A NOTE ON THE MEIR-KEELER
THEOREM IN THE CONTEXT OF b-METRIC SPACES

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Abstract:
In this note we consider the famous Meir-Keeler’s theorem in the context of b-metric spaces. Our result generalizes, improves, compliments, unifies and enriches several known ones in the existing literature. Also, our proof of Meir-Keeler’s theorem in the context of standard metric spaces is much shorter and nicer than the ones in (Čirić, 2003) and (Meir & Keeler, 1969, pp.326-329).

Keywords: b-metric space, b-complete, b-Cauchy, Meir-Keeler conditions, Picard sequence.

Definitions, notations and preliminaries

Let \((X, d)\) be a standard metric space and \(f : X \to X\) be a self-mapping. In the context of these spaces, the following (Meir-Keeler) conditions are well known: For each \(\varepsilon > 0\) there exists \(\delta = \delta(\varepsilon) > 0\) such that for all \(x, y \in X\) holds

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One says that the mapping $f$ defined on the standard metric space $(X, d)$ is contractive if $d(fx, fy) < d(x, y)$ holds, whenever $x \neq y$.

For more details, see (Čirić, 2003, pp.30-33, pp.56-58).

In 1969, Meir-Keeler proved the following:

**Theorem 1** (Meir & Keeler, 1969, pp.326-329, Theorem) Let $(X, d)$ be a complete metric space and let $f$ be a self-mapping on $X$ satisfying (1). Then $f$ has a unique fixed point, say $u \in X$, and for each $x \in X$, $\lim_{n \to \infty} f^n x = u$.

Inspired by the above Meir-Keeler theorem, Čirić proved the following, slightly more general result:

**Theorem 2** (Čirić, 2003, Theorem 2.5) Let $(X, d)$ be a complete metric space and let $f$ be a self-mapping on $X$ satisfying (2). Then $f$ has a unique fixed point, say $u \in X$, and for each $x \in X$, $\lim_{n \to \infty} f^n x = u$.

The example which follows shows that Čirić’s result is a proper generalization of the famous Meir-Keeler theorem:

**Example 3** Let $X = [0, 1] \cup \{3n - 1\}_{n \in \mathbb{N}} \cup \{3n + \frac{1}{3n}\}_{n \in \mathbb{N}}$ be a subset of real numbers with the Euclidean metric and let $f$ be a self-mapping on $X$ defined by

$$fx = 0, \text{ if } 0 \leq x \leq 1 \text{ and } x \in [3n - 1]_{n \in \mathbb{N}},$$

$$fx = 1, \text{ if } x \in \left\{3n + \frac{1}{3n}\right\}_{n \in \mathbb{N}}.$$
Then one can verify that \( f \) satisfies (2) while it does not satisfy Meir-Keeler condition (1). For all details, see (Čirić, 2003, p.33).

**Remark 1** Both previous theorems are true if the self-mapping \( f : X \to X \) satisfies condition (3).

Bakhtin (Bakhtin, 1989, pp.26-37) and Czerwik (Czerwik, 1993, pp.5-11) introduced \( b \)-metric spaces (as a generalization of metric spaces) and proved the contraction principle in this context. In the last period, many authors have obtained fixed point results for single-valued or set-valued functions, in the context of \( b \)-metric spaces. Now we give the definition of a \( b \)-metric space:

**Definition 1.1** (Bakhtin, 1989, pp.26-37), (Czerwik, 1993, pp.5-11)

Let \( X \) be a nonempty set and let \( s \geq 1 \) be a given real number. The function \( d : X \times X \to [0, \infty) \) is said to be a \( b \)-metric, and only if, for all \( x, y, z \in X \) the following conditions hold:

\begin{enumerate}
  \item \( d(x, y) = 0 \) if, and only if, \( x = y \);
  \item \( d(x, y) = d(y, x) \);
  \item \( d(x, z) \leq s[d(x, y) + d(y, z)] \).
\end{enumerate}

A triplet \( (X, d, s \geq 1) \) is called a \( b \)-metric space with the coefficient \( s \).

It should be noted that the class of \( b \)-metric spaces is effectively larger than that of standard metric spaces, since a \( b \)-metric is a metric when \( s = 1 \). The following example shows that, in general, a \( b \)-metric does not necessarily need to be a metric (Chandok et al, 2017, pp.331-345), (Došenović et al, 2017, pp.851-865), (Dubey et al, 2014), (Dung & Hang, 2018, pp.298-304), (Faraji & Nourouzi, 2017, pp.77-86), (Jovanović et al, 2010), (Jovanović, 2016), (Kir & Kiziltunc, 2016, pp.13-16), (Kirk & Shahzad, 2014).

**Example 4** Let \( (X, \rho) \) be a standard metric space, and \( d(x, y) = (\rho(x, y))^p, p > 1 \) is a real number. Then \( d \) is a \( b \)-metric with \( s = 2^{p-1} \), but \( d \) is not a standard metric on \( X \).

Otherwise, for more concepts such as \( b \)-convergence, \( b \)-completeness, \( b \)-Cauchy and \( b \)-closed set in \( b \)-metric spaces, we refer...

The following two lemmas are very significant in the theory of a fixed point in the context of $b$-metric spaces.

**Lemma 1.2** (Jovanović et al, 2010, p.15, Lemma 3.1) Let \( \{a_n\}_{n \in \mathbb{N} \cup \{0\}} \) be a sequence in a $b$-metric space \((X,d,s \geq 1)\) such that \(d(a_n, a_{n+1}) \leq kd(a_{n-1}, a_n)\) for some \(k \in \left[0, \frac{1}{s}\right]\), and each \(n = 1,2,...\). Then \(\{a_n\}\) is a $b$-Cauchy sequence in a $b$-metric space \((X,d,s \geq 1)\).

**Lemma 1.3** (Miculescu & Mihail, 2017, pp.1-11, Lemma 2.2) Let \(\{a_n\}_{n \in \mathbb{N} \cup \{0\}}\) be a sequence in a $b$-metric space \((X,d,s \geq 1)\) such that \(d(a_n, a_{n+1}) \leq kd(a_{n-1}, a_n)\) for some \(k \in [0,1]\), and each \(n = 1,2,...\). Then \(\{a_n\}\) is a $b$-Cauchy sequence in a $b$-metric space \((X,d,s \geq 1)\).

**Remark 2** In (Došenović et al, 2017, pp.851-865), it is proven that the previous lemmas are equivalent.

Since in general a $b$-metric is not necessarily continuous, many papers related with $b$-metric spaces used the following lemmas to prove the main results.

**Lemma 1.4** (Aghajani et al, 2014, pp.941-960, Lemma 2.1) Let \((X,d,s \geq 1)\) be a $b$-metric space. Suppose that \(\{a_n\}\) and \(\{b_n\}\) are $b$-convergent to \(a\) and \(b\), respectively. Then
\[
\frac{1}{s^2}d(a,b) \leq \liminf_{n \to \infty} d(a_n,b_n) \leq \limsup_{n \to \infty} d(a_n,b_n) \leq s^2d(a,b).
\]
In particular, if \(a = b\), then we have \(\lim_{n \to \infty} d(a_n, b_n) = 0\). Moreover, for each \(c \in X\), we have

\[
\frac{1}{s} d(a, c) \leq \lim \inf_{n \to \infty} d(a_n, c) \leq \lim \sup_{n \to \infty} d(a_n, c) \leq sd(a, c).
\]

**Lemma 1.5** (Paunović et al, 2017, pp.4162-4174, Lemma 2.3) Let \((X, d, s \geq 1)\) be a \(b\)-metric space and \(\{a_n\}\) a sequence in \(X\) such that

\[
\lim_{n \to \infty} d(a_n, a_{n+1}) = 0.
\]

If \(\{a_n\}\) is not \(b\)-Cauchy, then there exist \(\varepsilon > 0 \) and two sequences \(\{m(k)\}\) and \(\{n(k)\}\) of positive integers such that the following items hold:

\[
\varepsilon \leq \lim \inf_{k \to \infty} d(a_{m(k)}, a_{n(k)}) \leq \lim \sup_{k \to \infty} d(a_{m(k)}, a_{n(k)}) \leq \varepsilon s,
\]

\[
\frac{\varepsilon}{s} \leq \lim \inf_{k \to \infty} d(a_{m(k)}, a_{n(k)+1}) \leq \lim \sup_{k \to \infty} d(a_{m(k)}, a_{n(k)+1}) \leq \varepsilon s^2,
\]

\[
\frac{\varepsilon}{s^2} \leq \lim \inf_{k \to \infty} d(a_{m(k)+1}, a_{n(k)}) \leq \lim \sup_{k \to \infty} d(a_{m(k)+1}, a_{n(k)}) \leq \varepsilon s^3.
\]

In particular, if \(s = 1\) and \(\{a_n\}\) is not a \(b\)-Cauchy sequence, then there exists \(\varepsilon > 0\) as well as two sequences \(\{m(k)\}\) and \(\{n(k)\}\) of positive integers such that the sequences

\[
d(a_{m(k)}, a_{n(k)}) \to \varepsilon^+ \quad \text{as} \quad k \to \infty.
\]

**Main result**

Now, according to the last Lemma (the condition \(s = 1\)), we formulate and prove the following result:

**Theorem 5** Let \((X, d)\) be a complete metric space and let \(f\) be a contractive self-mapping on \(X\) satisfying the next condition:

Given \(\varepsilon > 0\), there exists \(\delta > 0\) such that for all \(x, y \in X\)

\[
\varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(fx, fy) \leq \varepsilon.
\]
Then \( f \) has a unique fixed point, say \( u \in X \), and for each \( x \in X \), \( \lim_{n \to \infty} f^n x = u \).

Proof. Let \( x_0 \) in \( X \) be arbitrary. Consider the sequence of iterates \( \{f^n x_0\}_{n=0}^{\infty} \). If \( d(f^n x_0, f^{n+1} x_0) = d(f^n x_0, ff^n x_0) = 0 \) for some \( n \in N \), then \( a_n = f^n x_0 \) is a fixed point of \( f \). Assume now that \( d(f^n x_0, f^{n+1} x_0) > 0 \) for all \( n \in N \). Since \( f \) is contractive, the sequence \( \{d(f^n x_0, f^{n+1} x_0)\}_{n=0}^{\infty} \) is strictly decreasing. Therefore, there exists the limit of this sequence, say \( \varepsilon \), and \( d(f^n x_0, f^{n+1} x_0) > \varepsilon \) for all \( n \in N \). Assume that \( \varepsilon > 0 \). In this case, by hypothesis, there exists a suitable \( \delta = \delta(\varepsilon) > 0 \) such that (5) holds. From the definition of \( \varepsilon \), it follows that there is \( n \in N \) such that

\[
\varepsilon \leq d(f^n x_0, f^{n+1} x_0) < \varepsilon + \delta.
\]

According to (5), we get that

\[
d(f^n x_0, f^{n+1} x_0) = d(f^{n+1} x_0, f^{n+2} x_0) \leq \varepsilon,
\]

a contradiction. Therefore \( \lim_{n \to \infty} d(f^n x_0, f^{n+1} x_0) = 0 \).

Now we show that \( \{f^n x_0\}_{n=0}^{\infty} \) is a Cauchy sequence. If this is not the case, by applying Lemma 1.5 to the sequence \( \{f^n x_0\}_{n=0}^{\infty} \), we get that there exist \( \varepsilon > 0 \) and two sequences of positive integers \( \{m(k)\} \) and \( \{n(k)\} \) such that \( n(k) > m(k) > k \), and sequences (4) tend to \( \varepsilon^+ \) as \( k \to \infty \). Using the condition (5) with \( x = a_{m(k)}, y = a_{n(k)} \) and the \( \delta = \delta(\varepsilon) > 0 \), one obtains that there exists a positive integer \( l \) such that for each \( k \geq l \), we have

\[
\varepsilon \leq d(a_{m(k)}, a_{n(k)}) = d(fa_{m(k)-1}, fa_{n(k)-1}) < \varepsilon + \delta \text{ implies } d(fa_{m(k)}, fa_{n(k)}) \leq \varepsilon.
\]

This contradicts the fact that

\[
d(fa_{m(k)}, fa_{n(k)}) = d(a_{m(k)+1}, a_{n(k)+1}) \to \varepsilon^+ \text{ as } k \to \infty.
\]

Hence, \( \{f^n x_0\}_{n=0}^{\infty} \) is a Cauchy sequence.

The proof is further as in (Čirić, 2003) and (Meir & Keeler, 1969, pp.326-329).
To our knowledge, it is not known whether Meir-Keeler’s and Ćirić’s theorems hold in the context of a $b$-metric space. Also, there is no known example that confirms that conditions (1) or (2) or (3) holds in the context of $b$-metric spaces but that $f$ does not have a fixed point.

However, with a stronger condition than (1), we have the positive result. Hence, our main result is the following:

**Theorem 6** Let $(X,d,s > 1)$ be a $b$-complete $b$-metric space and let $f$ self-mapping on $X$ satisfy the following condition:

Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$
\varepsilon \leq d(x,y) < \varepsilon + \delta \text{ implies } sd(fx,fy) < \varepsilon,
$$

where $a > 0$ is given.

Then $f$ has a unique fixed point, say $u \in X$, and for each $x \in X$, \(\lim_{n \to \infty} f^nx = u\).

**Proof.** It is clear that for all $x, y \in X$ we obtain

$$
d(fx,fy) \leq kd(x,y),
$$

where $k = \frac{1}{s^a} \in [0,1)$.

Let $a_0 \in X$ be an arbitrary point. Define the sequence $\{a_n\}$ by $a_{n+1} = fa_n$ for all $n \geq 0$. If $a_n = a_{n+1}$ for some $n$, then $a_n$ is a fixed point (unique) of $f$ and the results follows.

So, suppose that $a_n \neq a_{n+1}$ for all $n \geq 0$. From the condition (8), we obtain

$$
d(a_n,a_{n+1}) \leq kd(a_{n-1},a_n).
$$

Further, according to (Miculescu & Mihail, 2017, pp.1-11, Lemma 2.2.) we obtain that $\{a_n\}$ is a $b$-Cauchy sequence in a $b$-metric space $(X,d)$. By the $b$-completeness of $(X,d)$, there exists $u \in x$ such that

$$
\lim_{n \to \infty} a_n = u.
$$

Finally, (8) and (10) imply that $fu = u$, i.e. $u$ is a unique fixed point of $f$ in $X$. 
For the following facts and definitions, we refer to (Aghajani et al, 2014, pp.941-960), (Jovanović, 2016) and (Kirk & Shahzad, 2014) and the references therein.

**Definition 2.1** Let \( f \) and \( g \) be self-mappings of a nonempty set \( X \) such that \( f(X) \subset g(X) \). Let \( x_0 \in X \) be an arbitrary point. Then \( fx_0 \in g(X) \), so we can assume that \( fx_0 = gx_1 = y_0 \) (say) for some \( x_1 \in X \). Again, \( fx_1 \in g(X) \), so we can choose \( x_2 \in X \) such that \( fx_1 = gx_2 = y_1 \) (say). Similarly, we can construct two sequences \( \{x_n\} \) and \( \{y_n\} \) such that \( y_n = fx_n = gx_{n+1} \) for all \( n \geq 0 \). Here the sequence \( \{y_n\} \) is called a corresponding Jungck sequence for the point \( x_0 \in X \).

**Definition 2.2** Let \( f \) and \( g \) be the self-mappings of a nonempty set \( X \). If \( z = fx = gx \) for some \( x \) in \( X \), then \( x \) is called a coincidence point of \( f \) and \( g \), and \( z \) is called a point of coincidence of \( f \) and \( g \). The mappings \( f \) and \( g \) are called weakly compatible if they commute at their coincidence points.

**Lemma 2.3** Let \( f \) and \( g \) be the weakly compatible self-maps of a nonempty set \( X \). If \( f \) and \( g \) have a unique point of coincidence \( z = fx = gx \), then \( z \) is the unique common fixed point of \( f \) and \( g \).

Now, we announce the following result which generalizes Theorem 5 in several directions:

**Theorem 7** Let \( (X,d,s>1) \) be a \( b \)-complete \( b \)-metric space and let \( f,g : X \to X \) be two self-maps such that \( f(X) \subset g(X) \), one of these two subsets of \( X \) being \( b \)-complete. Suppose the following conditions hold:

1. For each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
   \[ \varepsilon \leq d(gx,gy) < \varepsilon + \delta \implies s^\delta d(fx,fy) < \varepsilon \]
2. and \( fx = fy \) whenever \( gx = gy \),

where \( a > 0 \) is given.
Then $f$ and $g$ have a unique point of coincidence, say $z \in X$. Moreover, for each $x_0 \in X$, the corresponding Jungck sequence $\{y_n\}$ can be chosen such that $\lim_{n \to \infty} y_n = z$. In addition, if $f$ and $g$ are weakly compatible, then they have a unique common fixed point.

Finally, we have an open question:

**Prove or disprove the following:**

- Let $(X,d,s > 1)$ be a $b$-complete $b$-metric space and $f,g : X \to X$ be two given mappings such that $f(X) \subset g(X)$, one of these two subsets of $X$ being $b$-complete. Assume that the following conditions hold:
  
  for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon \leq d(gx,gy) < \varepsilon + \delta$ implies $d(fx, fy) < \varepsilon$ and $fx = fy$, whenever $gx = gy$.

Then $f$ and $g$ have a unique point of coincidence, say $z \in X$. Moreover, if $f$ and $g$ are weakly compatible, then they have a unique common fixed point.

**References**


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Кључне речи: b-метрички простор, b-комплетан, b-Cauchy-јев, Meir-Keeler-ови услови, Picard-ов низ.