Results on Finite Time and Practical Stability of Continuous Time Delay Systems

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This paper gives sufficient conditions for the attractive practical and finite time stability of linear continuous time delay systems of the form $\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau)$.

When we consider finite time stability, these new, delay dependent conditions are derived using the approach based on Lyapunov-Krassovski functional.

In this case the functional need not to have: a) properties of positivity in whole state space and b) negative derivatives along system trajectories.

Key words: continuous time delay systems, finite time stability, practical stability

1. CLASS OF SYSTEMS TO BE CONSIDERED

The problem of investigation of time delay systems has been exploited over many years.

Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc.

The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability.

Consequently, the problem of stability analysis for this class of systems has been one of the main interests for many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

When the general time delay systems are considered, in the existing stability criteria, mainly two ways of approach have been adopted.

Namely, one direction is to contrive the stability

condition which does not include the information on the delay, and the other is the method which takes it into account.

The former case is often called the delay independent criteria and it generally provides simple algebraic conditions.

Their stability deserves great attention.

The former will need to create these components, incorporating the applicable criteria that follow.

2. STABILITY CONCEPT

In practice one is not only interested in system stability (e.g. in sense of Lyapunov), but also in bounds of system trajectories.

A system could be stable but completely useless because it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain sub-sets of state-space, which are a priori defined in a given problem.

Besides, it is of particular significance to concern the behavior of dynamical systems only over a finite time interval. These bound properties of system responses, i. e. the solution of system models, are very important from the engineering point of view. Realizing this fact, numerous definitions of the so-called technical and practical stability were introduced.

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Roughly speaking, these definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and allowable perturbation of system response. In the engineering applications of control systems, this fact becomes very important and sometimes crucial, for the purpose of characterizing in advance, in quantitative manner, possible deviations of system response.

Thus, the analysis of these particular bound properties of solutions is an important step, which precedes the design of control signals, when finite time or practical stability control is concerned.

Due to limited space we shall be only concerned with the countionos case.

Motivated by the brief discussion on practical stability in [1] various notations of stability over finite time interval, for continual-time systems and constant set trajectory bounds have been introduced.

Further development of these results were due to many other authors.

3. CHRONOLOGICAL PREVIEW OF THE PREVIOUS RESULTS

In the short overview, that follows, we shall be familiar only with results achieved for continuous linear time delay systems (LCTDS) in the area of non-Lyapunov stability.

3.1. Practical and Finite Time Stability – Time Delay Systems

In the context of finite or practical stability for particular class of nonlinear singularly perturbed multiple time delay systems various results were, the first time, obtained in [2].

It seems that their definitions are very similar to those in [2-5] clearly adopted for time delay systems. It should be noticed that those definitions significantly differ from definitions presented by the authors of this paper.

In the context of finite time and practical stability for linear continuous time delay systems, various results were first obtained in [6-8].

In [6] and [9] some basic results from the area of finite time and practical stability were extended to the particular class of linear continuous time delay systems.

Stability sufficient conditions dependent on delay, expressed in terms of time delay fundamental system matrix, have been derived.

Matrix measure approach has been, for the first time, applied in [6-8] for the analysis of practical and finite time stability of linear time delayed systems.

Based on Coppel's inequality and introducing matrix measure approach one provides a very simple

delay-dependent sufficient conditions of practical and finite time stability with no need for time delay fundamental matrix calculation.

Another approach, based on very well known Bellman-Gronwall Lemma, was applied in [10], to provide new, more efficient sufficient delay-dependent conditions for checking finite and practical stability of continuous systems with state delay.

Collection of all previous results and contributions were presented in [11] with overall comments and slightly modified Bellman-Gronwall approach.

Finally, modified Bellman-Gronwall principle, has been extended to the particular class of continuous non-autonomous time delayed systems operating over the finite time interval, [12].

4. NOTATION AND PRELIMINARIES

In our paper we present two quite different approaches to this problem and continue our investigation in usual way.

Namely, our first result is expressed directly in terms of eigenvalues of basic system matrices and naturally occurring in the system model and avoid the need to introduce any canonical form, or transformation into statements of a theorem.

In the second case the geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions.

This fact makes possible the construction of Lyapunov and non-Lyapunov stability theory even for the (LCTDS) in that sense that attractive property is equivalent to the existence of symmetric, positive definite solutions to a general form of Lyapunov matrix equation incorporating conditions which refer to boundedness of solutions.

First method is based on a classical approach mostly used in deriving sufficient delay independent conditions for finite time stability.

In the second case a new definition is introduced, based on the attractivity properties of system solution, which can be treated as something analogous to quasi-contractive stability, [4-5].

5. NOTATION AND PRELIMINARIES

$\mathbb{R}$	Real vector space
$\mathbb{C}$	Complex vector space
I	Identity matrix
$F = (f_{ij}) \in \square^{n \times n}$	Real matrix
F^T	Transpose of matrix F

$F > 0$	Positive definite matrix
$F \geq 0$	Positive semi definite matrix
$\lambda(F)$	Eigenvalue of matrix F

$\|F\|$ Euclidean matrix norm of $F = \sqrt{\lambda_{\max}(F^T F)}$.

Generally, the nonlinear differential control systems with time delay can be written as:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t-\tau), \mathbf{u}(t)), \quad t \geq 0 \\ \mathbf{x}(t) &= \boldsymbol{\psi}(t), \quad -\tau \leq t \leq 0\end{aligned}, \quad (1)$$

Where $\mathbf{x}(t) \in \mathbb{R}^n$ is a state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ is a control vector, $\boldsymbol{\psi} \in C = C([- \tau, 0], \mathbb{C}^n)$ is an admissible initial state functional, $C = C([- \tau, 0], \mathbb{C}^n)$ is the *Banach* space of continuous functions mapping the interval $[- \tau, 0]$ into $\mathbb{C}^n$ with topology of uniform convergence.

Vector function satisfies:

$$\mathbf{f}(\cdot): \mathfrak{T} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \quad (2)$$

and is assumed to be smooth enough to assure existence and uniqueness of solutions over a time interval

$$\mathfrak{T} = [t_0, (t_0 + T)] \in \mathbb{R} \quad (3)$$

as well as the continuous dependence of the solutions denoted as $\mathbf{x}(t, t_0, \mathbf{x}_0)$ with respect to t and initial data.

Quantity T may be either a positive real number or symbol $+\infty$, so that finite time stability and practical stability can be treated simultaneously, respectively.

In general, it is not required that

$$\mathbf{f}(t, \mathbf{0}, \mathbf{0}) = \mathbf{0}, \quad (4)$$

for autonomous system, which means that the origin of the state space is not necessarily required to be an equilibrium state.

Let $\mathbb{R}^n$ denote state space of the system (1) and $\|(\cdot)\|$ Euclidean norm.

Let $V: \mathfrak{T} \times \mathbb{R}^n \rightarrow \mathbb{R}$, be tentative aggregate function, so that $V(t, \mathbf{x}(t))$ is bounded for and for which $\|\mathbf{x}(t)\|$ is also bounded.

Define Eulerian derivative of $V(t, \mathbf{x}(t))$ along the trajectory of system (1), with

$$\dot{V}(t, \mathbf{x}(t)) = \frac{\partial V(t, \mathbf{x}(t))}{\partial t} + [\text{grad} V(t, \mathbf{x}(t))]^T \mathbf{f}(\cdot) \quad (5)$$

For time-invariant sets it is assumed: $S(\cdot)$ is a bounded, open set.

Let S_β be a given set of all allowable states of system for $\forall t \in \mathfrak{T}$.

Set S_α , $S_\alpha \subset S_\beta$ denotes set of all allowable initial states.

Sets S_α , S_β are connected and a priori known.

$\lambda(\cdot)$ denotes the eigenvalues of a matrix $(\cdot)$.

$\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues, respectively.

6. SOME PREVIOUS RESULTS

6.1 Introduction and general considerations

Consider a linear continuous singular system with state delay, described by:

$$\dot{\mathbf{x}}(t) = A_0 \mathbf{x}(t) + A_1 \mathbf{x}(t-\tau), \quad (6.a)$$

with known vector valued function of initial conditions:

$$\mathbf{x}(t) = \boldsymbol{\psi}_x(t), \quad -\tau \leq t \leq 0, \quad (6.b)$$

where A_0 and A_1 are constant matrices with appropriate dimensions.

Stability definitions

Definition 1. System (6) is stable w.r.t. $\{\alpha, \beta, -\tau, T, \|\mathbf{x}\|\}$, $\alpha \leq \beta$ if for any trajectory $\mathbf{x}(t)$ condition $\|\mathbf{x}_0\| < \alpha$ implies $\|\mathbf{x}(t)\| < \beta$ $\forall t \in [-\Delta, T]$, $\Delta = \tau_{\max}$, [2].

Definition 2. Autonomous system (6) is contractively stable w.r.t. $\{\alpha, \beta, \gamma, -\tau, T, \|\mathbf{x}\|\}$, $\gamma < \alpha < \beta$, iff for any trajectory $\mathbf{x}(t)$ condition $\|\mathbf{x}_0\| < \alpha$ implies:

(i) Stability with respect to $\{\alpha, \beta, -\tau, T, \|\mathbf{x}\|\}$,

(ii) There exist $t^* \in]0, T[$ such that $\|\mathbf{x}(t)\| < \gamma$ for all $\forall t \in]t^*, T[$, [2].

Definition 3. System (6.a) satisfying initial condition (6.b) is finite time stable w.r.t. to $\{\zeta(t), \beta, \mathfrak{T}\}$ iff:

$$\|\boldsymbol{\psi}_x(t)\| < \zeta(t),$$

implies:

$$\|x(t)\|_2 < \beta, \quad t \in \mathfrak{T},$$

$\zeta(t)$ being scalar function with the property $0 < \zeta(t) \leq \alpha$, $-\tau \leq t \leq 0$, where α is a real positive number such that $\beta \in \mathfrak{R}$ and $\beta > \alpha$, [6-9], see Fig.1.

Figure 1 - Illustration of the preceding Definition

B. Dependent delay stability conditions

Stability theorems

Theorem 1. The autonomous system (6.a) with the initial function (6.b) is finite time stable w.r.t. $\{\alpha, \beta, \tau, \mathfrak{T}\}$, if the following condition is satisfied:

$$\|\Phi(t)\|_2 < \frac{\sqrt{\beta/\alpha}}{1 + \tau \|A_1\|_2}, \quad \forall t \in \mathfrak{T} \quad (7)$$

where $\|(\cdot)\|$ is Euclidean norm and $\Phi(t)$ is fundamental matrix of system (6), see [2], [9].

When $\tau = 0$ or $\|A_1\| = 0$, the problem is reduced to the case of the ordinary linear systems [13].

Theorem 2. The autonomous system (6.a) with initial function (6.b) is finite time stable with respect to $\{\alpha, \beta, \tau, T\}$ if the following condition is satisfied:

$$e^{\mu_2(A_0)t} < \frac{\sqrt{\beta/\alpha}}{1 + \tau \|A_1\|_2}, \quad \forall t \in \mathfrak{T} \quad (8)$$

where $\|(\cdot)\|$ denotes the Euclidean norm, [7].

Theorem 3. The autonomous system (6.a) with the initial function (6.b) is finite time stable with respect to $\{\alpha, \beta, \tau, \mathfrak{T}, \mu_2(A_0) \neq 0\}$ if the following condition is satisfied [4]:

$$e^{\mu_2(A_0)t} < \frac{\beta/\alpha}{1 + \mu_2^{-1}(A_0) \cdot \|A_1\|_2 \cdot (1 - e^{-\mu_2(A_0)\tau})}, \quad \forall t \in \mathfrak{T} \quad (9)$$

[7].

Theorem 4. The autonomous system (6.a) with the initial function (6.b) is finite time stable with respect to $\{\sqrt{\alpha}, \sqrt{\beta}, \tau, \mathfrak{T}, \mu(A_0) = 0\}$, if the following condition is satisfied

$$1 + \tau \|A_1\|_2 < \sqrt{\beta/\alpha}, \quad \forall t \in \mathfrak{T}, \quad (10) \quad [8]$$

C. Independent delay stability conditions

Stability theorems

Results that will be presented in the sequel enables one to check finite time stability of the autonomous system to be considered, namely the system given by (6.a) and (6.b), without finding the fundamental matrix or corresponding matrix measure. (6) can be rewritten in its general form as:

$$\begin{aligned} x(t_0 + \vartheta) &= \psi_x(\vartheta) \\ -\tau &\leq \vartheta \leq 0 \\ \psi_x(\vartheta) &\in C[-\tau, 0] \end{aligned} \quad (11)$$

where t_0 is the initial time of observation of the system (6) and $C[-\tau, 0]$ is a Banach space of continuous functions over a time interval of length τ , mapping the interval $[(t-\tau), t]$ into $\mathfrak{R}^n$ with the norm defined in the following manner:

$$\|\psi\|_C = \max_{-\tau \leq \vartheta \leq 0} \|\psi(\vartheta)\| \quad (12)$$

It is assumed that the usual smoothness conditions are present so that there is no difficulty with questions of existence, uniqueness, and continuity of solutions with respect to initial data.

Moreover one can write:

$$x(t_0 + \vartheta) = \psi_x(\vartheta) \quad (13)$$

as well as:

$$\dot{x}(t_0) = f(t_0, \psi_x(\vartheta)) \quad (14)$$

Theorem 5. System (6.a) with initial function (6.b) is finite time stable with respect to $\{\alpha, \beta, t_0, \mathfrak{T}\}$, if the following condition is satisfied:

$$(1 + (t - t_0) \sigma_{\max}(\cdot))^2 e^{2(t-t_0)\sigma_{\max}(\cdot)} < \frac{\beta}{\alpha}, \quad \forall t \in \mathfrak{T} \quad (15)$$

$\sigma_{\max}(\cdot)$ being the largest singular value of matrix $(\cdot)$, namely [10-11]:

$$\sigma_{\max}(\cdot) = \sigma_{\max}(A_0) + \sigma_{\max}(A_1) \quad (16)$$

In the case when in the Theorem 5: $A_1 = 0$, e.g. A_1 is null matrix, we have the result similar to that presented in [13].

7. MAIN RESULT

Finite time stability

D) Delay dependent stability conditions Stability definition

Definition 4. The system (6.a) satisfying initial condition (6.b) is finite time stable w.r.t. to $\{\alpha, \beta, T\}$ if:

$$\sup_{t \in [-\tau, 0]} \Psi_x^T(t) \Psi_x(t) \leq \alpha \Rightarrow \mathbf{x}^T(t) \mathbf{x}(t) < \beta, \quad \forall t \in \mathfrak{T}$$

Stability theorems

Theorem 6. Suppose $\mu(-A_0) < 0$.

System (6.a) with initial function (6.b), is finite time stable with respect to $\{t_0, \mathfrak{T}, \alpha, \beta, \|\cdot\|^2\}$, $\alpha < \beta$ if:

$$(1 + \tau \|A_1\|)^2 e^{2\mu_2(A_0)t} < \frac{\beta}{\alpha}, \quad \forall t \in \mathfrak{T} \quad (17)$$

$\mu_2(A_0)$ being any matrix measure.

To prove *finite time stability* one should start with the solution of (6.a) with given initial function (6.b).

To this end, one can get:

$$\mathbf{x}(t) = \Phi(t) \Psi(0) + \int_{-\tau}^0 \Phi(t - \vartheta - \tau) A_1 \Psi(\vartheta) d\vartheta \quad (18)$$

Using the above equation:

$$\begin{aligned} \mathbf{x}^T(t) \mathbf{x}(t) &= \Psi^T(0) \Phi^T(t) \Phi(t) \Psi(0) \\ &+ \Psi^T(0) \Phi^T(t) \left(\int_{-\tau}^0 \Phi(t - \eta - \tau) A_1 \Psi(\eta) d\eta \right) \\ &+ \left(\int_{-\tau}^0 \Psi^T(\vartheta) A_1^T \Phi^T(t - \vartheta - \tau) d\vartheta \right) \Phi(t) \Psi(0) \\ &+ \left(\int_{-\tau}^0 \Psi^T(\vartheta) A_1^T \Phi^T(t - \vartheta - \tau) d\vartheta \right) \times \\ &\times \left(\int_{-\tau}^0 \Phi(t - \eta - \tau) A_1 \Psi(\eta) d\eta \right) \end{aligned} \quad (19)$$

and the following abbreviations:

$$\Phi(t) \Psi(0) = \mathbf{a}(t) \in \mathbb{R}^{n \times 1} \quad (20)$$

$$\Phi(t - \vartheta - \tau) A_1 \Psi(\vartheta) = \mathbf{b}(t, \vartheta) \in \mathbb{R}^{n \times 1} \quad (21)$$

it is more than obvious, that if one introduces:

$$\int_{-\tau}^0 \mathbf{b}(t, \vartheta) d\vartheta = \mathbf{c}(t) \in \mathbb{R}^{n \times 1} \quad (22)$$

then (53) becomes:

$$\begin{aligned} \mathbf{x}^T(t) \mathbf{x}(t) &= \Psi^T(0) \Phi^T(t) \Phi(t) \Psi(0) \\ &+ \mathbf{a}^T(t) \mathbf{c}(t) + \mathbf{c}^T(t) \mathbf{a}(t) + \mathbf{c}^T(t) \mathbf{c}(t) \end{aligned} \quad (23)$$

The very well known result, from the theory of quadratic forms, gives:

$$\Psi^T(0) (\Phi^T(t) \Phi(t)) \Psi(0) \leq \lambda_{\max}(t) \Psi^T(0) \Psi(0), \quad (24)$$

where:

$$\lambda_{\max}(t) = \max \sigma \{ \Phi^T(t) \Phi(t) \}, \quad (25)$$

and $\sigma(\cdot)$ denotes spectral operator.

It is easy to see that: $\mathbf{a}^T(t) \mathbf{c}(t) = f(t) \in \mathbb{R}^1$ and $\mathbf{c}^T(t) \mathbf{a}(t) = f(t) \in \mathbb{R}^1$ so it follows: $\mathbf{c}^T(t) \mathbf{a}(t) = \mathbf{c}(t) \mathbf{a}^T(t)$.

Now, one can write:

$$\mathbf{x}^T(t) \mathbf{x}(t) \leq \Lambda(t) \Psi^T(0) \Psi(0) + 2\mathbf{a}^T(t) \mathbf{c}(t) + \|\mathbf{c}(t)\|^2 \quad (26)$$

Moreover:

$$f(t) \leq |f(t)|, \quad \left| \int_a^b \varphi(t) dt \right| \leq \int_a^b |\varphi(t)| dt, \quad (27)$$

so, it follows:

$$\mathbf{a}^T(t) \mathbf{c}(t) \leq \|\Psi^T(0)\| \cdot \|\Phi^T(t)\| \cdot \int_{-\tau}^0 \|\mathbf{b}(t, \vartheta)\| d\vartheta \quad (28)$$

Eq. (23) may be now rewritten as:

$$\begin{aligned} \mathbf{x}^T(t) \mathbf{x}(t) &\leq \lambda_{\max}(t) \Psi^T(0) \Psi(0) \\ &+ 2 \|\Psi^T(0)\| \cdot \|\Phi^T(t)\| \cdot \|A_1\| \times \\ &\times \int_{-\tau}^0 \|\Phi(t - \vartheta - \tau)\| \cdot \|\Psi(\vartheta)\| d\vartheta \\ &+ \|A_1\|^2 \left(\int_{-\tau}^0 \|\Phi(t - \vartheta - \tau)\| \cdot \|\Psi(\vartheta)\| d\vartheta \right)^2 \end{aligned} \quad (29)$$

However, if:

$$\rho(t) \leq \|\Phi(t - \vartheta - \tau)\| \cdot \|\Psi(\vartheta)\| \leq \gamma(t), \quad \forall \vartheta \in [-\tau, 0] \quad (30)$$

then:

$$\rho(t) \tau \leq \int_{-\tau}^0 \|\Phi(t - \vartheta - \tau)\| \cdot \|\Psi(\vartheta)\| d\vartheta \leq \gamma(t) \tau \quad (31)$$

Under the basic assumption, it is easy to show, that if $\mu(-A_0) < 0$, then:

$$\begin{aligned} \|\Phi(t - \vartheta - \tau)\|_{\vartheta \in [-\tau, 0]} &\leq \|\Phi(t)\| \\ \|\Psi(\vartheta)\|_{\vartheta \in [-\tau, 0]} &< \sqrt{\alpha} \end{aligned} \quad (32)$$

as well as:

$$\|\Psi(0)\| < \sqrt{\alpha}, \quad \|\Phi^T(t)\| = \|\Phi(t)\| \quad (33)$$

so (26) becomes:

$$\begin{aligned} \mathbf{x}^T(t) \mathbf{x}(t) &\leq \lambda_{\max}(t) \Psi^T(0) \Psi(0) \\ &+ 2 \|\Psi^T(0)\| \cdot \|\Phi^T(t)\| \|A_1\| \tau \sqrt{\alpha} \\ &+ \|\Phi^T(t)\|^2 \|A_1\|^2 \tau^2 \alpha \end{aligned} \quad (34)$$

Also, based on:

$$\lambda_{\max}(t) \leq \|\Phi^T(t)\Phi(t)\| \leq \|\Phi^T(t)\| \cdot \|\Phi(t)\| = \|\Phi(t)\|^2 \quad (35)$$

it follows:

$$\begin{aligned} \mathbf{x}^T(t)\mathbf{x}(t) &\leq \|\Phi(t)\|^2 \Psi^T(0)\Psi(0) \\ &\quad + 2\|\Psi^T(0)\| \cdot \|\Phi^T(t)\|^2 \|A_1\| \tau \sqrt{\alpha} \\ &\quad + \|\Phi^T(t)\|^2 \|A_1\|^2 \tau^2 \alpha \end{aligned} \quad (36)$$

Using the known *Coppell's* inequality:

$$\|\Phi(t)\| = \|e^{A_0 t}\| \leq e^{\mu(A_0)t} \quad (37)$$

and applying it to the inequality (36), having in mind that:

$$\Psi^T(0)\Psi(0) < \alpha, \quad \|\Psi^T(0)\| = \|\Psi(0)\| < \sqrt{\alpha} \quad (38)$$

one can get:

$$\begin{aligned} \mathbf{x}^T(t)\mathbf{x}(t) &\leq e^{2\mu_2(A_0)t} \alpha \\ &\quad + 2e^{2\mu_2(A_0)t} \|A_1\| \tau \alpha + e^{2\mu_2(A_0)t} \|A_1\|^2 \tau^2 \alpha \\ &\leq \alpha e^{2\mu_2(A_0)t} (1 + \tau \cdot \|A_1\|)^2 \end{aligned} \quad (39)$$

Applying the basic conditions (17), of Theorem 6, on the preceding inequality, one can get:

$$\mathbf{x}^T(t)\mathbf{x}(t) < \left(\frac{\sqrt{\beta/\alpha}}{1 + \tau \cdot \|A_1\|} \right)^2 \alpha (1 + \tau \cdot \|A_1\|)^2 < \beta, \quad \forall t \in \mathbb{R} \quad (40)$$

what have to be proved. Q.E.D.

Practical stability

Delay dependent stability conditions

Stability definition

Definition 5. System (6) with initial condition (7), is attractive practically stable with respect to $\{\alpha, \beta, T\}$

if $\sup_{t \in [-\tau, 0]} \Phi_x^T(t)\Phi_x(t) \leq \alpha$ implies: $\mathbf{x}^T(t)\mathbf{x}(t) < \beta$,

$t \in [0, T]$ with property that: $\lim_{t \rightarrow \infty} \mathbf{x}^T(t)\mathbf{x}(t) \rightarrow 0$.

In this section, some novel delay dependent stability conditions were analyzed. Prior to the derivation of the stability conditions, the related results are presented. The detailed derivation of these results was reported in [14].

Theorem 7. The system is described as in (6). If there exists a positive definite Hermitian self-adjoint

matrix P_0 for any given positive definite Hermitian self-adjoint matrix Q such that:

$$P_0(A_0 + P_1(0)) + (A_0 + P_1(0))^* P_0 = -Q \quad (41)$$

for $\kappa \in [0, \tau]$, $P_1(\kappa)$ satisfying:

$$\dot{P}_1(\kappa) = (A_0 + P_1(0))P_1(\kappa) \quad (42)$$

with $P_1(\tau) = A_1$ boundary condition and $P_1(\tau) = 0$, then system (6) is asymptotically stable.

It can be concluded that the key factor in the design of the Lyapunov function correlated to system (6) is the existence of at least one solution $P_1(t)$ of (42)

fulfilling the condition $P_1(\tau) = A_1$.

Consequently, it is required that the non-linear algebraic matrix equation:

$$e^{(A_0 + P_1(0))\tau} P_1(0) = A_1 \quad (43)$$

has minimum one solution for $P_1(0)$.

Based on Theorem 7, the asymptotic stability of the system can be determined knowing only one instead of all solutions of the particular nonlinear matrix equation.

With the use of a counterexample, it was proven in [15] that the above theorem is not correct in any case since it does not take into account all possible solutions of (42), i.e. (43).

Remark 1. When the following novel matrix is defined:

$$R \triangleq A_0 + P_1(0) \quad (44)$$

then condition (41) is transformed as:

$$P_0 R + R^* P_0 = -Q \quad (45)$$

Equation (45) presents a previously defined Lyapunov's equation valid for the systems with no time lag.

Condition (45) is valid iff R is a matrix that is stable, that is:

$$\operatorname{Re} \lambda_i(R) < 0 \quad (46)$$

is fulfilled, [16].

Remark 2. Expression (43) developed using matrix R . The same equation can be presented in its modified form as:

$$R - A_0 - e^{-R\tau} A_1 = 0 \quad (47)$$

and:

$$\det(R - A_0 - e^{-R\tau} A_1) = 0 \quad (48)$$

[16].

Replacing a matrix R with a scalar s in (46), the characteristic equation of the system was recalculated:

$$f(s) = \det(sI - A_0 - e^{-s\tau} A_1) = 0 \quad (49)$$

The following was:

$$\Sigma = \{s \mid f(s) = 0\} \quad (50)$$

a certain set of all solutions of the system, [16].

Consequently, a novel formulation of Theorem 7 is proposed as:

Theorem 8. Assuming that there exist the solution vector $P_1(0)$ of (43). Then, the system is asymptotically stable *iff* any of the following conditions are correct:

a) There exists matrix $P_0 = P_0^* > 0$ for any matrix $Q = Q^* > 0$ such that (8) is valid for *all* solutions $P_1(0)$ of (43).

b) Condition $\operatorname{Re} \lambda_i(R) < 0$ is valid for all solutions R of equation (47), [16].

Remark 3. A statement of Theorem 8 requires that the corresponding conditions are satisfied for any solution $P_1(0)$ of (43), or for any solution R of (47).

Remark 4. Analyzing the previous theorems, a question can be asked. Is it feasible and how one can calculate all solutions $P_1(0)$ of (10), or R of (14)?

This problem cannot be numerically solved directly since the number of solutions $P_1(0)$ or R is not previously known.

Furthermore, the proposed equations can pose a infinite number of solutions. However, to examine the stability effectively, the numerical formulation should be analyzed using the maximal solvent of (47) as in [16].

Definition 6. All roots λ_m of characteristic equation (49) of system (6) that satisfy the condition: $\operatorname{Re} \lambda_m = \max \operatorname{Re} s$, $s \in \Sigma$ are denoted to as the maximal root or eigenvalue of the same system., [16].

Definition 7. All solvents $R_{\max}$ of (47) with the spectrums congaing maximal eigenvalue λ_m of system (6) are denoted as the maximal solvents of (47), [16].

Based on Remark 4 and Definitions 6 and 4, Theorem 8 can be reformulated in the following modus.

Theorem 9. Let us assume that there exists the maximal solvent $R_{\max}$ of (47). In that case, system (6) is asymptotically stable if and only if any of two equivalent statements are fulfilled:

a) For any given matrix $Q = Q^* > 0$ there is a matrix $P_0 = P_0^* > 0$ such that (45) is fulfilled for solvent $R_{\max}$.

b) $\operatorname{Re} \lambda_i(R_{\max}) < 0$, [16].

After the previous definitions and theorems, our main result on the attractive practical (technical) stability of the analyzed system can be formulated.

Theorem 9. System (6) having initial conditions (7) is attractive practically stable with respect to $\{\alpha, \beta, T, \mu(-A_0) < 0\}$, $\alpha < \beta$, if the condition (45-50) are fulfilled:

$$a) (1 + \tau \|A_1\|)^2 e^{2\mu_2(A_0)t} < \frac{\beta}{\alpha}, \quad \forall t \in \mathfrak{I} \quad (51)$$

where:

$$\mu(A_0) = \frac{1}{2} \lambda_{\max}(A_0 + A_0^T) \quad (52)$$

where $\mu(\cdot)$ is a matrix measure.

b) There exists maximal solvent $R_{\max}$ of:

$$R - A_0 - e^{-R\tau} A_1 = 0 \quad (53)$$

$$c) \operatorname{Re} \lambda_i(R_{\max}) < 0. \quad (54)$$

8. NUMERICAL EXAMPLE

Our main contribution is exposed throughout results of Theorem 6.

Example Given a system of the form:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-1) \\ A_0 &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \end{aligned} \quad (55)$$

In order to verify stability properties of the system (55), w.r.t. $\{\alpha = 2, \beta = 70, T\}$ the system operation is simulated under the conditions $\psi(t) = [1 \ 1]^T$, $t \in [-\tau, 0]$

It is obvious that:

$$\begin{aligned} \psi^T(t) \psi(t) &= 2 = \alpha, \quad t \in [-\tau, 0] \\ \mu(-A_0) &= -1 \end{aligned}$$

Figure 2 show the the initial response the square norm of state vector $\mathbf{x}(t)$, eg. $\mathbf{x}^T(t)\mathbf{x}(t)$ of the system to be considered.

It is observed that the system (55) is not asymptotically stable.

Figure 2 - System trajectory

It is easy to see that, for $\beta = 70$, $T_{actual} = 0.8[s]$.

$$\mu(A_0) = \frac{1}{2} \lambda_{\max}(A_0 + A_0^T) = 2$$

$$\mu(-A_0) = \frac{1}{2} \lambda_{\max}(-A_0 + (-A_0^T)) = -1$$

$$\|A_1\| = \sqrt{\lambda_{\max}(A_1^T A_1)} = 2$$

Maximal estimated time interval of finite time stability can be achieved throughout basic formula:

$$\left(1 + \tau \|A_1\|\right)^2 e^{2\mu(A_0)T_{est}^{\max}} = 3^2 \times e^{2 \times 2 \times T_{est}^{\max}} < \frac{70}{2},$$

with:

$$T_{est}^{\max} = 0,576[s] < T_{actual} = 0.8[s],$$

and is less conservative than that given by Theorem 3, without any limitations, giving only $T_{est}^{\max} = 0,34[s]$.

8. CONCLUSION

Quite new sufficient delay-dependent criteria for finite time stability of linear continuous time delay systems are derived, motivated by some ideas earlier presented in [2], [4-6] and [17].

It can be shown that this result is less conservative than some those established before.

A numerical example is included to show advantage of method proposed.

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REZIME

NOVI REZULTATI U IZUČAVANJU STABILNOSTI NA KONAČNOM VREMENSKOM INTERVALU I PRAKTIČNE STABILNOSTI VREMENSKI NEPREKIDNIH SISTEMA SA KAŠNENJEM

U ovom radu su izloženi dovoljni uslovi atraktivne praktične stabilnosti, kao i uslovi stabilnosti na konačnom vremenskom intervalu vremenski neprekidnih Sistema sa kašnjenjem koji su predstavljeni

svojim modelom u obliku: $\dot{\mathbf{x}}(t) = A_0 \mathbf{x}(t) + A_1 \mathbf{x}(t - \tau)$

Kada se razmatrala stabilnost na konačnom vremenskom intervalu, ovi novi, uslovi zavisni od iznosa čisto vremenskog kašnjenja, izvedeni su na bazi korišćenja Lyapunov-Krassovsk-og funkcionala.

U ovom slučaju, funkcional ne mora da poseduje: a) pozitivnu određenost u celom prostoru stanja i b) niti negativnu određenost njegovog izvoda duž trajektorija sistema.

Ključne reči: *vremenski neprekidni sistemi sa kašnjenjem, stabilnost na konačnom vremenskom interval, praktična stabilnost*