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ON CERTAIN MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY USING A DIFFERENTIAL OPERATOR

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Abstract. In this paper, we introduce the subclass $S_j(n,p,q,\alpha)$ of analytic and p-valent functions with negative coefficients defined by new operator D_p^n. In this paper we give some properties of functions in the class $S_j(n,p,q,\alpha)$ and obtain numerous sharp results including (for example) coefficient estimates, distortion theorem, radii of close-to-convexity, starlikeness and convexity and modified Hadamard products of functions belonging to the class $S_j(n,p,q,\alpha)$. Finally, several applications involving an integral operator and certain fractional calculus operators are also considered.

1. Introduction

Let $T(j,p)$ denote the class of functions of the form

$$f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k \quad (a_k \geq 0;\ p,j \in N = \{1,2,\dots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in T(j,p)$ is said to be p-valently starlike of order α if it satisfies the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U;\ 0 \leq \alpha < p;\ p \in N). \tag{1.2}$$

We denote by $T_j^*(p,\alpha)$ the class of all p-valently starlike functions of order α. Also a function $f(z) \in T(j,p)$ is said to be p-valently convex of order α if it satisfies the inequality

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U;\ 0 \leq \alpha < p;\ p \in N). \tag{1.3}$$

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We denote by $C_j(p,\alpha)$ the class of all p-valently convex functions of order α. We note that (see for example Duren [5] and Goodman [6])

$$f(z) \in C_j(p,\alpha) \Longleftrightarrow \frac{zf'(z)}{p} \in T_j^*(p,\alpha) \quad (0 \le \alpha < p;\ p \in N). \tag{1.4}$$

The classes $T_j^*(p,\alpha)$ and $C_j(p,\alpha)$ were studied by Owa [12].

For each $f(z) \in T(j,p)$, we have (see [3])

$$f^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=j+p}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q} \quad (q \in N_0 = N \cup \{0\};\ p > q). \tag{1.5}$$

For a function $f(z)$ in $T(j,p)$, we define

$$\begin{aligned} D_p^0 f^{(q)}(z) &= f^{(q)}(z), \\ D_p^1 f^{(q)}(z) &= Df^{(q)}(z) = \frac{z}{(p-q)} (f^{(q)}(z))' = \frac{z}{(p-q)} f^{(1+q)}(z) \\ &= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=j+p}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k-q}{p-q}\right) a_k z^{k-q}, \end{aligned} \tag{1.6}$$

$$\begin{aligned} D_p^2 f^{(q)}(z) &= D(D_p^1 f^{(q)}(z)) \\ &= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=j+p}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k-q}{p-q}\right)^2 a_k z^{k-q}, \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} D_p^n f^{(q)}(z) &= D(D_p^{n-1} f^{(q)}(z)) \quad (n \in N) \\ &= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=j+p}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k-q}{p-q}\right)^n a_k z^{k-q} \\ &\qquad (p, j \in N;\ q \in N_0;\ p > q). \end{aligned} \tag{1.8}$$

We note that, by taking $q = 0$ and $p = 1$, the differential operator $D_1^n = D^n$ was introduced by Salagean [13].

With the help of the differential operator D_p^n, we say that a function $f(z)$ belonging to $T(j,p)$ is in the class $S_j(n,p,q,\alpha)$ if and only if

$$\operatorname{Re}\left\{ \frac{z(D_p^n f^{(q)}(z))'}{D_p^n f^{(q)}(z)} \right\} > \alpha \quad (p \in N;\ q, n \in N_0) \tag{1.9}$$

for some $\alpha\,(0 \le \alpha < p-q, p > q)$ and for all $z \in U$.

We note that, by specializing the parameters j, p, n, q and α, we obtain the following subclasses studied by various authors:

(i) $S_j(0,p,q,\alpha) = S_j(p,q,\alpha)$ and $S_j(1,p,q,\alpha) = C_j(p,q,\alpha)$ (Chen et al. [3]);

(ii) $S_j(n,1,0,\alpha) = P(j,\alpha,n)$ $(j \in N;\ n \in N_0;\ 0 \le \alpha < 1)$ (Aouf and Srivastava [1]);

(iii) $S_1(n,1,0,\alpha) = T(n,\alpha)$ $(n \in N_0;\ 0 \le \alpha < 1)$ (Hur and Oh [7]);

(iv) $S_j(0,p,0,\alpha) = \begin{cases} T_j^*(p,\alpha) & \text{(Owa [12])}, \\ T_\alpha(p,j) & \text{(Yamakawa [20])} \end{cases} \quad (p, j \in N;\ 0 \le \alpha < p).$

(v) $S_j(1,p,0,\alpha) = \begin{cases} C_j(p,\alpha) & \text{(Owa [12])}, \\ CT_\alpha(p,j) & \text{(Yamakawa [20])} \end{cases} \quad (p, j \in N;\ 0 \le \alpha < p).$

(vi) $S_1(0,p,0,\alpha) = T^*(p,\alpha)$ and $S_1(1,p,0,\alpha) = C(p,\alpha)$ $(p \in N;\ 0 \le \alpha < p)$ (Owa [11] and Salagean et al. [14]);

(vii) $S_j(0,1,0,\alpha) = T_\alpha(j)$ and $S_j(1,1,0,\alpha) = C_\alpha(j)$ $(j \in N;\ 0 \le \alpha < 1)$ (Srivastava et al. [19]);

(viii) $S_j(n,p,0,\alpha) = S_j(n,p,\alpha)$ $(p, j \in N;\ n \in N_0;\ 0 \le \alpha < p)$, where $S_j(n,p,\alpha)$ represents the class of functions $f(z) \in T(j,p)$ satisfying the inequality

$$\mathrm{Re}\left\{ \frac{z\left(D_p^n f(z)\right)'}{D_p^n f(z)} \right\} > \alpha \qquad (z \in U). \tag{1.10}$$

In our present paper, we shall make use of the familiar integral operator $J_{c,p}$ defined by (cf. [2] , [8] and [9]; see also [18])

$$(J_{c,p}f)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{1.11}$$

$(f \in T(j,p);\ c > -p;\ p \in N)$ as well as the fractional calculus operator D_z^μ for which it is well known that (see for details [10] and [16]; see also Section 5 below)

$$D_z^\mu\{z^\rho\} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-\mu)} z^{\rho-\mu} \quad (\rho > -1;\ \mu \in R) \tag{1.12}$$

in terms of Gamma functions.

2. Coefficient estimates

Theorem 1. *Let the function $f(z)$ be defined by* (1.1). *Then $f(z) \in S_j(n,p,q,\alpha)$ if and only if*

$$\sum_{k=j+p}^{\infty} \left(\frac{k-q}{p-q} \right)^n (k-q-\alpha)\delta(k,q) a_k \le (p-q-\alpha)\delta(p,q) \tag{2.1}$$

$(0 \le \alpha < p-q;\ p, j \in N;\ q, n \in N_0;\ p > q)$ *where*

$$\delta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} p(p-1)\cdot \ldots \cdot (p-q+1), & q \ne 0, \\ 1, & q = 0. \end{cases} \tag{2.2}$$

Proof. Assume that inequality (2.1) holds true. Then we find that

$$\left|\frac{z\left(D_p^n f^{(q)}(z)\right)'}{D_p^n f^{(q)}(z)} - (p-q)\right| \leq \frac{\sum\limits_{k=j+p}^{\infty}(k-p)(\frac{k-q}{p-q})^n \delta(k,q) a_k \left|z\right|^{k-p}}{\delta(p,q) - \sum\limits_{k=j+p}^{\infty}(\frac{k-q}{p-q})^n \delta(k,q) a_k \left|z\right|^{k-p}}$$

$$\leq \frac{\sum\limits_{k=j+p}^{\infty}(k-p)(\frac{k-q}{p-q})^n \delta(k,q) a_k}{\delta(p,q) - \sum\limits_{k=j+p}^{\infty}(\frac{k-q}{p-q})^n \delta(k,q) a_k} \leq p-q-\alpha.$$

This shows that the values of the function

$$\Phi(z) = \frac{z\left(D_p^n f^{(q)}(z)\right)'}{D_p^n f^{(q)}(z)} \tag{2.3}$$

lie in a circle which is centered at $w = (p-q)$ and whose radius is $(p-q-\alpha)$. Hence $f(z)$ satisfies the condition (1.9).

Conversely, assume that the function $f(z)$ is in the class $S_j(n,p,q,\alpha)$. Then we have

$$\operatorname{Re}\left\{\frac{z\left(D_p^n f^{(q)}(z)\right)'}{D_p^n f^{(q)}(z)}\right\} =$$

$$\operatorname{Re}\left\{\frac{(p-q)\delta(p,q) - \sum\limits_{k=j+p}^{\infty}(k-q)\left(\frac{k-q}{p-q}\right)^n \delta(k,q) a_k z^{k-p}}{\delta(p,q) - \sum\limits_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n \delta(k,q) a_k z^{k-p}}\right\} > \alpha, \tag{2.4}$$

for some α $(0 \leq \alpha < p-q)$, $p, j \in N$, $q, n \in N_0$, $p > q$ and $z \in U$. Choose values of z on the real axis so that $\Phi(z)$ given by (2.3) is real. Upon clearing the denominator in (2.4) and letting $z \to 1^-$ through real values, we can see that

$$(p-q)\delta(p,q) - \sum_{k=j+p}^{\infty}(k-q)\left(\frac{k-q}{p-q}\right)^n \delta(k,q) a_k \geq$$

$$\geq \alpha\left\{\delta(p,q) - \sum_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n \delta(k,q) a_k\right\}. \tag{2.5}$$

Thus we have the inequality (2.1). ∎

COROLLARY 1. *Let the function $f(z)$ defined by* (1.1) *be in the class $S_j(n,p,q,\alpha)$. Then*

$$a_k \leq \frac{(p-q-\alpha)\delta(p,q)}{\left(\frac{k-q}{p-q}\right)^n (k-q-\alpha)\delta(k,q)} \quad (k \geq j+p;\ p,j \in N;\ q,n \in N_0;\ p > q). \tag{2.6}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)}{\left(\frac{k-q}{p-q}\right)^n (k-q-\alpha)\delta(k,q)} z^k \tag{2.7}$$

$(k \geq j+p;\ p,j \in N;\ q,n \in N_0;\ p > q)$.

Remark 1. (i) Putting $n = 0$ in Theorem 1, we obtain the result obtained by Chen et al. [3, Theorem 1].

(ii) Putting $n = 1$ in Theorem 1, we obtain the result obtained by Chen et al. [3, Theorem 2].

3. Distortion theorem

Theorem 3. *If the function $f(z)$ defined by* (1.1) *is in the class $S_j(n,p,q,\alpha)$, then*

$$\left\{\frac{p!}{(p-m)!} - \frac{(p-q-\alpha)\delta(p,q)(j+p-q)!}{(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)(j+p-m)!}|z|^j\right\}|z|^{p-m}$$
$$\leq \left|f^{(m)}(z)\right| \leq$$
$$\left\{\frac{p!}{(p-m)!} + \frac{(p-q-\alpha)\delta(p,q)(j+p-q)!}{(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)(j+p-m)!}|z|^j\right\}|z|^{p-m} \tag{3.1}$$

$(z \in U;\ 0 \leq \alpha < p-q;\ p,j \in N;\ q,n,m \in N_0;\ p > \max\{q,m\})$. *The result is sharp for the function $f(z)$ given by*

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)}{(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)} z^{j+p} \tag{3.2}$$

$(p,j \in N;\ q,n \in N_0;\ p > q)$.

Proof. In view of Theorem 1, we have

$$\frac{(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)}{(p-q-\alpha)\delta(p,q)(j+p)!}\sum_{k=j+p}^{\infty} k!a_k$$
$$\leq \sum_{k=j+p}^{\infty}\frac{(\frac{k-q}{p-q})^n(k-q-\alpha)\delta(k,q)}{(p-q-\alpha)\delta(p,q)}a_k \leq 1$$

which readily yields

$$\sum_{k=j+p}^{\infty} k!a_k \leq \frac{(p-q-\alpha)\delta(p,q)(j+p-q)!}{(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)}. \tag{3.3}$$

Now, by differentiating both sides of (1.1) m times, we have

$$f^{(m)}(z) = \frac{p!}{(p-m)!}z^{p-m} - \sum_{k=j+p}^{\infty}\frac{k!}{(k-m)!}a_k z^{k-m} \tag{3.4}$$

$(k \geq j+p;\ p,j \in N;\ q,m \in N_0;\ p > \max\{q,m\})$ and Theorem 2 follows from (3.3) and (3.4). ■

REMARK 2. (i) Putting $n=0$ in Theorem 2, we obtain the result obtained by Chen et al. [3, Theorem 7].

(ii) Putting $n=1$ in Theorem 2, we obtain the result obtained by Chen et al. [3, Theorem 8].

4. Radii of close-to-convexity, starlikeness and convexity

THEOREM 3. *Let the function $f(z)$ defined by* (1.1) *be in the class $S_j(n,p,q,\alpha)$. Then*

(i) *$f(z)$ is p-valently close-to-convex of order φ $(0\le\varphi<p)$ in $|z|<r_1$, where*

$$r_1=\inf_k\left\{\frac{(\frac{k-q}{p-q})^n(k-q-\alpha)\delta(k,q)}{(p-q-\alpha)\delta(p,q)}\left(\frac{p-\varphi}{k}\right)\right\}^{\frac{1}{k-p}} \tag{4.1}$$

$(k\ge j+p;\ p,j\in N;\ q,n\in N_0;\ p>q)$.

(ii) *$f(z)$ is p-valently starlike of order φ $(0\le\varphi<p)$ in $|z|<r_2$, where*

$$r_2=\inf_k\left\{\frac{(\frac{k-q}{p-q})^n(k-q-\alpha)\delta(k,q)}{(p-q-\alpha)\delta(p,q)}\left(\frac{p-\varphi}{k-\varphi}\right)\right\}^{\frac{1}{k-p}} \tag{4.2}$$

$(k\ge j+p;\ p,j\in N;\ q,n\in N_0;\ p>q)$.

(iii) $f(z)$ is p-valently convex of order φ $(0\le\varphi<p)$ in $|z|<r_3$, where

$$r_3=\inf_k\left\{\frac{(\frac{k-q}{p-q})^n(k-q-\alpha)\delta(k,q)}{(p-q-\alpha)\delta(p,q)}\,\frac{p(p-\varphi)}{k(k-\varphi)}\right\}^{\frac{1}{k-p}} \tag{4.3}$$

$(k\ge j+p;\ p,j\in N;\ q,n\in N_0;\ p>q)$. *Each of these results is sharp for the function $f(z)$ given by* (2.7).

Proof. It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le p-\varphi \quad (|z|<r_1;\ 0\le\varphi<p;\ p\in N), \tag{4.4}$$

$$\left|\frac{zf'(z)}{f(z)}-p\right|\le p-\varphi \quad (|z|<r_2;\ 0\le\varphi<p;\ p\in N), \tag{4.5}$$

and that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|\le p-\varphi \quad (|z|<r_3;\ 0\le\varphi<p;\ p\in N) \tag{4.6}$$

for a function $f(z)\in S_j(n,p,q,\alpha)$, where r_1,r_2 and r_3 are defined by (4.1), (4.2) and (4.3), respectively. The details involved are fairly straightforward and may be omitted. ∎

5. Modified Hadamard products

For the functions

$$f_\nu(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,\nu} z^k \quad (a_{k,\nu} \geq 0;\ \nu = 1,2) \tag{5.1}$$

we denote by $(f_1 \circledast f_2)(z)$ the modified Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, where

$$(f_1 \circledast f_2)(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,1} \cdot a_{k,2} z^k. \tag{5.2}$$

Theorem 4. *Let the functions $f_\nu(z)$ $(\nu = 1,2)$ defined by* (5.1) *be in the class $S_j(n,p,q,\alpha)$. Then $(f_1 \circledast f_2)(z) \in S_j(n,p,q,\gamma)$, where*

$$\gamma = (p-q) - \frac{j(p-q-\alpha)^2\delta(p,q)}{(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)^2\delta(j+p,q) - (p-q-\alpha)^2\delta(p,q)}. \tag{5.3}$$

The result is sharp for the functions $f_\nu(z)$ $(\nu = 1,2)$ given by

$$f_\nu(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)}{(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)} z^{j+p} \tag{5.4}$$

$(p, j \in N;\ q, n \in N_0;\ p > q;\ \nu = 1,2)$.

Proof. Employing the technique used earlier by Schild and Silverman [15], we need to find the largest γ such that

$$\sum_{k=j+p}^{\infty} \frac{(\frac{k-q}{p-q})^n(p-q-\gamma)\delta(k,q)}{(p-q-\gamma)\delta(p,q)} a_{k,1} \cdot a_{k,2} \leq 1 \tag{5.5}$$

$(f_\nu(z) \in S_j(n,p,q,\alpha),\ \nu = 1,2)$. Since $f_\nu(z) \in S_j(n,p,q,\alpha)$ $(\nu = 1,2)$, we readily see that

$$\sum_{k=j+p}^{\infty} \frac{(\frac{k-q}{p-q})^n(p-q-\alpha)\delta(k,q)}{(p-q-\alpha)\delta(p,q)} a_{k,\nu} \leq 1 \quad (\nu = 1,2). \tag{5.6}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=j+p}^{\infty} \frac{(\frac{k-q}{p-q})^n(k-q-\alpha)\delta(k,q)}{(p-q-\alpha)\delta(p,q)} \sqrt{a_{k,1} \cdot a_{k,2}} \leq 1. \tag{5.7}$$

Thus we only need to show that

$$\frac{(k-q-\gamma)}{(p-q-\gamma)} a_{k,1} \cdot a_{k,2} \leq \frac{(k-q-\alpha)}{(p-q-\alpha)} \sqrt{a_{k,1} \cdot a_{k,2}} \tag{5.8}$$

$(k \geq j+p;\ p, j \in N)$, or, equivalently, that

$$\sqrt{a_{k,1}.a_{k,2}} \leq \frac{(p-q-\gamma)(k-q-\alpha)}{(p-q-\alpha)(k-q-\gamma)} \tag{5.9}$$

$(k \geq j+p;\ p, j \in N)$. Hence, in the light of inequality (5.7), it is sufficient to prove that

$$\frac{(p-q-\alpha)\delta(p,q)}{(\frac{k-q}{p-q})^n(k-q-\alpha)\delta(k,q)} \leq \frac{(p-q-\gamma)(k-q-\alpha)}{(p-q-\alpha)(k-q-\gamma)} \tag{5.10}$$

$(k \geq j+p;\ p, j \in N)$. It follows from (5.10) that

$$\gamma \leq (p-q) - \frac{(k-p)(p-q-\alpha)^2\delta(p,q)}{(\frac{k-q}{p-q})^n(k-q-\alpha)^2\delta(k,q)-(p-q-\alpha)^2\delta(p,q)} \tag{5.11}$$

$(k \geq j+p;\ p, j \in N)$.

Now, defining the function $G(k)$ by

$$G(k) = (p-q) - \frac{(k-p)(p-q-\alpha)^2\delta(p,q)}{(\frac{k-q}{p-q})^n(k-q-\alpha)^2\delta(k,q)-(p-q-\alpha)^2\delta(p,q)} \tag{5.12}$$

$(k \geq j+p;\ p, j \in N)$, we see that $G(k)$ is an increasing function of k. Therefore, we conclude that

$$\gamma \leq G(j+p) = (p-q) - \frac{j(p-q-\alpha)^2\delta(p,q)}{(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)^2\delta(j+p,q)-(p-q-\alpha)^2\delta(p,q)}, \tag{5.13}$$

which evidently completes the proof of Theorem 4. ∎

Putting $n = 0$ and $n = 1$ in Theorem 4, we obtain

Corollary 2. *Let the functions $f_\nu(z)$ $(\nu = 1,2)$ defined by* (5.1) *be in the class $S_j(p,q,\alpha)$. Then $(f_1 \circledast f_2)(z) \in S_j(p,q,\gamma)$, where*

$$\gamma = (p-q) - \frac{j(p-q-\alpha)^2\delta(p,q)}{(j+p-q-\alpha)^2\delta(j+p,q)-(p-q-\alpha)^2\delta(p,q)}. \tag{5.14}$$

The result is sharp.

Remark 3. We note that the result obtained by Chen et al. [3, Theorem 5] is not correct. The correct result is given by (5.14).

Corollary 3. *Let the functions $f_\nu(z)(\nu = 1,2))$ defined by* (5.1) *be in the class $C_j(p,q,\alpha)$. Then $(f_1 \circledast f_2)(z) \in C_j(p,q,\gamma)$, where*

$$\gamma = (p-q) - \frac{j(p-q-\alpha)^2\delta(p,q+1)}{(j+p-q-\alpha)^2\delta(j+p,q+1)-(p-q-\alpha)^2\delta(p,q+1)}. \tag{5.15}$$

The result is sharp.

Remark 4. We note that the result obtained by Chen et al. [3, Theorem 6] is not correct. The correct result is given by (5.15).

Using arguments similar to those in the proof of Theorem 4, we obtain the following results.

Theorem 5. *Let the functions* $f_1(z)$, *resp.* $f_2(z)$ *defined by* (5.1) *be in the class* $S_j(n,p,q,\alpha)$, *resp.* $S_j(n,p,q,\tau)$. *Then* $(f_1 \circledast f_2)(z) \in S_j(n,p,q,\zeta)$, *where*

$$\zeta = (p-q) - \frac{j(p-q-\alpha)(p-q-\tau)\delta(p,q)}{(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)(j+p-q-\tau)\delta(j+p,q) - \Omega\delta(p,q)}, \tag{5.16}$$

where

$$\Omega = (p-q-\alpha)(p-q-\tau). \tag{5.17}$$

The result is the best possible for the functions

$$f_1(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)}{(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)} z^{j+p} \quad (p,j \in N;\ q,n \in N_0;\ p>q) \tag{5.18}$$

$$f_2(z) = z^p - \frac{(p-q-\tau)\delta(p,q)}{(\frac{j+p-q}{p-q})^n(j+p-q-\tau)\delta(j+p,q)} z^{j+p} \quad (p,j \in N;\ q,n \in N_0;\ p>q). \tag{5.19}$$

Theorem 6. *Let the functions* $f_\nu(z)$ $(\nu = 1,2)$ *defined by* (5.1) *be in the class* $S_j(n,p,q,\alpha)$. *Then the function*

$$h(z) = z^p - \sum_{k=j+p}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{5.20}$$

belongs to the class $S_j(n,p,q,\xi)$, *where*

$$\xi = (p-q) - \frac{2j(p-q-\alpha)^2\delta(p,q)}{(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)^2\delta(j+p,q) - 2(p-q-\alpha)^2\delta(p,q)}. \tag{5.21}$$

The result is the sharp for the functions $f_\nu(z)$ $(\nu = 1,2)$ *defined by* (5.4).

6. Applications of fractional calculus

Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in the literature rather extensively (cf., e.g., [4], [10], [17] and [18]; see also the various references cited therein). For our present investigation, we recall the following definitions.

Definition 1. The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}}\, d\zeta \quad (\mu > 0), \tag{6.1}$$

where the function $f(z)$ is analytic in a simply-connected domain of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Definition 2. The fractional derivative of order μ is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}}\, d\zeta \quad (0 \le \mu < 1), \tag{6.2}$$

where the function $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $n+\mu$ is defined, for a function $f(z)$, by

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n}\left\{D_z^{\mu} f(z)\right\} \quad (0 \leq \mu < 1;\ n \in N_0). \tag{6.3}$$

In this section, we shall investigate the growth and distortion properties of functions in the class $S_j(n,p,q,\alpha)$, involving the operators $J_{c,p}$ and D_z^{μ}. In order to derive our results, we need the following lemma given by Chen et al. [4].

Lemma 1. [4] *Let the function $f(z)$ be defined by* (1.1). *Then*

$$D_z^{\mu}\left\{(J_{c,p}f)(z)\right\} = \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} z^{p-\mu} - \sum_{k=j+p}^{\infty} \frac{(c+p)\Gamma(p+1)}{(c+k)\Gamma(p+1-\mu)} a_k z^{k-\mu} \tag{6.4}$$

$(\mu \in R;\ c > -p;\ p, j \in N)$ and

$$J_{c,p}(D_z^{\mu}\left\{f(z)\right\}) = \frac{(c+p)\Gamma(p+1)}{(c+p-\mu)\Gamma(p+1-\mu)} z^{p-\mu} - \sum_{k=j+p}^{\infty} \frac{(c+p)\Gamma(k+1)}{(c+k-\mu)\Gamma(k+1-\mu)} a_k z^{k-\mu} \tag{6.5}$$

$(\mu \in R;\ c > -p;\ p, j \in N)$, provided that no zeros appear in the denominators in (6.4) *and* (6.5).

Theorem 7. *Let the function $f(z)$ defined by* (1.1) *be in the class $S_j(n,p,q,\alpha)$. Then*

$$\left|D_z^{-\mu}\left\{(J_{c,p}f)(z)\right\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} - \frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)(\frac{j+p-q}{p-q})^n (j+p-q-\alpha)\delta(j+p,q)} |z|^j\right\} |z|^{p+\mu} \tag{6.6}$$

$(z \in U;\ 0 \leq \alpha < p-q;\ \mu > 0;\ c > -p;\ p, j \in N;\ q, n \in N_0;\ p > q)$ and

$$\left|D_z^{-\mu}\left\{(J_{c,p}f)(z)\right\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} + \frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)(\frac{j+p-q}{p-q})^n (j+p-q-\alpha)\delta(j+p,q)} |z|^j\right\} |z|^{p+\mu} \tag{6.7}$$

$(z \in U;\ 0 \leq \alpha < p-q;\ \mu > 0;\ c > -p;\ p, j \in N;\ q, n \in N_0;\ p > q)$. Each of the assertions (6.6) *and* (6.7) *is sharp.*

Proof. In view of Theorem 1, we have

$$\frac{(\frac{j+p-q}{p-q})^n (j+p-q-\alpha)\delta(j+p,q)}{(p-q-\alpha)\delta(p,q)} \sum_{k=j+p}^{\infty} a_k \leq \sum_{k=j+p}^{\infty} \frac{(\frac{k-q}{p-q})^n (k-q-\alpha)\delta(k,q)}{(p-q-\alpha)\delta(p,q)} a_k \leq 1, \tag{6.8}$$

which readily yields

$$\sum_{k=j+p}^{\infty} a_k \leq \frac{(p-q-\alpha)\delta(p,q)}{(\frac{j+p-q}{p-q})^n (j+p-q-\alpha)\delta(j+p,q)}. \tag{6.9}$$

Consider the function $F(z)$ defined in U by

$$\begin{aligned} F(z) &= \frac{\Gamma(p+1+\mu)}{\Gamma(p+1)} z^{-\mu} D_z^{-\mu} \{(J_{c,p}f)(z)\} \\ &= z^p - \sum_{k=j+p}^{\infty} \frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)\Gamma(p+1)} a_k z^k \\ &= z^p - \sum_{k=j+p}^{\infty} \Phi(k) a_k z^k \quad (z \in U) \end{aligned}$$

where

$$\Phi(k) = \frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)\Gamma(p+1)} \quad (k \geq j+p;\ p,j \in N;\ \mu > 0). \tag{6.10}$$

Since $\Phi(k)$ is a decreasing function of k when $\mu > 0$, we get

$$0 < \Phi(k) \leq \Phi(j+p) = \frac{(c+p)\Gamma(j+p+1)\Gamma(p+1+\mu)}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)} \tag{6.11}$$

$(c > -p;\ p,j \in N;\ \mu > 0)$. Thus, by using (6.9) and (6.11), we deduce that

$$|F(z)| \geq |z|^p - \Phi(j+p)\,|z|^{j+p} \sum_{k=j+p}^{\infty} a_k \geq |z|^p - \frac{(c+p)\Gamma(j+p+1)\Gamma(p+1+\mu)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)}\,|z|^{j+p}$$

$(z \in U)$ and

$$|F(z)| \leq |z|^p + \Phi(j+p)\,|z|^{j+p} \sum_{k=j+p}^{\infty} a_k \leq |z|^p + \frac{(c+p)\Gamma(j+p+1)\Gamma(p+1+\mu)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)}\,|z|^{j+p}$$

$(z \in U)$, which yield the inequalities (6.6) and (6.7) of Theorem 7. The equalities in (6.6) and (6.7) are attained for the function $f(z)$ given by

$$D_z^{-\mu}\{(J_{c,p}f)(z)\} = \left\{ \frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} - \frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)} z^j \right\} z^{p+\mu} \tag{6.12}$$

or, equivalently, by

$$(J_{c,p}f)(z) = z^p - \frac{(c+p)(p-q-\alpha)\delta(p,q)}{(c+j+p)(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)} z^{j+p}. \tag{6.13}$$

Thus we complete the proof of Theorem 7. ■

Using arguments similar to those in the proof of Theorem 7, we obtain the following result.

THEOREM 8. *Let the function $f(z)$ defined by* (1.1) *be in the class $S_j(n,p,q,\alpha)$. Then*

$$|D_z^{\mu}\{(J_{c,p}f)(z)\}| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} - \frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1-\mu)(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)}|z|^j\right\}|z|^{p-\mu} \tag{6.14}$$

$(z \in U;\ 0 \leq \alpha < p-q;\ 0 \leq \mu < 1;\ c > -p;\ p,j \in N;\ q,n \in N_0;\ p > q)$ *and*

$$|D_z^{\mu}\{(J_{c,p}f)(z)\}| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} + \frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1-\mu)(\frac{j+p-q}{p-q})^n(j+p-q-\alpha)\delta(j+p,q)}|z|^j\right\}|z|^{p-\mu} \tag{6.15}$$

$(z \in U;\ 0 \leq \alpha < p-q;\ 0 \leq \mu < 1;\ c > -p;\ p,j \in N;\ q,n \in N_0;\ p > q)$. *Each of the assertions* (6.14)*, and* (6.15) *is sharp.*

REMARK 5. Putting $n = 0$ and $n = 1$ in Theorem 7 and Theorem 8, we obtain the corresponding results for the classes $S_j(p,q,\alpha)$ and $C_j(p,q,\alpha)$, respectively.

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