SOME FIXED POINT THEOREMS IN $b_2$-METRIC SPACES

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Abstract:
In this paper, we first prove a result that gives a sufficient condition for the convergence of the sequences in the $b_2$-metric space. Next, we give some fixed point theorems in the $b_2$-metric space. Some of our results are the corresponding generalizations of the known results in the $b_2$-metric space, which is confirmed by some examples.

Keywords: fixed points, common fixed points, 2-metric space, $b_2$-metric space.

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Introduction and Preliminaries

The applications of fixed point theorems are very important in diverse disciplines of mathematics, engineering and economics. The origin of the fixed theory is dated to the last quarter of the nineteenth century. The work of S. Banach in 1922 known as the Banach contraction principle is the starting point of the metric fixed point theory.

More on fixed point results and contractive conditions, the reader can find in (Čirić, 2003), (Agarwal et al, 2015), (Kirk & Shahzad, 2014).

**Theorem 1.1** (Banach contraction principle) Let \((X,d)\) be a complete metric space. Let \(T\) be a contractive mapping on \(X\), that is, one for which there is a \(\lambda \in [0,1)\) satisfying
\[
d(Tx,Ty) \leq \lambda d(x,y)
\]
for all \(x,y \in X\). Then, there exists a unique fixed point \(x \in X\) of \(T\).

This theorem is a forceful tool in the nonlinear analysis. It has many applications and has been extended by a great number of authors. Although the famous Banach contraction principle was proved in the metric space, after 1990 many new modifications of the definition of the metric space appeared.

From now on, \(\mathbb{R}\) and \(\mathbb{N}\) will denote the set of real numbers and natural numbers, respectively. Let us recall the definitions of the b-metric spaces, the rectangular b-metric spaces, the 2-metric spaces, and the \(b_2\)-metric spaces.

In the papers of Bakhtin (Bakhtin, 1989, pp.26-37) and Czerwik (Czerwik, 1993, pp.5-11), the notion of the b-metric space was introduced and some fixed point theorems for single-valued and multi-valued mappings in the b-metric spaces were proved.

**Definition 1.2** Let \(X\) be a nonempty set and \(s \geq 1\) a given real number. The function \(d : X \times X \to [0, \infty)\) is said to be a b-metric if for all \(x,y,z \in X\) the following conditions are satisfied:

- \(d(x,y) = 0\) if and only if \(x = y\);
- \(d(x,y) = d(y,x)\);
- \(d(x,z) \leq s[d(x,y) + d(y,z)]\).

A triplet \((X,d,s)\) is called a b-metric space with the coefficient \(s\).

In the paper (George et al, 2015, pp.1005-1013), the authors introduced the concept of a rectangular b-metric space, which is not necessarily Hausdorff and which generalizes the concept of the metric space, the rectangular metric space (RMS) and the b-metric space.
**Definition 1.3** (George et al, 2015, pp.1005-1013) Let $X$ be a nonempty set and $s \geq 1$ a given real number. The function $d : X \times X \to [0, \infty)$ is said to be a rectangular b-metric if the following conditions are satisfied:

(RbM1) $d(x, y) = 0$ if and only if $x = y$;
(RbM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(RbM3) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$.

A triplet $(X, d, s)$ is called a rectangular b-metric space with the coefficient $s$ (in short RbMS).

Also in (George et al, 2015, pp.1005-1013), the concept of convergence in such spaces is similar to that of standard metric spaces.

In the paper (Gähler, 1963, pp.115-118), Gähler introduced the concept of the 2-metric space.

**Definition 1.4** (Gähler, 1963, pp.115-118) Let $X$ be a nonempty set and the mapping $d : X \times X \times X \to \mathbb{R}$ satisfies:

1. For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.
2. If at least two of three points $x, y, z$ are the same, then $d(x, y, z) = 0$.
3. The symmetry:
   \[d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)\]
   for all $x, y, z \in X$.
4. The rectangle inequality:
   \[d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t)\]
   for all $x, y, z, t \in X$.

Then $d$ is called a 2-metric on $X$ and $(X, d)$ is called a 2-metric space.

Many generalizations of the concept of metric spaces are established, and several papers are published on the topic of the b-metric spaces (see (Aleksić et al, 2018), (Aydi, 2016, pp.2417-2433), (Czerwik, 1993, pp.5-11), (Czerwik, 1998, pp.263-276), (Dung & Le Hang, 2016, pp.267-284), (Miculescu & Mihail, 2017, pp.2153-2163) and others), of the rectangular b-metric spaces, see (George et al, 2015, pp.1005-1013), (Mitrović & Radenović, 2017, pp.3087-3095), (Mitrović & Radenović, 2017, pp.401-407) and others) and of the 2-metric spaces, see (Ahmed, 2009, pp.2914-2920), (Aliouche & Simpson, 2012, pp.668-690), (Deshpande & Chouhan, 2011,
In the paper (Mustafa et al, 2014), the notion of the $b_2$-metric space was introduced and some fixed point theorems in the $b_2$-metric spaces proved.

**Definition 1.5** (Mustafa et al, 2014) Let $X$ be a nonempty set, $s \geq 1$ be a real number and let $d : X \times X \times X \to \mathbb{R}$ satisfies:

1. For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.
2. If at least two of three points $x, y, z$ are the same, then $d(x, y, z) = 0$.
3. The symmetry:
   \[ d(x, y, z) = d(y, x, z) = d(z, x, y) = d(z, y, x) \]
   for all $x, y, z \in X$.
4. The rectangle inequality:
   \[ d(x, y, z) \leq s[d(x, y, t) + d(y, z, t) + d(z, x, t)] \]
   for all $x, y, z, t \in X$.

Then $d$ is called a $b_2$-metric on $X$ and $(X, d, s)$ is called a $b_2$-metric space.

**Remark 1.6** Note that, $d(x, y, z) \geq 0$ for all $x, y, z \in X$. Applying the rectangle inequality, we get

\[ d(x, y, z) \leq s[d(z, y, x) + d(x, z, y) + d(x, y, z)]. \]

By (2) and the symmetry of $d$, we obtain $d(x, y, z) \geq 0$.

Note that a 2-metric space is included in the class of the $b_2$-metric spaces with the coefficient $s = 1$.

**Example 1** (Mustafa et al, 2014)

1. Let $X = [0, +\infty)$ and $d(x, y, z) = (xy + yz + zx)^p$ if $x \neq y \neq z \neq x$, and otherwise $d(x, y, z) = 0$, where $p \geq 1$ is a real number.

From convexity of the function \( f(x) = x^p \) for $x \geq 0$, then by Jensen inequality we have
\[ (a + b + c)^p \leq 3^{p-1}(a^p + b^p + c^p). \]
So, \((X, d, s)\) is a \(b_2\)-metric space with \(s \leq 3^{p-1}\).

2. Let a mapping \(d : \mathbb{R}^3 \to [0, +\infty)\) be defined by
\[
d(x, y, z) = \min\{|x - y|, |y - z|, |z - x|\}.
\]
Then \(d\) is a \(b_2\)-metric on \(\mathbb{R}\), i.e., the following inequality holds:
\[
d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t),
\]
for all \(x, y, z, t \in \mathbb{R}\). From the convexity of the function \(f(x) = x^p\) on \([0, +\infty)\) for \(p \geq 1\), we obtain that
\[
d^p(x, y, z) = \min\{|x - y|, |y - z|, |z - x|\}^p
\]
is a \(b_2\)-metric on \(\mathbb{R}\) with \(s \leq 3^{p-1}\).

**Definition 1.7** (Mustafa et al., 2014) Let \(\{x_n\}\) be a sequence in a \(b_2\)-metric space \((X, d, s)\).

1. \(\{x_n\}\) is said to be \(b_2\)-convergent to \(x \in X\), written as
\[
\lim_{n \to \infty} x_n = x,
\]
if for all \(a \in X\), \(\lim_{n \to \infty} d(x_n, x, a) = 0\).

2. \(\{x_n\}\) is said to be a \(b_2\)-Cauchy sequence in \(X\) if for all \(a \in X\), \(\lim_{m,n \to \infty} d(x_n, x_m, a) = 0\).

3. \((X, d)\) is said to be \(b_2\)-complete if every \(b_2\)-Cauchy sequence is a \(b_2\)-convergent sequence.

**Definition 1.8** (Mustafa et al., 2014) Let \((X, d, s)\) be a \(b_2\)-metric spaces and let \(f : X \to X\) be a mapping. Then \(f\) is said to be \(b_2\)-continuous at a point \(z \in X\) if for a given \(\varepsilon > 0\), there exists \(\delta > 0\) such that \(x \in X\) and \(d(z, x, a) < \delta\) for all \(a \in X\) imply that \(d(fz, fx, a) < \varepsilon\). The mapping \(f\) is \(b_2\)-continuous on \(X\) if it is \(b_2\)-continuous at all \(z \in X\).

**Remark 1.9** Let \((X, d)\) be \(b_2\)-metric spaces. Then a mapping \(f : X \to X\) is \(b_2\)-continuous at a point \(x \in X\) if and only if it is \(b_2\)-sequentially continuous at \(x\), that is, whenever \(\{x_n\}\) is \(b_2\)-convergent to \(x\), \(\{fx_n\}\) is \(b_2\)-convergent to \(f(x)\).

This paper is to derive theorems of Banach, Reich and Jungck in the \(b_2\)-metric spaces. Also, we obtain some results in partially ordered \(b_2\)-metric spaces.
One sequence convergence test in the $b_{2}$-metric space

The inequality given in the next Lemma 2.1 is key to proving our main results.

**Lemma 2.1** Let $(X,d,s)$ be a $b_{2}$-metric space and $\{x_n\}$ a sequence in $X$. Suppose that $\lambda \in (0,1)$ and let $c$ be a real nonnegative number such that
\[
d(x_m, x_n, a) \leq \lambda d(x_{m-1}, x_{n-1}, a) + c(\lambda^m + \lambda^n)
\]  
for all $m, n \in \mathbb{N}$ and $a \in X$. Then $\{x_n\}$ is a $b_{2}$-Cauchy sequence in $X$.

**Proof.** Let $m, n \in \mathbb{N}$, $a \in X$ and $p \in \mathbb{N}$ fixed such that $p > -2 \log_2 s$.

From (2.1) we have that
\[
d(x_{m+k}, x_{n+k}, a) \leq \lambda^k d(x_m, x_n, a) + ck\lambda^k (\lambda^m + \lambda^n)
\]  
for all $m, n, k \in \mathbb{N}$ and $a \in X$. We have
\[
d(x_m, x_n, a) \leq s \left[ d(x_{m+p}, x_n, a) + d(x_m, x_{m+p}, a) + d(x_m, x_n, x_{m+p}) \right]
\]
\[
+ cm\lambda^m (1 + \lambda^p) + \lambda^n d(x_0, x_n, x_p) + c m\lambda^m (1 + \lambda^p) \]
\[
\leq s \left[ d(x_{m+p}, x_n, a) + \lambda^m d(x_0, x_n, a) + \lambda^n d(x_0, x_n, x_p) \right]
\]
\[
+ 4scm\lambda^m .
\]

Next, as it is
\[
d(x_0, x_n, x_p) \leq s \left[ d(x_{n+p}, x_n, x_p) + d(x_0, x_{n+p}, x_p) + d(x_0, x_n, x_{n+p}) \right]
\]
\[
\leq s \left[ \lambda^p d(x_p, x_0, x_0) + cn\lambda^n (1 + \lambda^p) + \lambda^d d(x_0, x_n, x_p) \right]
\]
\[
+ cp\lambda^p (1 + \lambda^n) + \lambda^n d(x_0, x_0, x_p) + cn\lambda^n (1 + \lambda^p) \]
\[
\leq 2cs \left( 2n\lambda^p + p \right),
\]

and
\[
d(x_{m+p}, x_n, a) \leq s \left[ d(x_{m+p}, x_n, a) + d(x_{m+p}, x_{n+p}, a) + d(x_m, x_{n+p}, a) \right]
\]
\[
\leq s \left[ \lambda^d d(x_p, x_0, a) + cn\lambda^n (1 + \lambda^p) + \lambda^p d(x_m, x_n, a) \right]
\]
\[
+ cp\lambda^p (1 + \lambda^n) + \lambda^n d(x_m, x_n, a) + cp\lambda^p (1 + \lambda^n) \]
\[
\leq s \left[ \lambda^p d(x_p, x_0, a) + \lambda^p d(x_m, x_n, a) \right]
\]
\[
+ 2cs \left[ p(\lambda^m + \lambda^n) + n\lambda^p \right].
\]
we obtain the following
\[
d(x_m, x_n, a) \leq s^2 \left[ \lambda^n d(x_p, x_0, a) + \lambda^p d(x_m, x_n, a) + s \lambda^m d(x_0, x_p, a) \right] \\
+ 2cs^2 \left[ p(\lambda^m + \lambda^n) + n\lambda^n \right] + 2s^2 c\lambda^m (2n\lambda^n + p).
\]

So,
\[
\left(1 - s^2 \lambda^p \right) d(x_m, x_n, a) \leq s^2 \lambda^n d(x_p, x_0, a) + s \lambda^m d(x_0, x_p, a) \\
+ 2cs^2 \left[ 2p\lambda^m + (p + n)\lambda^n + 2n\lambda^{m+n} \right],
\]

how is it \( p > -2 \log_\lambda s \) we have \( 1 - s^2 \lambda^p > 0 \), therefore, we obtain that \( \{x_n\} \) is a \( b_2 \)-Cauchy.

**Lemma 2.2** Let \( (X, d, s) \) be a \( b_2 \)-metric space and \( \{x_n\} \) a sequence in \( X \). Suppose that \( \lambda \in [0,1) \) such that
\[
d(x_m, x_n, a) \leq \lambda d(x_{m-1}, x_{n-1}, a) \quad \text{for all} \ m, n \in \mathbb{N}, \ a \in X.
\]

Then
\[
d(x_m, x_n, a) \leq \frac{s^2 (\lambda^m + \lambda^n)}{1 - s^2 p} d(x_0, x_p, a),
\]
for all \( m, n, p \in \mathbb{N}, \ p > -2 \log_\lambda s \) and \( a \in X \).

**Proof.** It follows directly from Lemma 2.1, if we put \( c = 0 \), see (2.4).

**Remark 2.3** Note that both Lemma 2.1 and Lemma 2.2 improve the result of Lemma 1.6. in (Fadail et al, 2015, pp.533-548).

**A Theorem of Jungck in the \( b_2 \)-metric space**

The following Theorem is the version of the Theorem of Jungck (Jungck, 1976, pp.261-263) in \( b_2 \)-metric spaces.

**Theorem 3.1** Let \( T \) and \( I \) be commuting mappings of a \( b_2 \)-complete \( b_2 \)-metric space \( (X, d, s) \) into itself satisfying the inequality
\[
d(Tx, Ty, a) \leq \lambda d(Ix, Iy, a)
\]
(3.1)
for all \( x, y, a \in X \), where \( 0 < \lambda < 1 \). If the range of \( I \) contains the range of \( T \) and if \( I \) is \( \beta_2 \)-continuous, then \( T \) and \( I \) have a unique common fixed point.

**Proof.** Let \( x_0 \in X \) be arbitrary. Then \( Tx_0 \) and \( Ix_0 \) are well defined. Since \( Tx_0 \in I(X) \), there is any \( x_1 \in X \) such that \( Ix_1 = Tx_0 \). In general, if \( x_n \) is chosen, then we choose a point \( x_{n+1} \in X \) such that \( Ix_{n+1} = Tx_n \). We show that \( \{Ix_n\} \) is a \( \beta_2 \)-Cauchy sequence. From (3.1) we have

\[
d(Ix_m, Ix_n, a) = d(Tx_{m-1}, Tx_{n-1}, a) \leq \lambda d(Ix_{m-1}, Ix_{n-1}, a).
\]

So,

\[
d(Ix_m, Ix_n, a) \leq \lambda d(Ix_{m-1}, Ix_{n-1}, a), \text{ for all } m, n \in \mathbb{N}, \ a \in X.
\] (3.2)

From Lemma 2.2 we obtain

\[
d(Ix_m, Ix_n, a) \leq \frac{s^2(\lambda^m + \lambda^n)}{1-s^2} d(Ix_0, Ix_p, a),
\] (3.3)

for all \( m, n \in \mathbb{N}, \ a \in X \) and for some \( p \in \mathbb{N} \) such that it is \( p > -2 \log_\lambda s \). Thus, we obtain that \( \{Ix_n\} \) is a \( \beta_2 \)-Cauchy sequence in \( X \).

By the \( \beta_2 \)-completeness of \( X \), there exists \( u \in X \) such that

\[
\lim_{n \to \infty} Ix_n = \lim_{n \to \infty} Tx_{n-1} = u.
\]

Now, since \( I \) is \( \beta_2 \)-continuous, (3.1) implies that both \( I \) and \( T \) are \( \beta_2 \)-continuous. Since \( T \) and \( I \) commute, we obtain

\[
Iu = I(\lim_{n \to \infty} Tx_n) = \lim_{n \to \infty} ITx_n = \lim_{n \to \infty} TLx_n = T(\lim_{n \to \infty} Ix_n) = Tu.
\]

Let \( v = Iu = Tu \). We get \( Tv = Tlu = ITu = Iv \). If \( Tu \neq Tv \), from (3.1) we obtain

\[
d(Tu, Tv, a) \leq \lambda d(Iu, Iv, a) = \lambda d(Tu, Tv, a) < d(Tu, Tv, a),
\]

a contradiction. So we have \( Tu = Tv \), and finally we obtain \( Tv = Iv = v \) i.e. \( v \) is a common fixed point for \( T \) and \( I \). Condition (3.1) implies that \( v \) is the unique common fixed point.

**Theorem 3.2** Let \( T \) and \( I \) be commuting mappings of a complete 2-metric space \((X, d, s)\) into itself satisfying the inequality
\[ d(Tx,Ty,a) \leq \lambda d(Ix,Iy,a) \]  \hspace{1cm} (3.4)

for all \( x, y, a \in X \), where \( 0 < \lambda < 1 \). If the range of \( I \) contains the range of \( T \) and \( I \) is continuous, then \( T \) and \( I \) have a unique common fixed point.

From Theorem 3.1, we obtain the following variant of the Banach theorem in \( b_2 \)-metric spaces.

**Theorem 3.3** Let \( (X,d,s) \) be a \( b_2 \)-complete \( b_2 \)-metric space and \( T : X \to X \) a mapping satisfying:
\[ d(Tx,Ty,a) \leq \alpha d(x,y,a) \]  \hspace{1cm} (3.5)

for all \( x, y, a \in X \), where \( \alpha \in [0,1) \). Then \( T \) has a unique fixed point.

**Proof.** Put \( I(x) = x, x \in X \) in Theorem 3.1.

We obtain the following result as a consequence of Theorem 3.1 if we put \( K = \frac{1}{\lambda} \) and \( T \) to be the identity map, i.e. \( T(x) = x, x \in X \).

**Theorem 3.4** Let \( I \) be a continuous onto mapping of a \( b_2 \)-complete \( b_2 \)-metric space \( (X,d,s) \). If there exists \( K > 1 \) such that
\[ d(Ix,Iy,a) \geq Kd(x,y,a) \]
for all \( x, y, a \in X \), then \( I \) has a unique fixed point.

**The Reich theorem in \( b_2 \)-metric spaces**

The following theorem is the analogue of the Reich contraction principle (Reich, 1971, pp.121-124) in the \( b_2 \)-metric space.

**Theorem 4.1** Let \( (X,d,s) \) be a \( b_2 \)-complete \( b_2 \)-metric space and \( T : X \to X \) be a mapping satisfying:
\[ d(Tx,Ty,a) \leq \alpha d(x,y,a) + \beta d(x,Tx,a) + \gamma d(y,Ty,a) \]  \hspace{1cm} (4.1)

for all \( x, y, a \in X \), where \( \alpha, \beta, \gamma \) are nonnegative constants with \( \alpha + \beta + \gamma < 1 \) and \( \min \{\beta, \gamma\} < \frac{1}{s} \). Then \( T \) has a unique fixed point.
Proof. Let \( x_0 \in X \) be arbitrary. Define the sequence \( \{x_n\} \) by \( x_{n+1} = Tx_n \) for all \( n \geq 0 \). From condition (4.1) we have that

\[
d(x_{n+1}, x_n, a) \leq \alpha d(x_n, x_{n-1}, a) + \beta d(x_n, x_{n+1}, a) + \gamma d(x_{n-1}, x_n, a).
\]

Therefore,

\[
d(x_{n+1}, x_n, a) \leq \frac{\alpha + \gamma}{1 - \beta} d(x_n, x_{n-1}, a). \tag{4.2}
\]

Put \( r = \frac{\alpha + \gamma}{1 - \beta} \). We have that \( r \in [0,1) \). It follows from (4.2) that

\[
d(x_{n+1}, x_n, a) \leq r^n d(x_1, x_0, a) \quad \text{for all} \quad n \geq 1. \tag{4.3}
\]

From conditions (4.1) and (4.3), we obtain

\[
d(x_m, x_n, a) \leq \alpha d(x_{m-1}, x_{n-1}, a) + \beta d(x_{m-1}, x_m, a) + \gamma d(x_{n-1}, x_n, a)
\leq \alpha d(x_{m-1}, x_{n-1}, a) + \beta r^{n-m} d(x_0, x_1, a) + \gamma r^{n-1} d(x_0, x_1, a)
= \alpha d(x_{m-1}, x_{n-1}, a) + (\beta r^{n-m} + \gamma r^{n-1}) d(x_0, x_1, a)
\]

From this, together with Lemma 2.1 (we can put \( \{x_n\} \) is Cauchy. By the \( b_2 \)-completeness of \( (X, d, s) \) there exists \( x^* \in X \) such that

\[
\lim_{n \to \infty} x_n = x^*. \tag{4.4}
\]

Now we obtain that \( x^* \) is the unique fixed point of \( T \). Namely, we have
\[ d(x^*, Tx^*, a) \leq s[d(x_{n+1}, Tx^*, a) + d(x^*, x_{n+1}, a) + d(x^*, Tx^*, x_{n+1})] \]
\[ = s[d(Tx_n, Tx^*, a) + d(x^*, x_{n+1}, a) + d(x^*, Tx^*, x_n)] \]
\[ \leq s[\alpha d(x_n, x^*, a) + \beta d(x_n, x_{n+1}, a) + \gamma d(x^*, Tx^*, a) + d(x^*, x_{n+1}, a) + \alpha d(x^*, x^*, x_n) + \beta d(x^*, x^*, Tx^*) + \gamma d(x^*, x_n, x_{n+1})] \]

and

\[ d(Tx^*, x^*, a) \leq s[d(x_{n+1}, x^*, a) + d(Tx^*, x_{n+1}, a) + d(Tx^*, x^*, x_{n+1})] \]
\[ = s[d(x_{n+1}, x^*, a) + d(Tx^*, x_n, a) + d(Tx^*, x^*, x_n)] \]
\[ \leq s[d(x_{n+1}, x^*, a) + \alpha d(x^*, x_n, a) + \beta d(x^*, x^*, a) + \gamma d(x^*, x^*, x_n) + \alpha d(x^*, x^*, x_n) + \beta d(x^*, x^*, Tx^*) + \gamma d(x^*, x_n, x_{n+1})] \]

Since \( \lim_{n \to \infty} d(x^*, x_n, a) = 0 \), \( \lim_{n \to \infty} d(x_n, x_{n+1}, a) = 0 \), \( \lim_{n \to \infty} d(x_n, x_{n+1}, x^*) = 0 \) and \( \min(\beta, \gamma) < \frac{1}{s} \), we have \( d(x^*, Tx^*, a) = 0 \) for all \( a \in X \) i.e., \( Tx^* = x^* \) (Axiom (1) in Definition 1.5).

For uniqueness, let \( y^* \) be another fixed point of \( T \). Then it follows from (4.1) that

\[ d(x^*, y^*, a) = d(Tx^*, Ty^*, a) \leq \alpha d(x^*, y^*, a) + \beta d(x^*, Tx^*, a) + \gamma d(y^*, Ty^*, a) \]
\[ = \alpha d(x^*, y^*, a) < d(x^*, y^*, a) \]

is a contradiction. Therefore, we must have \( d(x^*, y^*, a) = 0 \), i.e., \( x^* = y^* \).

From Theorem 4.1, we obtain the following variant of the Kannan theorem (Kannan, 1968, pp.71-76) in \( b_2 \)-metric spaces.

**Theorem 4.2** Let \( (X, d, s) \) be a \( b_2 \)-complete \( b_2 \)-metric space and \( T : X \to X \) be a mapping satisfying:

\[ d(Tx, Ty, a) \leq \beta d(x, Tx, a) + \gamma d(y, Ty, a) \] (4.5)
for all $x, y, a \in X$, where $\beta, y$ nonnegative constants with $\beta + y < 1$ and $\min\{\beta, y\} < \frac{1}{s}$. Then $T$ has a unique fixed point.

A result in partial order $b_2$-metric space

Let $F_s$ denote the class of all functions $\beta : [0, \infty) \to \left[0, \frac{1}{s}\right]$ satisfying

the following condition:

$$\beta(t_n) \to \frac{1}{s} \text{ as } n \to \infty \text{ implies } t_n \to 0 \text{ as } n \to \infty.$$  

In the paper (Mustafa et al, 2014), Mustafa et al. obtain the following result in partially ordered $b_2$-metric spaces.

**Theorem 5.1** (Mustafa et al, 2014, Theorem 1) Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a $b_2$-metric $d$ on $X$ such that $(X, d, s)$ is a $b_2$-complete $b_2$-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f x_0$. Suppose that

$$sd(f x, f y, a) \leq \beta(d(x, y, a))M(x, y, a)$$  

for all $a \in X$ and for all comparable elements $x, y \in X$, where

$$M(x, y, a) = \max\left\{d(x, y, a), \frac{d(f x, a)d(y, f y, a)}{1 + d(f x, f y, a)}\right\}.$$  

If $f$ is $b_2$-continuous, then $f$ has a fixed point. Moreover, the set of fixed points of $f$ is well ordered if and only if $f$ has one and only one fixed point.

In the further, we consider that $M(x, y, a)$ is given as in Theorem 5.1.

In the paper (Fadail et al, 2015, pp.533-548), Fadail et al. generalize, complement and improve Theorem 5.1 in several directions.

**Theorem 5.2** (Fadail et al, 2015, pp.533-548, Theorem 2.1) Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a $b_2$-metric $d$ on $X$ such that $(X, d, s)$ is a $b_2$-complete $b_2$-metric space with $s > 1$. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f x_0$. Suppose that
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\[
s^\varepsilon d(fx, fy, a) \leq \beta(d(x, y, a))M(x, y, a)
\]

(5.2)

for all \(a \in X\) and for all comparable elements \(x, y \in X\), where \(\varepsilon \in (0,1]\). If \(f\) is \(b^2\)-continuous then \(f\) has a fixed point. Moreover, the set of fixed points of \(f\) is well ordered if and only if \(f\) has one and only one fixed point.

Using Lemma 2.1, we get the following result.

**Theorem 5.3** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a \(b^2\)-metric \(d\) on \(X\) such that \((X, d)\) is a \(b^2\)-complete \(b^2\)-metric space with \(s > 1\). Let \(f : X \to X\) be an increasing mapping with respect to \(\preceq\) such that there exists an element \(x_0 \in X\) with \(x_0 \preceq fx_0\).

Suppose that exists \(\lambda \in (0,1]\) such that

\[
d(fx, fy, a) \leq \lambda M(x, y, a),
\]

(5.3)

for all \(a \in X\) and for all comparable elements \(x, y \in X\). If \(f\) is \(b^2\)-continuous, then \(f\) has a fixed point. Moreover, the set of the fixed points of \(f\) is well ordered if and only if \(f\) has one and only one fixed point.

**Proof.** Since \(x_0 \preceq fx_0\) and \(f\) is an increasing function, we obtain that

\[
x_0 \preceq fx_0 \preceq f^2x_0 \preceq \cdots \preceq f^nx_0 \preceq f^{n+1}x_0 \preceq \cdots.
\]

Since \(x_n \preceq x_{n+1}\), where \(x_{n+1} = fx_n, n \in \mathbb{N}\), from (5.3) we obtain

\[
d(x_n, x_{n+1}, a) \leq \lambda \max\left\{d(x_{n-1}, x_n, a)
\frac{d(x_{n-1}, x_n, a)d(x_{n-1}, x_n, a)}{1+d(x_{n-1}, x_n, a)}\right\}
\]

\[
\leq \lambda d(x_{n-1}, x_n, a).
\]

Using induction, we conclude that

\[
d(x_n, x_{n+1}, a) \leq \lambda d(x_{n-1}, x_n, a),
\]

(5.4)

for all \(n \in \mathbb{N}, a \in X\). Further from conditions (5.3), we have

\[
d(x_m, x_n, a) \leq \lambda \max\left\{d(x_{m-1}, x_{n-1}, a)
\frac{d(x_{m-1}, x_{n-1}, a)d(x_{m-1}, x_{n-1}, a)}{1+d(x_{m-1}, x_{n-1}, a)}\right\}
\]

\[
\leq \lambda \max\{d(x_{m-1}, x_{n-1}, a), d(x_{m-1}, x_m, a)d(x_{m-1}, x_{n-1}, a)\}.
\]

Now, from inequality (5.4), it follows...
$d(x_m, x_n, a) \leq \lambda \max \{d(x_{m-1}, x_{n-1}, a), \lambda^{m-1}d(x_0, x_1, a)\lambda^{n-1}d(x_0, x_1, a)\}$

$\leq \lambda d(x_{m-1}, x_{n-1}, a) + \lambda^{m+n-1}d^2(x_0, x_1, a)$

$= \lambda d(x_{m-1}, x_{n-1}, a) + c(\lambda^m + \lambda^n)$

where $c = d^2(x_0, x_1, a)$. Now, because of Lemma 2.1, we get that $\{x_n\}$ is a $b_2$-Cauchy sequence in $(X, d)$. The rest of the proof is the same as in (Mustafa et al, 2014) (Steps IV and V).

Note that condition (5.1) implies (5.2) and condition (5.2) implies (5.3).

**Example 2** Let $X = \{(a,0): a \in [0, +\infty)\} \cup \{(0,2)\}$ and let $d(x, y, z)$ denote the square of the area of a triangle with the vertices $x, y, z \in X$, e.g.,

$d((a,0),(b,0),(0,2)) = (a - b)^2$.

Then $d$ is a $b_2$-metric with the parameter $s = 2$. Introduce an order in $X$ by

$(a,0) \leq (b,0) \iff a \geq b$,

with all other pairs of distinct points in $X$ incomparable.

Consider the mapping $f: X \to X$ given by

$f(a,0) = (\lambda a,0)$ for $a \in [0, +\infty)$ and $f(0,2) = (0,2)$,

and the function $\beta \in F_2$ given as

$\beta(t) = \frac{1 + t}{2 + 4t}$ for $t \in [0, +\infty)$.

Then $f$ is an increasing mapping with $(a,0) \leq f(a,0)$ for each $a \geq 0$.

1. If $\lambda = \frac{1}{3}$ then the assumptions of Theorem 5.1 are satisfied (Mustafa et al, 2014, Example 3).

2. If $\lambda = \frac{1}{2}$ then the assumptions of Theorem 5.2 are satisfied (Fadail et al, 2015, pp.533-548, Example 2.6), but we cannot apply Theorem 5.1.

3. If $\lambda \in \left(\frac{1}{2}, 1\right)$ then the assumptions of Theorem 5.3 are satisfied, but we cannot apply Theorem 5.2.
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НЕКЕ ТЕОРЕМЕ О ФИКСНОЈ ТАЧКИ У $b_2$-МЕТРИЧКИМ ПРОСТОРИМА

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Сажетак:

У овом раду прво је доказан резултат који даје довољан услов за конвергенцију низова у $b_2$-метричком простору. Такође, наведене су неке теореме о фиксној тачки у $b_2$-метричком простору. Неки резултати представљају одговарајуће генерализације познатих резултата у $b_2$-метричком простору, а примери су презентирани да то потврде.

Кључне речи: фиксне тачке, заједничке фиксне тачке, $2$-метрички простор, $b_2$-метрички простор.

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