

## ON SEMI-INVARIANT SUBMANIFOLDS OF A NEARLY KENMOTSU MANIFOLD WITH THE CANONICAL SEMI-SYMMETRIC SEMI-METRIC CONNECTION

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**Abstract.** We define the canonical semi-symmetric semi-metric connection in a nearly Kenmotsu manifold and we study semi-invariant submanifolds of a nearly Kenmotsu manifold endowed with the canonical semi-symmetric semi-metric connection. Moreover, we discuss the integrability of distributions on semi-invariant submanifolds of a nearly Kenmotsu manifold with the canonical semi-symmetric semi-metric connection.

### 1. Introduction

In [9], K. Kenmotsu introduced and studied a new class of almost contact manifolds called Kenmotsu manifolds. The notion of nearly Kenmotsu manifold was introduced by A. Shukla in [13]. Semi-invariant submanifolds in Kenmotsu manifolds were studied by N. Papaghuic [11] and M. Kobayashi [10]. Semi-invariant submanifolds of a nearly Kenmotsu manifolds were studied by M.M. Tripathi and S.S. Shukla in [14]. In this paper we study semi-invariant submanifolds of a nearly Kenmotsu manifold with the canonical semi-symmetric semi-metric connection.

Let  $\nabla$  be a linear connection in an  $n$ -dimensional differentiable manifold  $M$ . The torsion tensor  $T$  and the curvature tensor  $R$  of  $\nabla$  are given respectively by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$
$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

The connection  $\nabla$  is symmetric if the torsion tensor  $T$  vanishes, otherwise it is non-symmetric. The connection  $\nabla$  is a metric connection if there is a Riemannian metric  $g$  in  $M$  such that  $\nabla g = 0$ , otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

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In [8, 12], A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric linear connection. A linear connection  $\nabla$  is said to be a semi-symmetric connection if its torsion tensor  $T$  is of the form

$$T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where  $\eta$  is a 1-form.

Some properties of semi-invariant submanifolds, hypersurfaces and submanifolds with respect to semi-symmetric or quarter symmetric connections were studied in [1, 7], [2, 3] and [4] respectively.

This paper is organized as follows. In Section 2, we give a brief introduction of nearly Kenmotsu manifold. In Section 3, we show that the induced connection on semi-invariant submanifolds of a nearly Kenmotsu manifold with the canonical semi-symmetric semi-metric connection is also semi-symmetric semi-metric. In Section 4, we establish some lemmas on semi-invariant submanifolds and in Section 5, we discuss the integrability conditions of distributions on semi-invariant submanifolds of nearly Kenmotsu manifolds with the canonical semi-symmetric semi-metric connection.

## 2. Preliminaries

Let  $\bar{M}$  be  $(2m + 1)$ -dimensional almost contact metric manifold [6] with a metric tensor  $g$ , a tensor field  $\phi$  of type  $(1,1)$ , a vector field  $\xi$ , a 1-form  $\eta$  which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta\phi = 0, \quad \eta(\xi) = 1 \quad (2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (2.2)$$

for any vector fields  $X, Y$  on  $\bar{M}$ . If in addition to the above conditions we have  $d\eta(X, Y) = g(X, \phi Y)$ , the structure is said to be a contact metric structure.

The almost contact metric manifold  $\bar{M}$  is called a nearly Kenmotsu manifold if it satisfies the condition [13]

$$(\bar{\nabla}_X \phi)(Y) + (\bar{\nabla}_Y \phi)(X) = -\eta(Y)\phi X - \eta(X)\phi Y, \quad (2.3)$$

where  $\bar{\nabla}$  denotes the Riemannian connection with respect to  $g$ . If, moreover,  $M$  satisfies

$$(\bar{\nabla}_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad (2.4)$$

then it is called Kenmotsu manifold [9]. Obviously a Kenmotsu manifold is also a nearly Kenmotsu manifold.

**DEFINITION.** An  $n$ -dimensional Riemannian submanifold  $M$  of a nearly Kenmotsu manifold  $\bar{M}$  is called a semi-invariant submanifold if  $\xi$  is tangent to  $M$  and there exists on  $M$  a pair of distributions  $(D, D^\perp)$  such that [10]:

- (i)  $TM$  orthogonally decomposes as  $D \oplus D^\perp \oplus \langle \xi \rangle$ ,

- (ii) the distribution  $D$  is invariant under  $\phi$ , that is,  $\phi D_x \subset D_x$  for all  $x \in M$ ,
- (iii) the distribution  $D^\perp$  is anti-invariant under  $\phi$ , that is,  $\phi D_x^\perp \subset T_x^\perp M$  for all  $x \in M$ , where  $T_x M$  and  $T_x^\perp M$  are the tangent and normal spaces of  $M$  at  $x$ .

The distribution  $D$  (resp.  $D^\perp$ ) is called the horizontal (resp. vertical) distribution. A semi-invariant submanifold  $M$  is said to be an invariant (resp. anti-invariant) submanifold if we have  $D_x^\perp = \{0\}$  (resp.  $D_x = \{0\}$ ) for each  $x \in M$ . We also call  $M$  proper if neither  $D$  nor  $D^\perp$  is null. It is easy to check that each hypersurface of  $\bar{M}$  which is tangent to  $\xi$  inherits a structure of semi-invariant submanifold of  $\bar{M}$ .

Now, we remark that owing to the existence of the 1-form  $\eta$ , we can define the canonical semi-symmetric semi-metric connection  $\bar{\nabla}$  in any almost contact metric manifold  $(\bar{M}, \phi, \xi, \eta, g)$  by

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y - \eta(X)Y + g(X, Y)\xi \tag{2.5}$$

such that  $(\bar{\nabla}_X g)(Y, Z) = 2\eta(X)g(Y, Z) - \eta(Y)g(X, Z) - \eta(Z)g(X, Y)$  for any  $X, Y \in T\bar{M}$ . In particular, if  $\bar{M}$  is a nearly Kenmotsu manifold, then from (2.5) we have

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = -\eta(X)\phi Y - \eta(Y)\phi X. \tag{2.6}$$

**THEOREM 2.1.** *Let  $(\bar{M}, \phi, \xi, \eta, g)$  be an almost contact metric manifold and  $M$  be a submanifold tangent to  $\xi$ . Then, with respect to the orthogonal decomposition  $TM \oplus T^\perp M$ , the canonical semi-symmetric semi-metric connection  $\bar{\nabla}$  induces on  $M$  a connection  $\nabla$  which is semi-symmetric and semi-metric.*

*Proof.* With respect to the orthogonal decomposition  $TM \oplus T^\perp M$ , we have

$$\bar{\nabla}_X Y = \nabla_X Y + m(X, Y), \tag{2.7}$$

where  $m$  is a  $T^\perp M$ -valued symmetric tensor field on  $M$ . If  $\nabla^*$  denotes the induced connection from the Riemannian connection  $\bar{\bar{\nabla}}$ , then

$$\bar{\bar{\nabla}}_X Y = \nabla^*_X Y + h(X, Y), \tag{2.8}$$

where  $h$  is the second fundamental form. By the definition of semi-symmetric semi-metric connection

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y - \eta(X)Y + g(X, Y)\xi. \tag{2.9}$$

Now using above equations, we have

$$\nabla_X Y + m(X, Y) = \nabla^*_X Y + h(X, Y) - \eta(X)Y + g(X, Y)\xi.$$

Equating tangential and normal components from both the sides, we get

$$h(X, Y) = m(X, Y)$$

and

$$\nabla_X Y = \nabla^*_X Y - \eta(X)Y + g(X, Y)\xi.$$

Thus  $\nabla$  is also a semi-symmetric semi-metric connection. ■

Now, Gauss equation for  $M$  in  $(\bar{M}, \bar{\nabla})$  is

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.10}$$

and Weingarten formulas are given by

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N - \eta(X)N \tag{2.11}$$

for  $X, Y \in TM$  and  $N \in T^\perp M$ . Moreover, we have

$$g(h(X, Y), N) = g(A_N X, Y). \tag{2.12}$$

From now on, we consider a nearly Kenmotsu manifold  $\bar{M}$  and a semi-invariant submanifold  $M$ . Any vector  $X$  tangent to  $M$  can be written as

$$X = PX + QX + \eta(X)\xi, \tag{2.13}$$

where  $PX$  and  $QX$  belong to the distribution  $D$  and  $D^\perp$  respectively. For any vector field  $N$  normal to  $M$ , we put

$$\phi N = BN + CN, \tag{2.14}$$

where  $BN$  (resp.  $CN$ ) denotes the tangential (resp. normal) component of  $\phi N$ .

**DEFINITION.** A semi-invariant submanifold is said to be mixed totally geodesic if  $h(X, Z) = 0$  for all  $X \in D$  and  $Z \in D^\perp$ .

Using the canonical semi-symmetric semi-metric connection, the Nijenhuis tensor of  $\phi$  is expressed by

$$N(X, Y) = (\bar{\nabla}_{\phi X} \phi)(Y) - (\bar{\nabla}_{\phi Y} \phi)(X) - \phi(\bar{\nabla}_X \phi)(Y) + \phi(\bar{\nabla}_Y \phi)(X) \tag{2.15}$$

for any  $X, Y \in T\bar{M}$ .

From (2.6), we have

$$(\bar{\nabla}_{\phi X} \phi)(Y) = \eta(Y)X - \eta(X)\eta(Y)\xi - (\bar{\nabla}_Y \phi)\phi X. \tag{2.16}$$

Also,

$$(\bar{\nabla}_Y \phi)\phi X = ((\bar{\nabla}_Y \eta)(X))\xi + \eta(X)\bar{\nabla}_Y \xi - \phi(\bar{\nabla}_Y \phi)X. \tag{2.17}$$

By virtue of (2.15), (2.16) and (2.17), we get

$$\begin{aligned} N(X, Y) = & -\eta(Y)X - 3\eta(X)Y + 4\eta(X)\eta(Y)\xi + \eta(Y)\bar{\nabla}_X \xi \\ & - \eta(X)\bar{\nabla}_Y \xi + 2d\eta(X, Y)\xi + 4\phi(\bar{\nabla}_Y \phi)X \end{aligned} \tag{2.18}$$

for any  $X, Y \in T\bar{M}$ .

### 3. Basic lemmas

LEMMA 3.1. *Let  $M$  be a semi-invariant submanifold of a nearly Kenmotsu manifold  $\bar{M}$  with the canonical semi-symmetric semi-metric connection. Then*

$$2(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y]$$

for any  $X, Y \in D$ .

*Proof.* By Gauss formula we have

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X). \quad (3.1)$$

Also by use of (2.10) covariant differentiation yields

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X + \phi[X, Y]. \quad (3.2)$$

From (3.1) and (3.2), we get

$$(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y]. \quad (3.3)$$

Using  $\eta(X) = 0$  for each  $X \in D$  in (2.6), we get

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 0. \quad (3.4)$$

Adding (3.3) and (3.4) we get the result. ■

Similar computations also yield

LEMMA 3.2. *Let  $M$  be a semi-invariant submanifold of a nearly Kenmotsu manifold with the canonical semi-symmetric semi-metric connection. Then*

$$2(\bar{\nabla}_X \phi)Y = -A_{\phi Y} X + \nabla_X^\perp \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y]$$

for any  $X \in D$  and  $Y \in D^\perp$ .

LEMMA 3.3. *Let  $M$  be a semi-invariant submanifold of a nearly Kenmotsu manifold  $\bar{M}$  with the canonical semi-symmetric semi-metric connection. Then*

$$\begin{aligned} P\nabla_X \phi PY + P\nabla_Y \phi PX - PA_{\phi QY} X - PA_{\phi QX} Y \\ = -2\eta(Y)\phi PX - \eta(X)\phi PY + \phi P\nabla_X Y + \phi P\nabla_Y X \end{aligned} \quad (3.5)$$

$$\begin{aligned} Q\nabla_X \phi PY + Q\nabla_Y \phi PX - QA_{\phi QY} X - QA_{\phi QX} Y \\ = -\eta(Y)\phi QX - 2\eta(X)\phi QY + 2Bh(X, Y) \end{aligned} \quad (3.6)$$

$$\begin{aligned} h(X, \phi PY) + h(Y, \phi PX) + \nabla_X^\perp \phi QY + \nabla_Y^\perp \phi QX \\ = 2Ch(X, Y) + \phi Q\nabla_X Y + \phi Q\nabla_Y X \end{aligned} \quad (3.7)$$

$$\eta(\nabla_X \phi PY + \nabla_Y \phi PX - A_{\phi QY} X - A_{\phi QX} Y) = 0 \quad (3.8)$$

for all  $X, Y \in TM$ .

*Proof.* Differentiating (2.13) covariantly and using (2.10) and (2.11), we have

$$\begin{aligned}
 (\bar{\nabla}_X\phi)Y + \phi(\nabla_X Y) + \phi h(X, Y) &= P\nabla_X(\phi PY) + Q\nabla_X(\phi PY) \\
 &\quad - \eta(A_{\phi QY}X)\xi + \eta(\nabla_X\phi PY)\xi - PA_{\phi QY}X - QA_{\phi QY}X \\
 &\quad + \nabla_X^\perp\phi QY + h(X, \phi PY) + \eta(X)\phi PY. \tag{3.9}
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 (\bar{\nabla}_Y\phi)X + \phi(\nabla_Y X) + \phi h(Y, X) &= P\nabla_Y(\phi PX) + Q\nabla_Y(\phi PX) \\
 &\quad - \eta(A_{\phi QX}Y)\xi + \eta(\nabla_Y\phi PX)\xi - PA_{\phi QX}Y - QA_{\phi QX}Y \\
 &\quad + \nabla_Y^\perp\phi QX + h(Y, \phi PX) + \eta(Y)\phi PX. \tag{3.10}
 \end{aligned}$$

Adding (3.9) and (3.10) and using (2.6) and (2.14), we have

$$\begin{aligned}
 &- 2\eta(Y)\phi PX - 2\eta(Y)\phi QX - 2\eta(X)\phi PY - 2\eta(X)\phi QY + \phi P\nabla_X Y \\
 &\quad + \phi Q\nabla_X Y + \phi P\nabla_Y X + \phi Q\nabla_Y X + 2Bh(Y, X) + 2Ch(Y, X) \\
 &= P\nabla_X(\phi PY) + P\nabla_Y(\phi PX) + Q\nabla_Y(\phi PX) - PA_{\phi QY}X \\
 &\quad + Q\nabla_X(\phi PY) + \nabla_X^\perp\phi QY - PA_{\phi QX}Y - QA_{\phi QY}X \\
 &\quad - QA_{\phi QX}Y + \nabla_Y^\perp\phi QX + h(Y, \phi PX) + h(X, \phi PY) \\
 &\quad + \eta(\nabla_X\phi PY)\xi + \eta(\nabla_Y\phi PX)\xi - \eta(A_{\phi QX}Y)\xi - \eta(A_{\phi QY}X)\xi. \tag{3.11}
 \end{aligned}$$

Equations (3.5)–(3.8) follow by comparison of tangential, normal and vertical components of (3.11). ■

**DEFINITION.** The horizontal distribution  $D$  is said to be parallel with respect to the connection  $\nabla$  on  $M$  if  $\nabla_X Y \in D$  for all vector fields  $X, Y \in D$ .

**PROPOSITION 3.4.** *Let  $M$  be a semi-invariant submanifold of a nearly Kenmotsu manifold  $\bar{M}$  with the canonical semi-symmetric semi-metric connection. If the horizontal distribution  $D$  is parallel then  $h(X, \phi Y) = h(Y, \phi X)$  for all  $X, Y \in D$ .*

*Proof.* Since  $D$  is parallel, therefore,  $\nabla_X \phi Y \in D$  and  $\nabla_Y \phi X \in D$  for each  $X, Y \in D$ . Now from (3.6) and (3.7), we get

$$h(X, \phi Y) + h(Y, \phi X) = 2\phi h(X, Y). \tag{3.12}$$

Replacing  $X$  by  $\phi X$  in above equation, we have

$$h(\phi X, \phi Y) - h(Y, X) = 2\phi h(\phi X, Y). \tag{3.13}$$

Replacing  $Y$  by  $\phi Y$  in (3.12), we have

$$-h(X, Y) + h(\phi X, \phi Y) = 2\phi h(X, \phi Y). \tag{3.14}$$

Comparing (3.13) and (3.14), we have  $h(X, \phi Y) = h(\phi X, Y)$  for all  $X, Y \in D$ . ■

LEMMA 3.5. *Let  $M$  be a semi-invariant submanifold of a nearly Kenmotsu manifold  $\bar{M}$  with the canonical semi-symmetric semi-metric connection. Then  $M$  is mixed totally geodesic if and only if  $A_N X \in D$  for all  $X \in D$  and  $N \in T^\perp M$ .*

*Proof.* If  $A_N X \in D$ , then  $g(h(X, Y), N) = g(A_N X, Y) = 0$ , which gives  $h(X, Y) = 0$  for  $Y \in D^\perp$ . Hence  $M$  is mixed totally geodesic. ■

#### 4. Integrability conditions for distributions

THEOREM 4.1. *Let  $M$  be a semi-invariant submanifold of a nearly Kenmotsu manifold  $\bar{M}$  with the canonical semi-symmetric semi-metric connection. Then the following conditions are equivalent:*

- (i) *the distribution  $D \oplus \langle \xi \rangle$  is integrable,*
- (ii)  *$N(X, Y) \in D \oplus \langle \xi \rangle$  and  $h(X, \phi Y) = h(\phi X, Y)$  for any  $X, Y \in D \oplus \langle \xi \rangle$ .*

*Proof.* The condition  $N(X, Y) \in D \oplus \langle \xi \rangle$  for any  $X, Y \in D \oplus \langle \xi \rangle$  is equivalent to the following two

- (I)  $N(X, \xi) \in D \oplus \langle \xi \rangle$  for any  $X \in D$ ,
- (II)  $N(X, Y) \in D \oplus \langle \xi \rangle$  for any  $X, Y \in D$ .

In the first case, using Gauss formula and (2.6) in (2.18), we get

$$N(X, \xi) = 3X - 3\nabla_X \xi + 2d\eta(X, \xi)\xi - 3h(X, \xi) + 4\eta(\nabla_X \xi)\xi$$

and

$$N(X, \xi) \in D \oplus \langle \xi \rangle \Leftrightarrow Q(\nabla_X \xi) = 0, h(X, \xi) = 0.$$

Using again (2.6) and computing its normal component we get

$$h(\xi, \phi X) - \phi Q(\nabla_\xi X) - 2C(h(\xi, X)) - \phi Q(\nabla_X \xi) = 0.$$

Hence for any  $X \in D$

$$N(X, \xi) \in D \oplus \langle \xi \rangle \Rightarrow Q([X, \xi]) = 0, h(X, \xi) = 0. \tag{4.1}$$

In case (II), using Gauss formula in (2.18), we get

$$N(X, Y) = 2d\eta(X, Y)\xi + 4\phi(\nabla_Y \phi X) + 4\phi h(Y, \phi X) + 4h(Y, X) + 4\nabla_Y X - 4\eta(\nabla_Y X)\xi \tag{4.2}$$

for all  $X, Y \in D$ . From (4.2) we have that  $N(X, Y) \in (D \oplus \langle \xi \rangle)$  implies

$$\phi Q(\nabla_Y \phi X) + Ch(Y, \phi X) + h(Y, X) = 0$$

for all  $X, Y \in D$ . Replacing  $Y$  by  $\phi Z$ , where  $Z \in D$ , we get

$$\phi Q(\nabla_{\phi Z} \phi X) + Ch(\phi Z, \phi X) + h(\phi Z, X) = 0.$$

Interchanging  $X$  and  $Z$ , we have

$$\phi Q(\nabla_{\phi X} \phi Z) + Ch(\phi X, \phi Z) + h(\phi X, Z) = 0.$$

Subtracting above two equations, we have

$$\phi Q[\phi X, \phi Z] + h(Z, \phi X) - h(X, \phi Z) = 0.$$

Thus, we get, for any  $X, Y \in D$

$$N(X, Y) \in D \oplus \langle \xi \rangle \Rightarrow \phi Q([X, Y]) + h(\phi X, Y) - h(X, \phi Y) = 0. \tag{4.3}$$

Now, suppose that  $D \oplus \langle \xi \rangle$  is integrable so for any  $X, Y \in D \oplus \langle \xi \rangle$  we have  $N(X, Y) \in D \oplus \langle \xi \rangle$ , since  $\phi(D \oplus \langle \xi \rangle) \subset D$ . Moreover,  $h(X, \xi) = 0, h(X, \phi Y) = h(\phi X, Y)$  for any  $X, Y \in D$  and ii) is proven. Vice versa, if ii) holds, then from (4.1) and (4.3) we get the integrability of  $D \oplus \langle \xi \rangle$ . ■

LEMMA 4.2. *Let  $M$  be a semi-invariant submanifold of a nearly Kenmotsu manifold  $\bar{M}$  with the canonical semi-symmetric semi-metric connection. Then*

$$2(\bar{\nabla}_Y \phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^\perp \phi Z - \nabla_Z^\perp \phi Y - \phi[Y, Z]$$

for  $Y, Z \in D^\perp$ .

*Proof.* From Weingarten equation, we have

$$\bar{\nabla}_Y \phi Z - \bar{\nabla}_Z \phi Y = -A_{\phi Z}Y + A_{\phi Y}Z + \nabla_Y^\perp \phi Z - \nabla_Z^\perp \phi Y. \tag{4.4}$$

Also by covariant differentiation, we get

$$\bar{\nabla}_Y \phi Z - \bar{\nabla}_Z \phi Y = (\bar{\nabla}_Y \phi)Z - (\bar{\nabla}_Z \phi)Y + \phi[Y, Z]. \tag{4.5}$$

From (4.4) and (4.5) we have

$$(\bar{\nabla}_Y \phi)Z - (\bar{\nabla}_Z \phi)Y = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^\perp \phi Z - \nabla_Z^\perp \phi Y - \phi[Y, Z]. \tag{4.6}$$

From (2.6) we obtain

$$(\bar{\nabla}_Y \phi)Z + (\bar{\nabla}_Z \phi)Y = 0 \tag{4.7}$$

for any  $Y, Z \in D^\perp$ . Adding (4.6) and (4.7), we get

$$2(\bar{\nabla}_Y \phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^\perp \phi Z - \nabla_Z^\perp \phi Y - \phi[Y, Z]. \quad \blacksquare$$

PROPOSITION 4.3. *Let  $M$  be a semi-invariant submanifold of a nearly Kenmotsu manifold  $\bar{M}$  with the canonical semi-symmetric semi-metric connection. Then*

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y, Z]$$

for any  $Y, Z \in D^\perp$ .

*Proof.* Let  $Y, Z \in D^\perp$  and  $X \in TM$  then from (2.10) and (2.12), we have

$$2g(A_{\phi Z}Y, X) = -g(\bar{\nabla}_Y \phi X, Z) - g(\bar{\nabla}_X \phi Y, Z) + g((\bar{\nabla}_Y \phi)X + (\bar{\nabla}_X \phi)Y, Z).$$

By use of (2.6) and  $\eta(Y) = 0$  for  $Y \in D^\perp$ , we have

$$2g(A_{\phi Z}Y, X) = -g(\phi \bar{\nabla}_Y Z, X) + g(A_{\phi Y}Z, X).$$



Interchanging  $Y$  and  $Z$  and subtracting we get

$$g(3A_{\phi Y}Z - 3A_{\phi Z}Y - \phi P[Y, Z], X) = 0 \quad (4.8)$$

from which, for any  $Y, Z \in D^\perp$ ,

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y, Z]$$

follows. ■

**THEOREM 4.4.** *Let  $M$  be a semi-invariant submanifold of a nearly Kenmotsu manifold  $\bar{M}$  with the canonical semi-symmetric semi-metric connection. Then the distribution  $D^\perp$  is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = 0$$

for all  $Y, Z \in D^\perp$ .

*Proof.* Suppose that the distribution  $D^\perp$  is integrable. Then  $[Y, Z] \in D^\perp$  for any  $Y, Z \in D^\perp$ . Therefore,  $P[Y, Z] = 0$  and from (4.8), we get

$$A_{\phi Y}Z - A_{\phi Z}Y = 0. \quad (4.9)$$

Conversely, let (4.9) hold. Then by virtue of (4.8) we have  $\phi P[Y, Z] = 0$  for all  $Y, Z \in D^\perp$ . Since  $\text{rank } \phi = 2m$ , we have  $\phi P[Y, Z] = 0$  and  $P[Y, Z] \in D \cap \langle \xi \rangle$ . Hence  $P[Y, Z] = 0$ , which is equivalent to  $[Y, Z] \in D^\perp$  for all  $Y, Z \in D^\perp$  and  $D^\perp$  is integrable. ■

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