implies that $|V_2^f| \geq 2\delta - 1$ and the result follows. ■

**Theorem 11.** For a simple connected graph $G$ with $\delta \geq 2$ and $g(G) \geq 7$, $\gamma_R(G) \geq 2\Delta$. This bound is sharp for $g(G) = 7$.

**Proof.** Let $f = \langle V_0^f, V_1^f, V_2^f \rangle$ be a $\gamma_R(G)$-function such that $|V_1^f|$ is minimum and let $C$ be a cycle of $G$ with $g(G)$ edges. Suppose $v \in V(G)$ is a vertex with degree $\Delta$. By Theorem D, $V_1^f$ is an independent set of $G$ and $N(w_1) \cap N(w_2) = \emptyset$ if $w_1 \neq w_2$ for $w_1, w_2 \in V_1^f$. Consider $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$. For $v \notin V_2^f$, similar to the proof of Theorem 9, the statement follows. For $v \in V_2^f$, let $A = N[v] \cap V_2^f$ and $B = N(v) \cap V_1^f$. For $u \in B$, three cases may occur.

Case 1. $u$ has a neighbor in $V_2^f - \{v\}$. In this case, consider $x_u \in (V_2^f - \{v\}) \cap N(u)$.

Case 2. $u$ has no neighbor in $V_2^f - \{v\}$ and $u$ has some neighbor in $V_0^f$. For $y_u \in N(u) \cap V_0^f$, since $g(G) \geq 7$, $y_u \notin B$. In this case, let $x_u \in V_2^f \cap N(y_u)$.

Case 3. $u$ has no neighbor in $V_0^f \cup (V_2^f - \{v\})$ and $u$ has some neighbor in $V_1^f$. For $z_u \in V_1^f \cap N(u)$, since $G$ is connected and $\delta \geq 2$, $z_u$ has a neighbor in $V_0^f - \{u\}$, say $y_u$. On the other hand, $y_u$ has a neighbor in $V_2^f$, say $x_u$.

Since $g(G) \geq 7$, it is straightforward to verify that $A \cap \{x_u \mid u \in B\} = \emptyset$ and $x_u \neq x_u'$ when $u \neq u'$ and $u, u' \in B$. Thus, $|V_2^f| \geq \Delta$ that implies the statement.

The bound is sharp for the graph $G = (V, E)$, where $V = \{v, u, w, v_1, u_i, w_i \mid 1 \leq i \leq m\}$ and $E = \{vu, uw, w_1w_2, vv_1, v_iu_i, u_iw_i \mid 1 \leq i \leq m\}$ for $m \geq 2$ when $g(G) = 7$. ■

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