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ON VITALI SETS AND THEIR UNIONS

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Abstract. It is well known that any Vitali set on the real line $\mathbb{R}$ does not possess the Baire property. In this article we prove the following:

Let S be a Vitali set, S_r be the image of S under the translation of $\mathbb{R}$ by a rational number r and $\mathcal{F}= \{S_r : r \text{ is rational}\}$. Then for each non-empty proper subfamily $\mathcal{F}'$ of $\mathcal{F}$ the union $\cup\mathcal{F}'$ does not possess the Baire property.

Our starting point is the classical theorem by Vitali [3] stating the existence of subsets of the real line $\mathbb{R}$, called below Vitali sets, which are non-Lebesgue measurable. Recall that if one considers a Vitali set S on the real line $\mathbb{R}$ then each its image S_x under the translation of $\mathbb{R}$ by a real number x is also a Vitali set. Moreover, the family $\mathcal{F}= \{S_r : r \text{ is rational}\}$ is disjoint and its union is $\mathbb{R}$. Such decompositions of the real line $\mathbb{R}$ are used in measure theory to show the nonexistence of a certain type of measure.

Recall that a subset A of $\mathbb{R}$ possesses *the Baire property* if and only if there is an open set O and two meager sets M, N such that $A = (O \setminus M) \cup N$. Note that the Vitali sets do not possess the Baire property. This can be easily seen from the following known properties of Vitali sets:

(A) each Vitali set is not meager;

(B) each Vitali set does not contain the set $O \setminus M$ for any non-empty open subset O of $\mathbb{R}$ and any meager set M.

Really, if a Vitali set S possesses the Baire property then there is an open set O and two meager sets M, N such that $S = (O \setminus M) \cup N$. If $O = \emptyset$ then $S = N$, i.e. S is meager. But this is impossible by (A). So $O \neq \emptyset$ and $S \supset O \setminus M$. However this inclusion is impossible by (B). Hence S does not possess the Baire property. The first proof of (B) belongs to Vitali (cf. [2]), another one the reader can find in Proposition 3.1.

Let us say now that a subset A of $\mathbb{R}$ possesses *Vitali property* if there exist a non-empty open set O and a meager set M such that $A \supset O \setminus M$.

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We continue with the following question.

For what proper subfamilies $\mathcal{F}'$ of $\mathcal{F}$ do the unions $\bigcup\mathcal{F}'$ not possess the Baire property (resp. the Vitali property)?

Here we present characterizations for countable union of nonmeager sets of a hereditarily Lindelöf space R:

(1) to be the union $(O\setminus M)\cup N$, where O is an open set and M, N are meager sets of R (Theorem 2.1); as a corollary we show that for each non-empty proper subfamily $\mathcal{F}'$ of $\mathcal{F}$ the union $\bigcup\mathcal{F}'$ does not possess the Baire property (see Theorem 3.1); and

(2) to have a complement in R which contains the set $O\setminus M$, where O is a non-empty open set and M is a meager set of R (Theorem 2.2); as a corollary we get a characterization of those non-empty proper subfamilies $\mathcal{F}'$ of $\mathcal{F}$ which unions $\bigcup\mathcal{F}'$ possess the Vitali property (see Theorem 3.2);

1. Preliminary notations

Let $\mathbb{R}$ be the real line. For $x\in\mathbb{R}$ denote by T_x the translation of $\mathbb{R}$ by x, i.e. $T_x(y)=y+x$ for each $y\in\mathbb{R}$.

The equivalence relation E on $\mathbb{R}$ is defined as follows. For $x,y\in\mathbb{R}$, let xEy if and only if $x-y\in\mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Let us denote the equivalence classes by $E_\alpha, \alpha\in I$. Note that $|I|=c$. It is evident that for each $\alpha\in I$ and each $x\in E_\alpha$, $E_\alpha=T_x(\mathbb{Q})$. Hence every equivalence class E_α is dense in $\mathbb{R}$.

A Vitali set here is any subset S of $\mathbb{R}$ such that $|S\cap E_\alpha|=1$ for each $\alpha\in I$.

Let S be a Vitali set. For each $x\in\mathbb{R}$ put $S_x=T_x(S)$. Note that S_x is a Vitali set. Observe also that $S_0=S$ and $S_{x_2}=S_{x_2-x_1}(S_{x_1})$ for any $x_1,x_2\in\mathbb{R}$. Moreover, if $r_1,r_2\in\mathbb{Q}$ and $r_1\neq r_2$ then $S_{r_1}\cap S_{r_2}=\emptyset$ and

$$\bigcup_{r\in\mathbb{Q}} S_r=\mathbb{R}. \tag{*}$$

Recall also that a subset A of a topological space R is called *meager in* R if A is the union of countably many nowhere dense in R sets, otherwise A is called *nonmeager in* R. It is evident that if $B\subset A\subset R$ then B is meager in R whenever A is meager in R, and A is nonmeager in R whenever B is nonmeager in R.

Our terminology follows [1] and [2].

2. Countable unions of nonmeager sets in a hereditarily Lindelöf space

Let R be a topological space.

Definition 2.1. Let X be a nonmeager subset in R. Put $O_X=\mathrm{Int}_R(\mathrm{Cl}_R(X))$. Note that $O_X\neq\emptyset$. Define further

$X'=\{x\in X\cap O_X:$ there is an open nbd V of x such that $V\cap X$ is meager$\}$

and $X''=(X\cap O_X)\setminus X'$.

LEMMA 2.1. *Let X be a nonmeager set in R. Then the set $Y = X \cap (R \setminus \mathrm{Cl}_R O_X)$ is nowhere dense in R.*

Proof. Since $Y \subset X$ (resp. $Y \subset R \setminus \mathrm{Cl}_R O_X$), we have $\mathrm{Int}_R(\mathrm{Cl}_R(Y)) \subset O_X$ (resp. $\mathrm{Int}_R(\mathrm{Cl}_R(Y)) \subset R \setminus O_X$). Hence, $\mathrm{Int}_R(\mathrm{Cl}_R(Y)) = \emptyset$. ■

PROPOSIITON 2.1. *Let X be a nonmeager set in R and X' be Lindelöf. Then*

(i) the set X' is meager;

(ii) the set $X \setminus X''$ is meager and $X'' \neq \emptyset$;

(iii) for any open subset V of R such that $V \cap X'' \neq \emptyset$, the set $V \cap X''$ is nonmeager;

In particular, the set $X'' = X'' \cap R$ is nonmeager and so $O_{X''} \neq \emptyset$.

Proof. (i) For each $x \in X'$ there is an open neighbourhood V_x of x such that $V_x \cap X$ is meager. Note that $X' = \bigcup\{V_x \cap X' : x \in X'\}$. Since X' is Lindelöff, there exists a sequence $\{x_i\}_{i=1}^{\infty}$ of points of X' such that $\bigcup\{V_x \cap X' : x \in X'\} = \bigcup_{i=1}^{\infty}(V_{x_i} \cap X')$. Note that the set $V_{x_i} \cap X' \subset V_{x_i} \cap X$ is meager for each i. So the set X' is also meager.

(ii) Let us note that $X \setminus X'' = (X \cap (R \setminus \mathrm{Cl}_R O_X)) \cup (X \cap \mathrm{Bd}_R O_X) \cup X'$ and each term of the union is meager. Hence, the union $X \setminus X''$ is meager. Since X is nonmeager, we have $X'' \neq \emptyset$.

(iii) Since $V \cap X'' \neq \emptyset$, the set $V \cap X = V \cap (X'' \cup (X \setminus X'')) = (V \cap X'') \cup (V \cap (X \setminus X''))$ is nonmeager. By (ii) the set $V \cap (X \setminus X'')$ is meager. Hence, the set $(V \cap X'')$ is nonmeager. ■

COROLLARY 2.1. *Let X be a nonmeager set in R and X' be Lindelöf. Then*

(i) $X'' \subset \mathrm{Cl}_R(O_{X''})$;

(ii) the set $X \setminus O_{X''}$ is meager;

(iii) for any non-empty open subset V of $O_{X''}$ the set $V \cap X''$ is nonmeager.

Proof. Let us note that by Proposition 2.1 (iii) the set X'' is nonmeager and $O_{X''} \neq \emptyset$.

(i) Assume that $Y = X'' \setminus \mathrm{Cl}_R(O_{X''}) = X'' \cap (R \setminus \mathrm{Cl}_R(O_{X''}) \neq \emptyset$. It follows from Lemma 2.1 that the set Y is nowhere dense in R. Observe also that the set $R \setminus \mathrm{Cl}_R(O_{X''})$ is open. Hence by Proposition 2.1 (iii), Y must be nonmeager. This is a contradiction.

(ii) By (i) we have $X \cap (R \setminus \mathrm{Cl}_R(O_{X''})) \subset X \setminus X''$. Now it follows from Proposition 2.1 (ii) that $X \cap (R \setminus \mathrm{Cl}_R(O_{X''}))$ is meager. Hence, the set $X \setminus O_{X''} = (X \cap (R \setminus \mathrm{Cl}_R(O_{X''}))) \cup (X \cap \mathrm{Bd}_R(O_{X''}))$ is also meager.

(iii) Since $V \cap X'' \neq \emptyset$, it follows from Proposition 2.1 (iii) that $V \cap X''$ is nonmeager. ■

THEOREM 2.1. *Let R be a hereditarily Lindelöf topological space, $\mathcal{A}$ be a non-empty set with $|\mathcal{A}| \leq \aleph_0$ and $X(\alpha)$ be nonmeager in R for each $\alpha \in \mathcal{A}$. Then the* *set* $U = \bigcup\{(X(\alpha)) : \alpha \in \mathcal{A}\}$ *is equal to the union* $(O \setminus M) \cup N$ *for some open set* O *and some meager sets* M, N *iff* $O_{X''(\alpha)} \setminus U$ *is meager for each* $\alpha \in \mathcal{A}$.

Proof. "$\Rightarrow$". Let us assume that $U = (O \setminus M) \cup N$ for some open set O and some meager sets M, N, and the set $A(\alpha^*) = O_{X''(\alpha^*)} \setminus U$ is nonmeager for some $\alpha^* \in \mathcal{A}$. Since U is a nonmeager set, we have $O \neq \emptyset$.

It follows from Proposition 2.1 (iii) that $A''(\alpha^*)$ is nonmeager and so the set $V = O_{A''(\alpha^*)}$ is non-empty. Since $A''(\alpha^*) \subset A(\alpha^*) \subset O_{X''(\alpha^*)}$, we have $V = \mathrm{Int}_R(\mathrm{Cl}_R(A''(\alpha^*))) \subset \mathrm{Int}_R(\mathrm{Cl}_R(O_{X''(\alpha^*)})$. But $\mathrm{Int}_R(\mathrm{Cl}_R(O_{X''(\alpha^*)})) = \mathrm{Int}_R(\mathrm{Cl}_R(\mathrm{Int}_R(\mathrm{Cl}_R(X''(\alpha^*))))) = \mathrm{Int}_R(\mathrm{Cl}_R(X''(\alpha^*))) = O_{X''(\alpha^*)}$. So we get that $V \subset O_{X''(\alpha^*)}$. It follows from Corollary 2.1 (iii) that the set $X''(\alpha^*) \cap V$ is nonmeager.

Put $W = O \cap V$. Note that $W \neq \emptyset$. Really, if $O \cap V = \emptyset$ then we have that $X''(\alpha^*) \cap V \subset X(\alpha^*) \cap V \subset U \cap V = ((O \setminus M) \cup N) \cap V \subset N$. Recall that the set N is meager and the set $X''(\alpha^*) \cap V$ is nonmeager. This is a contradiction.

Since W is a non-empty subset of V, it follows from Corollary 2.1 (iii) that the set $A''(\alpha^*) \cap W$ is nonmeager. Note that $A''(\alpha^*) \subset A(\alpha^*) \subset R \setminus U$ and $W \subset O$. So $A''(\alpha^*) \cap W \subset O \setminus U = O \setminus ((O \setminus M) \cup N) \subset M$. Recall that the set M is meager. Consequently, the set $A''(\alpha^*) \cap W$ is also meager. This is again a contradiction. The necessity is proved.

"$\Leftarrow$". Let us put $O = \bigcup\{O_{X''(\alpha)} : \alpha \in \mathcal{A}\}$, $M = \bigcup\{O_{X''(\alpha)} \setminus U : \alpha \in \mathcal{A}\}$ and $N = \bigcup\{X(\alpha) \setminus O_{X''(\alpha)} : \alpha \in \mathcal{A}\}$. Note that the set O is non-empty, open and M, N are meager sets (see Corollary 2.1 (ii) for N). Observe also that $U = U \cap R = (U \cap O) \cup ((R \setminus O) \cap U)$, $U \cap O = O \setminus (O \setminus U)$, $M_1 = O \setminus U \subset M$, $N_1 = (R \setminus O) \cap U \subset N$. Thus $U = (O \setminus M_1) \cup N_1$, where the set O is non-empty and open, and the sets M_1, N_1 are meager. The sufficiency is also proved. ∎

Theorem 3.2. *Let* R *be a hereditarily Lindelöf topological space,* $\mathcal{A}$ *be a non-empty set with* $|\mathcal{A}| \leq \aleph_0$ *and* $X(\alpha)$ *be nonmeager in* R *for each* $\alpha \in \mathcal{A}$. *Then the set* $R \setminus (\bigcup\{(X(\alpha)) : \alpha \in \mathcal{A}\})$ *contains the difference* $O \setminus M$ *for some non-empty open set* O *and some meager set* M *if and only if* $\mathrm{Cl}_R(\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \neq R$.

Proof. Let us put $U = \bigcup\{(X(\alpha)) : \alpha \in \mathcal{A}\}$.

"$\Rightarrow$". Assume that $R \setminus U$ contains the set $O \setminus M$, where O is a non-empty open subset of R and M is a meager set.

So $\bigcup\{X(\alpha) \cap O : \alpha \in \mathcal{A}\} = (\bigcup\{X(\alpha) : \alpha \in \mathcal{A}\}) \cap O = U \cap O \subset M$. Note that for each $\alpha \in \mathcal{A}$ we have $X''(\alpha) \cap O \subset X(\alpha) \cap O \subset U \cap O$. Since M is meager, for each $\alpha \in \mathcal{A}$ the set $X''(\alpha) \cap O$ must be meager. But it follows from Proposition 2.1 (iii) that $\bigcup\{X''(\alpha) \cap O : \alpha \in \mathcal{A}\} = (\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \cap O = \emptyset$.

So $\mathrm{Cl}_R(\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \neq R$. The necessity is proved.

"$\Leftarrow$". Note that

$$R \setminus U = R \setminus \bigcup\{X(\alpha) : \alpha \in \mathcal{A}\} = R \setminus \bigcup\{X''(\alpha) \bigcup (X(\alpha) \setminus X''(\alpha)) : \alpha \in \mathcal{A}\} =$$
$$R \setminus ((\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \cup (\bigcup\{X(\alpha) \setminus X''(\alpha) : \alpha \in \mathcal{A}\})) =$$

$(R \setminus \bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \setminus \bigcup\{X(\alpha) \setminus X''(\alpha) : \alpha \in \mathcal{A}\} \supset$
$(R \setminus \mathrm{Cl}_R(\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\})) \setminus \bigcup\{X(\alpha) \setminus X''(\alpha) : \alpha \in \mathcal{A}\}$.

Since $\mathrm{Cl}_R(\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \neq R$, the set $O = R \setminus \mathrm{Cl}_R(\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \neq \emptyset$. Put $M = \bigcup\{X \setminus X''(\alpha) : \alpha \in \mathcal{A}\}$. Since $|\mathcal{A}| \leq \aleph_0$ and for each $\alpha \in \mathcal{A}$ the set $X \setminus X''(\alpha)$ is meager (see Proposition 2.1 (ii)), we have that even M is meager. Moreover, $R \setminus U \supset O \setminus M$. The sufficiency is also proved. ■

3. Unions of Vitali sets and the Vitali (resp. Baire) property

PROPOSITION 3.1. *Let $S(i)$ be a Vitali set for each $i \leq n$, where n is some integer ≥ 1. Then $\bigcup_{i=1}^n S(i)$ does not possess the Vitali property. (Hence $\bigcup_{i=1}^n S(i)$ does not possess the Baire property.)*

Proof. Put $U = \bigcup_{i=1}^n S(i)$. Assume that there is a non-empty open set O and a meager set M such that $U \supset O \setminus M$. Since $O \neq \emptyset$ and U consists of finitely many Vitali sets, $E_\alpha \cap (O \setminus U) \neq \emptyset$ for each $\alpha \in I$. So we can find a Vitali set S such that $S \subset O$ and $S \cap U = \emptyset$. Hence $S \subset M$. Note that S is nonmeager and M is meager. This is a contradiction. ■

REMARK 3.1. Since equality $(*)$ is valid, Proposition 3.1 can not be extended in general onto countable unions. However we can prove the following statement.

THEOREM 3.1. *If S is a Vitali set and $\mathcal{A}$ is a non-empty proper subset of $\mathbb{Q}$ then the set $U(S, \mathcal{A}) = \bigcup\{S_r : r \in \mathcal{A}\}$ does not possess the Baire property.*

Proof. Put $V = \mathrm{Int}_{\mathbb{R}}(\mathrm{Cl}_{\mathbb{R}}\, S'')$ and $V_r = T_r(V)$ for each $r \in \mathbb{Q}$. Note that $V_r = \mathrm{Int}_{\mathbb{R}}(\mathrm{Cl}_{\mathbb{R}}(S_r)'')$ for each $r \in \mathbb{Q}$. Recall that $V \neq \emptyset$ (see Proposition 2.1 (iii)) and hence $V_r \neq \emptyset$ for each $r \in \mathbb{Q}$.

Assume that $U(S, \mathcal{A})$ possesses the Baire property.

CLAIM. *For any $r_1 \in \mathcal{A}$ and $r_2 \in \mathbb{Q} \setminus \mathcal{A}$ we have $V_{r_1} \cap V_{r_2} = \emptyset$.*

Proof. Suppose that $W = V_{r_1} \cap V_{r_2} \neq \emptyset$. Since W is a non-empty open subset of V_{r_2} it follows from Corollary 2.1 (iii) that the set $W \cap (S_{r_2})''$ is nonmeager. Note also that $W \cap (S_{r_2})'' \subset V_{r_1} \cap S_{r_2} \subset V_{r_1} \setminus U(S, \mathcal{A})$. Hence by Theorem 2.1 the set $U(S, \mathcal{A})$ does not possess the Baire property. This is a contradiction. So $W = \emptyset$. The claim is proved.

We continue with the proof of the theorem. It follows from the claim that $(\bigcup\{V_{r_1} : r_1 \in \mathcal{A}\}) \cap (\bigcup\{V_{r_2} : r_2 \in \mathbb{Q} \setminus \mathcal{A}\}) = \emptyset$. But the family $\{V_r : r \in \mathbb{Q}\}$ is evidently an open cover of $\mathbb{R}$. Thus we have a contradiction with the connectedness of $\mathbb{R}$. The theorem is proved. ■

THEOREM 3.2. *If S is a Vitali set and $\mathcal{A}$ is a non-empty proper subset of $\mathbb{Q}$ then the set $\bigcup\{S_r : r \in \mathcal{A}\}$ possesses the Vitali property if and only if $\mathrm{Cl}_{\mathbb{R}}(\bigcup\{(S_r)'' : r \in \mathbb{Q} \setminus \mathcal{A}\}) \neq \mathbb{R}$.*

Proof. Note that $\bigcup\{S_r : r \in \mathcal{A}\} = \mathbb{R} \setminus \bigcup\{S_r : r \in \mathbb{Q} \setminus \mathcal{A}\}$. Apply then Theorem 2.2. ■

4. Concluding remarks

Some of our results can be considered as particular answers to the following general problem: *Let G be a topological group acting on a topological space X. Assume that $A \subset G$, $B \subset X$, and let $\mathcal{P}$ be a topological property. For what A, B does $A \cdot B$ possess (or not possess) the property $\mathcal{P}$?*

Really, in this paper we have studied the additive group $G = \mathbb{R}$ of the real numbers acting on the real line $X = \mathbb{R}$ by the natural way: $x \cdot y = x + y$, where $x \in G$ and $y \in X$, and the Baire property (respectively, the Vitali property). It is easy to see that some of our results can be rewritten as follows.

(a) If A is finite and B is the union of finitely many Vitali sets then $A \cdot B$ does not possess the Baire property (respectively, the Vitali property) (Proposition 3.1);

(b) if A is a non-empty proper subset of $\mathbb{Q}$ and B is a Vitali set then $A \cdot B$ does not possess the Baire property (Theorem 3.1);

(c) if A is a non-empty proper subset of $\mathbb{Q}$ and B is a Vitali set then $A \cdot B$ possess the Vitali property iff $\mathrm{Cl}_{\mathbb{R}}(\bigcup\{(r \cdot B)'' : r \in \mathbb{Q} \setminus \mathcal{A}\}) \neq \mathbb{R}$ (Theorem 3.2).

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