PROPERTY P IN MODULAR METRIC SPACES

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Abstract:

Introduction/purpose: The aim of this paper is to present the concept of the generalized ∅ –weak contractive condition involving various combinations of d(x,y) in modular metric spaces.

Methods: Conventional theoretical methods of functional analysis.

Results: This study presents the result of (Murthy & Vara Prasad, 2013) for a single-valued mapping satisfying a generalized ∅ –weak contractive condition involving various combinations of d(x,y). It is generalized in the setting of modular metric spaces, and then it is proved that this single-valued map satisfies the property P. In the end, an example is given in support of the result.

Conclusion: With proper generalisations, it is possible to formulate well-known results of classical metric spaces to the case of modular metric spaces.

Key words: Fixed point, ∅ –weak contraction, modular metric spaces, property P.

Introduction

One of trends in mathematical research is to refine the frameworks of the known theorems and their results. For instance, Polish mathematician Banach observed the first metric fixed point results in the setting of complete normed spaces. An immediate extension of this theorem was given by Caccioppoli who observed the characterization of the Banach fixed point theorem in the context of complete metric spaces. Afterwards, for various abstract spaces, several analogs of the Banach contraction principle have been reported. Among them, we can underline some of interesting abstract structures such as modular metric space, partial metric space, *b*-metric space, fuzzy metric space, probabilistic metric space, *G*-metric space, etc.

This paper will be restricted to the recently introduced generalization of a metric space, namely, a modular metric space. Chistyakov introduced the notion of modular metric spaces (Chistyakov, 2010a, 2010b) inspired partly by the classical linear modulars on function spaces. Informally speaking, whereas a metric on a set represents the nonnegative finite distances between any two points of the set, a modular on a set attributes a nonnegative (possibly, infinite valued) "field of (generalized) velocities": to each "time" λ>0 the absolute value of an average velocity $w_\lambda(x,y)$ is associated in such a way that in order to cover the "distance" between the points x, $y \in \psi$, it takes time λ to move from x to y with the velocity $w_\lambda(x,y)$. But the way we approached the concept of modular metric spaces is different. Indeed, we look at these spaces as a nonlinear version of the classical modular spaces introduced by H. Nakano (Nakano, 1950) on vector spaces and modular function spaces introduced by (Musielak, 1983) and (Orlicz, 1988a, 1988b). More about modular metric spaces can be read in (Hussain et al, 2011), (Paknazar & De la Sen, 2017) and (Paknazar & De la Sen, 2020).

In the formulation given by (Khamsi, 1996) and (Kozlowski, 1988), a modular on a vector space ψ is a function $m : \psi \rightarrow [0,+\infty)$ satisfying:
(1) $m(x) = 0$ if and only if $x = 0$,
(2) $m(ax) = m(x)$ for every $a \in R$ with $|a| = 1$,
(3) $m(ax + by) \leq m(x) + m(y)$ if $a, b \geq 0$ and $a + b = 1$.

A modular m is said to be convex if, instead of (3), it satisfies the stronger property:
(3*) $m(ax + by) \leq am(x) + bm(y)$ if $a, b \geq 0$ and $a + b = 1$.

Given a modular m on ψ, the modular space is defined by
$\psi_m = \{x \in \psi : m(ax) \to 0 \text{ as } a \to 0\}$.

It is possible to define a corresponding F-norm (or a norm when m is convex) on the modular space. The Orlicz spaces L^{Φ} are examples of this construction (Rao & Ren, 2002). The modular metric approach is more natural and has not been used extensively. For more on the metric fixed point theory, the reader may consult the book (Khamsi & Kirk, 2001) and for modular function spaces (Chistyakov, 2010a, 2010b) and (Chistyakov, 2008). Some recent work in modular metric spaces has been presented in (Mongkolkeha et al, 2011) and (Padcharoen et al, 2016). It has been almost a century since several mathematicians improved, extended and enriched the classical Banach contraction principle (Banach, 1922) in different directions along with variety of applications. In the sequel, we recall some basic concepts about modular metric spaces.

Throughout this paper, $\mathbb{N}$ will denote the set of natural numbers.
Let ψ be a nonempty set. Throughout this paper, for a function
$\omega: (0, +\infty) \times \psi \times \psi \to [0, +\infty)$, we write
$\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in \psi$.

DEFINITION 1. (Chistyakov, 2006) Let ψ be a nonempty set. A function $\omega: (0, +\infty) \times \psi \times \psi \to [0, +\infty)$ is said to be a metric modular on ψ if it satisfies, for all $x, y, z \in \psi$, the following conditions:
1) $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$,
2) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$,
3) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$.

If instead of (1) we have only the condition (1'):
$\omega_\lambda(x, x) = 0$ for all $x \in \psi$, $\lambda > 0$, then ω is said to be a pseudo modular (metric) on ψ.
An important property of the (metric) pseudo modular on the set ψ is that the mapping $\lambda \mapsto \omega_\lambda(x, y)$ is non increasing for all $x, y \in \psi$.

DEFINITION 2. (Chistyakov, 2006) Let ω be a pseudo modular on ψ. Fixed $x_0 \in \psi$. The set
$\psi_\omega = \psi_\omega(x_0) = \{x \in \psi : \omega_\lambda(x, x_0) \to 0 \text{ as } \lambda \to +\infty\}$ is said to be a modular metric space (around x_0).

DEFINITION 3. (Padcharoen et al, 2016)Let ψ_ω be a modular metric space.

1) The sequence $\{x_\eta\}$ in ψ_ω is said to be ω-convergent to $x\in\psi_\omega$ if and only if there exists a number $\lambda > 0$, possibly depending on $\{x_\eta\}$ and x, such that $\lim_{\eta\to+\infty} \omega_\lambda(x_\eta, x)= 0$.
2) The sequence $\{x_\eta\}$ in ψ_ω is said to be ω-Cauchy if there exists $\lambda > 0$, possibly depending on the sequence, such that $\omega_\lambda(x_m, x_\eta) \to 0$ as $m, \eta \to +\infty$.
3) A subset $C$of ψ_ω is said to be ω -complete if any ω-Cauchy sequence in C is a convergent sequence and its limit is in C.

DEFINITION 4.(Mongkolkeha et al, 2011) Let ω be a metric modular on ψ and ψ_ω be a modular metric space induced by ω. If ψ_ω is a ω-complete modular metric space and $\mathcal{T}: \psi_\omega \to \psi_\omega$be an arbitrary mapping, $\mathcal{T}$ is called a contraction if for each $x, y\in\psi_\omega$ and for all $\lambda > 0$ there exists $0 \leq \sigma < 1$ such that

$$\omega_\lambda(\mathcal{T}x, \mathcal{T}y) \leq \sigma\omega_\lambda(x, y)$$

Mongkolkeha et al, (2011) proved that if $\psi_\omega$is a ω - complete modular metric space, then contraction mapping $\mathcal{T}$ has a unique fixed point.

Main result

In this section, there is a generalization of the result proved by (Murthy & Vara Prasad, 2013):
Let $\mathcal{T}$ be a self-map of a complete metric space ψ satisfying:

(C_1) **:**

$$[1 + pd(x, y)]d^2(\mathcal{T}x, \mathcal{T}y) \leq pmax\begin{Bmatrix} \frac{1}{2}[d^2(x, \mathcal{T}x)d(y, \mathcal{T}y) + d(x, \mathcal{T}x)d^2(y, \mathcal{T}y)], \\ d(x, \mathcal{T}x)d(x, \mathcal{T}y)d(y, \mathcal{T}x), \\ d(x, \mathcal{T}y)d(y, \mathcal{T}x)d(y, \mathcal{T}y) \end{Bmatrix} + m(x, y) - \emptyset m(x, y).$$

Where,

(C_2) **:**

$$m(x, y) = \max\begin{Bmatrix} d^2(x, y), d(x, \mathcal{T}x)d(y, \mathcal{T}y), d(x, \mathcal{T}y)d(y, \mathcal{T}x), \\ \frac{1}{2}[d(x, \mathcal{T}x)d(x, \mathcal{T}y) + d(y, \mathcal{T}x)d(y, \mathcal{T}y)] \end{Bmatrix},$$

$p \geq 0$ is a real number and $\emptyset$: $[0,+\infty) \to [0,+\infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$.
Then $\mathcal{T}$ has a unique fixed point in ψ.

Now we will generalize the above result in the setting of modular metric spaces as follows:

Theorem 1. Let (ψ_ω, ω) be a complete modular metric space. Let $\mathcal{T}$ be a self-map of a complete modular metric space ψ_ω satisfying:
(C_3) **:**

$$[1 + p\omega_1(x, y)]\omega_1{}^2(\mathcal{T}x, \mathcal{T}y) \leq pmax\begin{Bmatrix} \frac{1}{2}[\omega_1{}^2(x, \mathcal{T}x)\omega_1(y, \mathcal{T}y) + \omega_1(x, \mathcal{T}x)\omega_1{}^2(y, \mathcal{T}y)], \\ \omega_1(x, \mathcal{T}x)\omega_2(x, \mathcal{T}y)\omega_1(y, \mathcal{T}x), \\ \omega_2(x, \mathcal{T}y)\omega_1(y, \mathcal{T}x)\omega_1(y, \mathcal{T}y) \end{Bmatrix} + m(x, y) - \emptyset m(x, y),$$

where,
(C_4) **:**

$$m(x, y) = \max\begin{Bmatrix} \omega_1{}^2(x, y), \omega_1(x, \mathcal{T}x)\omega_1(y, \mathcal{T}y), \omega_2(x, \mathcal{T}y)\omega_1(y, \mathcal{T}x), \\ \frac{1}{2}[\omega_1(x, \mathcal{T}x)\omega_2(x, \mathcal{T}y) + \omega_1(y, \mathcal{T}x)\omega_1(y, \mathcal{T}y)] \end{Bmatrix}$$

$p \geq 0$ is a real number and $\emptyset$: $[0, +\infty) \to [0, +\infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$.
Then $\mathcal{T}$ has a unique common fixed point in ψ_ω.

Proof. Let $x_0 \in \psi_\omega$ be an arbitrary point. Then we can find x_1 such that $x_1 = \mathcal{T}(x_0)$. For this x_1, we can find $x_2 \in \psi_\omega$ such that $x_2 = \mathcal{T}(x_1)$.

In general, one can choose $\{x_{\eta+1}\}$ in ψ_ω such that

$$x_{\eta+1} = \mathcal{T}(x_\eta),\ \eta = 0,1,2 \ldots \tag{1}$$

We may assume that $x_\eta \neq x_{\eta+1}$ for each η.
Since if there exists η such that $x_\eta = x_{\eta+1}$ then $x_\eta = x_{\eta+1} = \mathcal{T}(x_\eta)$,
We write $\alpha_\eta = d(x_\eta, x_{\eta+1})$.
Firstly, we prove that α_η is a non - increasing sequence and converges to 0.

Case I. If η is even, taking $x = x_{2\eta}$ and $y = x_{2\eta+1}$ in (C_3), we get

$$[1 + p\omega_1(x_{2\eta}, x_{2\eta+1})]\omega_1{}^2(\mathcal{T}x_{2\eta}, \mathcal{T}x_{2\eta+1})$$

$$\leq pmax\begin{Bmatrix} \frac{1}{2}\begin{bmatrix} \omega_1{}^2(x_{2\eta},\mathcal{T}x_{2\eta})\omega_1(x_{2\eta+1},\mathcal{T}x_{2\eta+1}) \\ +\ \omega_1(x_{2\eta},\mathcal{T}x_{2\eta})\omega_1{}^2(x_{2\eta+1},\mathcal{T}x_{2\eta+1}) \end{bmatrix}, \\ \omega_1(x_{2\eta},\mathcal{T}x_{2\eta})\omega_2(x_{2\eta},\mathcal{T}x_{2\eta+1})\omega_1(x_{2\eta+1},\mathcal{T}x_{2\eta}), \\ \omega_2(x_{2\eta},\mathcal{T}x_{2\eta+1})\omega_1(x_{2\eta+1},\mathcal{T}x_{2\eta})\omega_1(x_{2\eta+1},\mathcal{T}x_{2\eta+1}) \end{Bmatrix}$$

$$+\ m(x_{2\eta},x_{2\eta+1}) - \emptyset m(x_{2\eta},x_{2\eta+1}), \tag{2}$$

where,

$$m(x_{2\eta},x_{2\eta+1}) = \max\begin{Bmatrix} \omega_1{}^2(x_{2\eta},x_{2\eta+1}),\omega_1(x_{2\eta},\mathcal{T}x_{2\eta})\omega_1(x_{2\eta+1},\mathcal{T}x_{2\eta+1}), \\ \omega_2(x_{2\eta},\mathcal{T}x_{2\eta+1})\omega_1(x_{2\eta+1},\mathcal{T}x_{2\eta}), \\ \frac{1}{2}\begin{bmatrix} \omega_1(x_{2\eta},\mathcal{T}x_{2\eta})\omega_2(x_{2\eta},\mathcal{T}x_{2\eta+1}) \\ +\omega_1(x_{2\eta+1},\mathcal{T}x_{2\eta})\omega_1(x_{2\eta+1},\mathcal{T}x_{2\eta+1}) \end{bmatrix} \end{Bmatrix}. \tag{3}$$

Using (1) we get

$$[1+p\omega_1(x_{2\eta},x_{2\eta+1})]\omega_1{}^2(x_{2\eta+1},x_{2\eta+2})$$

$$\leq pmax\begin{Bmatrix} \frac{1}{2}\begin{bmatrix} \omega_1{}^2(x_{2\eta},x_{2\eta+1})\omega_1(x_{2\eta+1},x_{2\eta+2}) \\ +\ \omega_1(x_{2\eta},x_{2\eta+1})\omega_1{}^2(x_{2\eta+1},x_{2\eta+2}) \end{bmatrix}, \\ \omega_1(x_{2\eta},x_{2\eta+1})\omega_2(x_{2\eta},x_{2\eta+2})\omega_1(x_{2\eta+1},x_{2\eta+1}), \\ \omega_2(x_{2\eta},x_{2\eta+2})\omega_1(x_{2\eta+1},x_{2\eta+1})\omega_1(x_{2\eta+1},x_{2\eta+2}) \end{Bmatrix}$$

$$+\ m(x_{2\eta},x_{2\eta+1}) - \emptyset m(x_{2\eta},x_{2\eta+1})\ , \tag{4}$$

where,

$$m(x_{2\eta},x_{2\eta+1}) = \max\begin{Bmatrix} \omega_1{}^2(x_{2\eta},x_{2\eta+1}),\omega_1(x_{2\eta},x_{2\eta+1})\omega_1(x_{2\eta+1},x_{2\eta+2}), \\ \omega_2(x_{2\eta},x_{2\eta+2})\omega_1(x_{2\eta+1},x_{2\eta+1}), \\ \frac{1}{2}\begin{bmatrix} \omega_1(x_{2\eta},x_{2\eta+1})\omega_2(x_{2\eta},x_{2\eta+2}) \\ +\omega_1(x_{2\eta+1},x_{2\eta+1})\omega_1(x_{2\eta+1},x_{2\eta+2}) \end{bmatrix} \end{Bmatrix} \tag{5}$$

Now consider $\alpha_{2\eta} = \omega_1(x_{2\eta},x_{2\eta+1})$; then we have

$$[1+p\alpha_{2\eta}]\alpha_{2\eta+1}^2 \leq pmax\left\{\frac{1}{2}\left[\alpha_{2\eta}^2\alpha_{2\eta+1} +\ \alpha_{2\eta}\alpha_{2\eta+1}^2\right],0,0\right\} + m(x_{2\eta},x_{2\eta+1}) - \emptyset m(x_{2\eta},x_{2\eta+1})\ , \tag{6}$$

where, $m(x_{2\eta},x_{2\eta+1}) = \max\left\{\alpha_{2\eta}^2,\alpha_{2\eta}\alpha_{2\eta+1},0,\frac{1}{2}\left[\alpha_{2\eta}\omega_2(x_{2\eta},x_{2\eta+2})+0\right]\right\}$.

By triangular inequality and using the property of ∅, we get

$$\omega_2(x_{2\eta},x_{2\eta+2}) \leq \omega_1(x_{2\eta},x_{2\eta+1}) + \omega_1(x_{2\eta+1},x_{2\eta+2})$$

$$= \alpha_{2\eta} + \alpha_{2\eta+1}, \qquad (7)$$

and

$m(x_{2\eta}, x_{2\eta+1}) =$

$$m(x, y) \leq \max\left\{\alpha_{2\eta}^2, \alpha_{2\eta}\alpha_{2\eta+1}, 0, \frac{1}{2}\left[\alpha_{2\eta}\left(\alpha_{2\eta} + \alpha_{2\eta+1}\right) + 0\right]\right\}. \qquad (8)$$

If $\alpha_{2\eta} < \alpha_{2\eta+1}$, then (C_3) reduces to

$p\alpha_{2\eta+1}^2 \leq p\alpha_{2\eta+1}^2 - \emptyset\alpha_{2\eta+1}^2$, a contradiction.

Therefore, $\alpha_{2\eta+1}^2 \leq \alpha_{2\eta}^2 \Rightarrow \alpha_{2\eta+1} \leq \alpha_{2\eta}$.

Case II. In a similar way, if η is odd, then we can obtain $\alpha_{2\eta+2} < \alpha_{2\eta+1}$.
It follows that the sequence $\{\alpha_\eta\}$ is decreasing.
Let $\lim_{\eta\to+\infty} \alpha_\eta = r$, for some $r \geq 0$.
Suppose $r > 0$; then from the inequality (C_3) and (C_4), we have

$$\left[1 + p\omega_1(x_\eta, x_{\eta+1})\right]\omega_1{}^2(\mathcal{T}x_\eta, \mathcal{T}x_{\eta+1})$$

$$\leq pmax\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c}\omega_1{}^2(x_\eta, \mathcal{T}x_\eta)\omega_1(x_{\eta+1}, \mathcal{T}x_{\eta+1}) \\ +\omega_1(x_\eta, \mathcal{T}x_\eta)\omega_1{}^2(x_{\eta+1}, \mathcal{T}x_{\eta+1})\end{array}\right], \\ \omega_1(x_\eta, \mathcal{T}x_\eta)\omega_2(x_\eta, \mathcal{T}x_{\eta+1})\omega_1(x_{\eta+1}, \mathcal{T}x_\eta), \\ \omega_2(x_\eta, \mathcal{T}x_{\eta+1})\omega_1(x_{\eta+1}, \mathcal{T}x_\eta)\omega_1(x_{\eta+1}, \mathcal{T}x_{\eta+1}) \end{array}\right\}$$

$$+ m(x_\eta, x_{\eta+1}) - \emptyset m(x_\eta, x_{\eta+1}), \qquad (9)$$

where, $m(x_\eta, x_{\eta+1}) =$

$$\max\left\{\begin{array}{c} \omega_1{}^2(x_\eta, x_{\eta+1}), \omega_1(x_\eta, \mathcal{T}x_\eta)\omega_1(x_{\eta+1}, \mathcal{T}x_{\eta+1}), \\ \omega_2(x_\eta, \mathcal{T}x_{\eta+1})\omega_1(x_{\eta+1}, \mathcal{T}x_\eta), \\ \frac{1}{2}\left[\begin{array}{c}\omega_1(x_\eta, \mathcal{T}x_\eta)\omega_2(x_\eta, \mathcal{T}x_{\eta+1}) \\ +\omega_1(x_{\eta+1}, \mathcal{T}x_\eta)\omega_1(x_{\eta+1}, \mathcal{T}x_{\eta+1})\end{array}\right] \end{array}\right\}. \qquad (10)$$

Now by using (1) we get,

$$\left[1 + p\omega_1(x_\eta, x_{\eta+1})\right]\omega_1{}^2(x_{\eta+1}, x_{\eta+2})$$

$$\leq pmax\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c}\omega_1{}^2(x_\eta, x_{\eta+1})\omega_1(x_{\eta+1}, x_{\eta+2}) \\ +\omega_1(x_\eta, x_{\eta+1})\omega_1{}^2(x_{\eta+1}, x_{\eta+2})\end{array}\right], \\ \omega_1(x_\eta, x_{\eta+1})\omega_2(x_\eta, x_{\eta+2})\omega_1(x_{\eta+1}, x_{\eta+1}), \\ \omega_2(x_\eta, x_{\eta+2})\omega_1(x_{\eta+1}, x_{\eta+1})\omega_1(x_{\eta+1}, x_{\eta+2}) \end{array}\right\}$$

$$+ m(x_\eta, x_{\eta+1}) - \emptyset m(x_\eta, x_{\eta+1}), \qquad (11)$$

where, $m(x_\eta, x_{\eta+1}) = \max \left\{ \begin{matrix} {\omega_1}^2(x_\eta, x_{\eta+1}), \omega_1(x_\eta, x_{\eta+1})\omega_1(x_{\eta+1}, x_{\eta+2}), \\ \omega_2(x_\eta, x_{\eta+2})\omega_1(x_{\eta+1}, x_{\eta+1}), \\ \frac{1}{2}\begin{bmatrix} \omega_1(x_\eta, x_{\eta+1})\omega_2(x_\eta, x_{\eta+2}) \\ +\omega_1(x_{\eta+1}, x_{\eta+1})\omega_1(x_{\eta+1}, x_{\eta+2}) \end{bmatrix} \end{matrix} \right\}$

Using the triangular inequality and the property of $\emptyset$, and taking the limit $\eta \to +\infty$, we get

$[1+pr]r^2 \le pr^3 + r^2 - \emptyset(r^2)$. (12)

Then $\emptyset(r^2) \le 0$, since r is positive, then by the property of $\emptyset$, we get $r = 0$, and we conclude that

$\lim_{\eta\to+\infty} \alpha_\eta = \lim_{\eta\to+\infty} \omega_1(x_\eta, x_{\eta+1}) = r = 0$. (13)

Now we show that $\{x_\eta\}$ is a Cauchy sequence. For the given $\epsilon > 0$, we can find two sequences of positive integers $\{m(\sigma)\}$ and $\{\eta(\sigma)\}$ such that

$\omega_8(x_{m(\sigma)}, x_{\eta(\sigma)}) \ge \epsilon, \qquad \omega_{\frac{1}{4}}(x_{m(\sigma)}, x_{\eta(\sigma)-1}) < \epsilon$ (14)

and $\eta(\sigma) > m(\sigma) > \sigma$.

Now $\epsilon \le \omega_8(x_{m(\sigma)}, x_{\eta(\sigma)})$

$\le \omega_2(x_{m(\sigma)}, x_{\eta(\sigma)}) + \omega_1(x_{m(\sigma)}, x_{\eta(\sigma)})$

$\le \omega_{\frac{1}{2}}(x_{m(\sigma)}, x_{\eta(\sigma)-1}) + \omega_{\frac{1}{2}}(x_{\eta(\sigma)-1}, x_{\eta(\sigma)})$

$\le \omega_{\frac{1}{4}}(x_{m(\sigma)}, x_{\eta(\sigma)-1}) + \omega_{\frac{1}{2}}(x_{\eta(\sigma)-1}, x_{\eta(\sigma)})$

$\le \epsilon + \omega_{\frac{1}{2}}(x_{\eta(\sigma)-1}, x_{\eta(\sigma)})$

Letting $\sigma \to +\infty$, we get $\lim_{\sigma\to+\infty} \omega_2(x_{m(\sigma)}, x_{\eta(\sigma)}) = \lim_{\sigma\to+\infty} \omega_1(x_{m(\sigma)}, x_{\eta(\sigma)}) = \epsilon$

Again using the triangular inequality, we have

$\epsilon \le \omega_8(x_{m(\sigma)}, x_{\eta(\sigma)}) \le \omega_4(x_{m(\sigma)}, x_{\eta(\sigma)})$

$\le \omega_2(x_{\eta(\sigma)}, x_{\eta(\sigma)+1}) + \omega_2(x_{m(\sigma)}, x_{\eta(\sigma)+1})$. (15)

We get

$\epsilon - \omega_2(x_{\eta(\sigma)}, x_{\eta(\sigma)+1}) \le \omega_2(x_{m(\sigma)}, x_{\eta(\sigma)+1}) \le \omega_1(x_{m(\sigma)}, x_{\eta(\sigma)+1})$

$\le \omega_{\frac{1}{4}}(x_{m(\sigma)}, x_{\eta(\sigma)+1})$

$\le \omega_{\frac{1}{8}}(x_{m(\sigma)}, x_{\eta(\sigma)}) + \omega_{\frac{1}{8}}(x_{\eta(\sigma)}, x_{\eta(\sigma)+1})$.

Taking the limits as $\sigma \to +\infty$, we have

$\lim_{\sigma\to+\infty} \omega_1(x_{m(\sigma)}, x_{\eta(\sigma)+1}) = \lim_{\sigma\to+\infty} \omega_2(x_{m(\sigma)}, x_{\eta(\sigma)+1}) = \epsilon$. (16)

Now from the triangular inequality, we have

$\epsilon \le \omega_2(x_{m(\sigma)}, x_{\eta(\sigma)}) \le \omega_1(x_{m(\sigma)}, x_{m(\sigma)+1}) + \omega_1(x_{m(\sigma)+1}, x_{\eta(\sigma)})$

We get

ϵ -$\omega_1(x_{m(\sigma)}, x_{m(\sigma)+1}) \leq \omega_1(x_{m(\sigma)+1}, x_{\eta(\sigma)})$
$\leq \omega_{\frac{1}{2}}(x_{\eta(\sigma)}, x_{m(\sigma)-1}) + \omega_{\frac{1}{2}}(x_{m(\sigma)+1}, x_{m(\sigma)-1})$
$\leq \omega_{\frac{1}{2}}(x_{\eta(\sigma)}, x_{m(\sigma)-1}) + \omega_{\frac{1}{4}}(x_{m(\sigma)-1}, x_{m(\sigma)}) + \omega_{\frac{1}{4}}(x_{m(\sigma)}, x_{m(\sigma)+1})$.

Letting $\sigma \to +\infty$, we have $\lim_{\sigma\to+\infty} \omega_1(x_{m(\sigma)+1}, x_{\eta(\sigma)}) = \epsilon$ (17)

Again, from the triangular inequality, we have

$$\omega_8(x_{m(\sigma)}, x_{\eta(\sigma)}) \leq \omega_4(x_{\eta(\sigma)}, x_{\eta(\sigma)+1}) + \omega_4(x_{\eta(\sigma)+1}, x_{m(\sigma)})$$

We get

$\omega_8(x_{m(\sigma)}, x_{\eta(\sigma)}) \leq \omega_4(x_{\eta(\sigma)}, x_{\eta(\sigma)+1}) + \omega_2(x_{m(\sigma)+1}, x_{m(\sigma)}) + \omega_2(x_{m(\sigma)+1}, x_{\eta(\sigma)+1})$

$\omega_8(x_{m(\sigma)}, x_{\eta(\sigma)})$-$\omega_4(x_{\eta(\sigma)}, x_{\eta(\sigma)+1})$-$\omega_2(x_{m(\sigma)+1}, x_{m(\sigma)}) \leq \omega_2(x_{m(\sigma)+1}, x_{\eta(\sigma)+1})$
$\leq \omega_1(x_{m(\sigma)+1}, x_{m(\sigma)}) + \omega_1(x_{\eta(\sigma)+1}, x_{m(\sigma)})$.

Letting $\sigma \to +\infty$, we have $\lim_{\sigma\to+\infty} \omega_2(x_{m(\sigma)+1}, x_{\eta(\sigma)+1}) = \epsilon$ (18)

Since $\omega_2(x_{m(\sigma)+1}, x_{\eta(\sigma)+1}) \leq \omega_1(x_{m(\sigma)+1}, x_{\eta(\sigma)+1})$
$\leq \omega_{\frac{1}{2}}(x_{m(\sigma)+1}, x_{m(\sigma)}) + \omega_{\frac{1}{2}}(x_{m(\sigma)}, x_{\eta(\sigma)+1}) \leq \omega_{\frac{1}{2}}(x_{m(\sigma)}, x_{\eta(\sigma)+1})$
$\leq \omega_{\frac{1}{4}}(x_{m(\sigma)}, x_{m(\sigma)-1}) + \omega_{\frac{1}{4}}(x_{m(\sigma)-1}, x_{\eta(\sigma)+1})$
$\leq \omega_{\frac{1}{8}}(x_{m(\sigma)-1}, x_{\eta(\sigma)}) + \omega_{\frac{1}{8}}(x_{\eta(\sigma)+1}, x_{\eta(\sigma)})$.

Letting $\sigma \to +\infty$, we have $\lim_{\sigma\to+\infty} \omega_1(x_{m(\sigma)+1}, x_{\eta(\sigma)+1}) = \epsilon$. (19)

On putting $x = x_{m(\sigma)}$ and $y = x_{\eta(\sigma)}$ in (C_3), we get

$$\begin{aligned}&[1 + p\omega_1(x_{m(\sigma)}, x_{\eta(\sigma)})]{\omega_1}^2(\mathcal{T}x_{m(\sigma)}, \mathcal{T}x_{\eta(\sigma)})\\ &\leq pmax\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}{\omega_1}^2(x_{m(\sigma)}, \mathcal{T}x_{m(\sigma)})\omega_1(x_{\eta(\sigma)}, \mathcal{T}x_{\eta(\sigma)})\\ +\omega_1(x_{m(\sigma)}, \mathcal{T}x_{m(\sigma)}){\omega_1}^2(x_{\eta(\sigma)}, \mathcal{T}x_{\eta(\sigma)})\end{array}\right],\\ \omega_1(x_{m(\sigma)}, \mathcal{T}x_{m(\sigma)})\omega_2(x_{m(\sigma)}, \mathcal{T}x_{\eta(\sigma)})\omega_1(x_{\eta(\sigma)}, \mathcal{T}x_{m(\sigma)}),\\ \omega_2(x_{m(\sigma)}, \mathcal{T}x_{\eta(\sigma)})\omega_1(x_{\eta(\sigma)}, \mathcal{T}x_{m(\sigma)})\omega_1(x_{\eta(\sigma)}, \mathcal{T}x_{\eta(\sigma)})\end{array}\right\}\\ &+ m(x_{m(\sigma)}, x_{\eta(\sigma)}) - \emptyset m(x_{m(\sigma)}, x_{\eta(\sigma)})\end{aligned} \quad (20)$$

where,

$$m(x_{m(\sigma)}, x_{\eta(\sigma)}) = \max\left\{ \begin{array}{c} \omega_1{}^2(x_{m(\sigma)}, x_{\eta(\sigma)}), \\ \omega_1(x_{m(\sigma)}, \mathcal{T}x_{m(\sigma)})\omega_1(x_{\eta(\sigma)}, \mathcal{T}x_{\eta(\sigma)}), \\ \omega_2(x_{m(\sigma)}, \mathcal{T}x_{\eta(\sigma)})\omega_1(x_{\eta(\sigma)}, \mathcal{T}x_{m(\sigma)}), \\ \frac{1}{2}\left[\begin{array}{c} \omega_1(x_{m(\sigma)}, \mathcal{T}x_{m(\sigma)})\omega_2(x_{m(\sigma)}, \mathcal{T}x_{\eta(\sigma)}) \\ +\omega_1(x_{\eta(\sigma)}, \mathcal{T}x_{m(\sigma)})\omega_1(x_{\eta(\sigma)}, \mathcal{T}x_{\eta(\sigma)}) \end{array} \right] \end{array} \right\}.$$

Using (1), we obtain

$$[1 + p\omega_1(x_{m(\sigma)}, x_{\eta(\sigma)})]\omega_1{}^2(x_{m(\sigma)+1}, x_{\eta(\sigma)+1}) \leq p\max\left\{ \begin{array}{c} \frac{1}{2}\left[\begin{array}{c} \omega_1{}^2(x_{m(\sigma)}, x_{m(\sigma)+1})\omega_1(x_{\eta(\sigma)}, x_{\eta(\sigma)+1}) \\ +\omega_1(x_{m(\sigma)}, x_{m(\sigma)+1})\omega_1{}^2(x_{\eta(\sigma)}, x_{\eta(\sigma)+1}) \end{array} \right], \\ \omega_1(x_{m(\sigma)}, x_{m(\sigma)+1})\omega_2(x_{m(\sigma)}, x_{\eta(\sigma)+1})\omega_1(x_{\eta(\sigma)}, x_{m(\sigma)+1}), \\ \omega_2(x_{m(\sigma)}, x_{\eta(\sigma)+1})\omega_1(x_{\eta(\sigma)}, x_{m(\sigma)+1})\omega_1(x_{\eta(\sigma)}, x_{\eta(\sigma)+1}) \end{array} \right\}$$

$$+m(x_{m(\sigma)}, x_{\eta(\sigma)}) - \emptyset m(x_{m(\sigma)}, x_{\eta(\sigma)}) \quad (21)$$

where,

$$m(x_{m(\sigma)}, x_{\eta(\sigma)}) = \max\left\{ \begin{array}{c} \omega_1{}^2(x_{m(\sigma)}, x_{\eta(\sigma)}), \\ \omega_1(x_{m(\sigma)}, x_{m(\sigma)+1})\omega_1(x_{\eta(\sigma)}, x_{\eta(\sigma)+1}), \\ \omega_2(x_{m(\sigma)}, x_{\eta(\sigma)+1})\omega_1(x_{\eta(\sigma)}, x_{m(\sigma)+1}), \\ \frac{1}{2}\left[\begin{array}{c} \omega_1(x_{m(\sigma)}, x_{m(\sigma)+1})\omega_2(x_{m(\sigma)}, x_{\eta(\sigma)+1}) \\ +\omega_1(x_{\eta(\sigma)}, x_{m(\sigma)+1})\omega_1(x_{\eta(\sigma)}, x_{\eta(\sigma)+1}) \end{array} \right] \end{array} \right\}$$

Letting $\sigma \to +\infty$ and using (13) - (19), we get

$$[1 + p\epsilon]\epsilon^2 \leq p\max\left\{\frac{1}{2}[0 + 0], 0,0\right\} + \epsilon^2 - \emptyset(\epsilon^2)$$
$$= \epsilon^2 - \emptyset(\epsilon^2),$$

a contradiction.

Thus, $\{x_\eta\}$ is a Cauchy sequence in ψ_ω, since (ψ_ω, ω) is a complete modular metric space.

Therefore, $\{x_\eta\}$ converges to a point z and $x_{\eta+1} = \mathcal{T}(x_\eta)$ also converges to the same point z, $\lim_{\eta\to\infty} x_\eta = z$. (22)

Now, we will prove that z is a fixed point of $\mathcal{T}$.

For this, let $x = x_\eta$ and $y = z$ in (C_3), we get

$$[1 + p\omega_1(x_\eta, z)]\omega_1{}^2(\mathcal{T}x_\eta, \mathcal{T}z)$$

$$\leq pmax\begin{Bmatrix} \frac{1}{2}\left[{\omega_1}^2(x_\eta,\mathcal{T}x_\eta)\omega_1(z,\mathcal{T}z)+\omega_1(x_\eta,\mathcal{T}x_\eta){\omega_1}^2(z,\mathcal{T}z)\right], \\ \omega_1(x_\eta,\mathcal{T}x_\eta)\omega_2(x_\eta,\mathcal{T}z)\omega_1(z,\mathcal{T}x_\eta), \\ \omega_2(x_\eta,\mathcal{T}z)\omega_1(z,\mathcal{T}x_\eta)\omega_1(z,\mathcal{T}z) \end{Bmatrix}+$$

$$+m(x_\eta,z)-\emptyset m(x_\eta,z) \quad (23)$$

where,

$$m(x_\eta,z) = \max\begin{Bmatrix} {\omega_1}^2(x_\eta,z),\omega_1(x_\eta,\mathcal{T}x_\eta)\omega_1(z,\mathcal{T}z),\omega_2(x_\eta,\mathcal{T}z)\omega_1(z,\mathcal{T}x_\eta), \\ \frac{1}{2}\left[\omega_1(x_\eta,\mathcal{T}x_\eta)\omega_2(x_\eta,\mathcal{T}z)+\omega_1(z,\mathcal{T}x_\eta)\omega_1(z,\mathcal{T}z)\right] \end{Bmatrix}$$

Using (22) and (1), we have

$$[1+p\omega_1(z,z)]{\omega_1}^2(z,\mathcal{T}z)$$

$$\leq pmax\begin{Bmatrix} \frac{1}{2}[{\omega_1}^2(z,z)\omega_1(z,\mathcal{T}z)+\omega_1(z,z){\omega_1}^2(z,\mathcal{T}z)], \\ \omega_1(z,z)\omega_2(z,\mathcal{T}z)\omega_1(z,z), \\ \omega_2(z,\mathcal{T}z)\omega_1(z,z)\omega_1(z,\mathcal{T}z) \end{Bmatrix}$$

$$+m(x_\eta,z)-\emptyset m(x_\eta,z) \quad (24)$$

where,

$$m(x_\eta,z) = \max\begin{Bmatrix} {\omega_1}^2(z,z),\omega_1(z,z)\omega_1(z,\mathcal{T}z),\omega_2(z,\mathcal{T}z)\omega_1(z,z), \\ \frac{1}{2}[\omega_1(z,z)\omega_2(z,\mathcal{T}z)+\omega_1(z,z)\omega_1(z,\mathcal{T}z)] \end{Bmatrix} = 0.$$

Hence, ${\omega_1}^2(z,\mathcal{T}z) \leq 0 \Rightarrow \mathcal{T}z = z$.
So, $\mathcal{T}$ has a fixed point in ψ_ω.

Uniqueness:

To show that $\mathcal{T}$ can have only one common fixed point.
Suppose $x \neq y$ be two fixed points of $\mathcal{T}$.
Therefore, $x = \mathcal{T}x$ and $y = \mathcal{T}y$ from (C_3) ,we have

$$[1+p\omega_1(x,y)]{\omega_1}^2(x,y) \leq pmax\{0,0,0\}+m(x,y)-\emptyset m(x,y)$$
$$\leq (1-\emptyset){\omega_1}^2(x,y)$$
$$\Rightarrow {\omega_1}^2(x,y)\,[1+\omega_1(x,y)-1+\emptyset] \leq 0$$
$$\Rightarrow {\omega_1}^2(x,y) = 0$$
$$\Rightarrow x = y.$$

This completes the proof.

Corollary 1. Let $\mathcal{T}$ be a mapping of a complete modular metric space (ψ_ω,ω) into itself satisfying the condition

$${\omega_1}^2(\mathcal{T}x,\mathcal{T}y) \leq m(x,y)-\emptyset m(x,y)$$

where,

$$m(x,y) = \max\left\{\begin{matrix}\omega_1{}^2(x,y), \omega_1(x,\mathcal{T}x)\omega_1(y,\mathcal{T}y), \\ \omega_2(x,\mathcal{T}y)\omega_1(y,\mathcal{T}x), \\ \frac{1}{2}[\omega_1(x,\mathcal{T}x)\omega_2(x,\mathcal{T}y) + \omega_1(y,\mathcal{T}x)\omega_1(y,\mathcal{T}y)]\end{matrix}\right\}.$$

For all $x, y \in \psi$ and $\emptyset: [0,+\infty) \to [0,+\infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$. Then $\mathcal{T}$ has a unique fixed point in ψ_ω.

Proof. Put $p = 0$ in Theorem 1 and we have the required result.

Example 1. Let $\psi = \mathbb{R}$. We define the mapping ω: (0,1) × $\mathbb{R}$ × $\mathbb{R}$ →[0,1] by $\omega_\lambda(x, y) = \frac{|x-y|}{1+\lambda}$ for all x, $y \in \mathbb{R}$ and λ> 0. Then it is obvious that $\mathbb{R}_\omega$ is a complete modular metric space. Define $\mathcal{T}$: $\mathbb{R}_\omega \to \mathbb{R}_\omega$ by $\mathcal{T}x = \frac{x}{4}$ and $\emptyset: [0,+\infty) \to [0,+\infty)$ by $\emptyset(t) = \frac{t}{3}$, for any values of $p > 0$ and $x, y \in \psi$.
Then it is easy to verify the inequalities (C_3) and (C_4) hold.
Hence from Theorem 1, the mapping $\mathcal{T}$ has a unique fixed point 0.
Moreover, it is 0 ∈ $\mathbb{R}_\omega$.

Property P

In this section, we will show that the maps satisfying (C_3) and (C_4) possess the property P.

Let us denote the set of all fixed points of a self –mapping $\mathcal{T}$ from X into itself by F($\mathcal{T}$), that is, F($\mathcal{T}$)= { $z \in X$: $\mathcal{T}z = z$ }. It is clearly that if z is a fixed point of $\mathcal{T}$, then it is also a fixed point of $\mathcal{T}^n$ for each $n \in \mathbb{N}$, that is, F($\mathcal{T}$)⊂F($\mathcal{T}^n$) if F($\mathcal{T}$) ≠ ϕ. However, converse is false.

Indeed the mapping $\mathcal{T}$: $\mathbb{R} \to \mathbb{R}$ defined by $\mathcal{T}x = \frac{1}{2} - x$ has a unique fixed point, that is, F ($\mathcal{T}$) = $\left\{\frac{1}{4}\right\}$, but every $x \in \mathbb{R}$ is a fixed point for $\mathcal{T}^2$.
If F($\mathcal{T}$) = F($\mathcal{T}^n$), for each $n \in \mathbb{N}$, then we say that $\mathcal{T}^n$ has no periodic points.
(Jeong & Rhoades, 2005) examined a number of situations in which the fixed point sets for maps and their iterates are the same. They state that a map $\mathcal{T}$ has the property P if F($\mathcal{T}$) = F($\mathcal{T}^n$) for each $n \in \mathbb{N}$.

Theorem 2.

Under the condition of Theorem 1, $\mathcal{T}$ has the property P.

Proof. From Theorem 1, $\mathcal{T}$ has a fixed point. Therefore, $F(\mathcal{T}^n) \neq \phi$ for each $n \in \mathbb{N}$. Fix $n > 1$ and assume that $p \in F(\mathcal{T}^n)$.
We wish to show that $p \in F(\mathcal{T})$.
Suppose that $p \neq \mathcal{T}p$.
Using the inequality (C_3), we have

$$[1 + p\omega_1(\mathcal{T}^{n-1}p, \mathcal{T}^n p)]\omega_1{}^2(\mathcal{T}\mathcal{T}^{n-1}p, \mathcal{T}\mathcal{T}^n p)$$
$$\leq \quad pmax\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c}\omega_1{}^2(\mathcal{T}^{n-1}p, \mathcal{T}\mathcal{T}^{n-1}p)\omega_1(\mathcal{T}^n p, \mathcal{T}\mathcal{T}^n p) + \\ \omega_1(\mathcal{T}^{n-1}p, \mathcal{T}\mathcal{T}^{n-1}p)\omega_1{}^2(\mathcal{T}^n p, \mathcal{T}\mathcal{T}^n p)\end{array}\right], \\ \omega_1(\mathcal{T}^{n-1}p, \mathcal{T}\mathcal{T}^{n-1}p)\omega_2(\mathcal{T}^{n-1}p, \mathcal{T}\mathcal{T}^n p)\omega_1(\mathcal{T}^n p, \mathcal{T}\mathcal{T}^{n-1}p), \\ \omega_2(\mathcal{T}^{n-1}p, \mathcal{T}\mathcal{T}^n p)\omega_1(\mathcal{T}^n p, \mathcal{T}\mathcal{T}^{n-1}p)\omega_1(\mathcal{T}^n p, \mathcal{T}\mathcal{T}^n p)\end{array}\right\}$$
$$+ \ m(\mathcal{T}^{n-1}p, \mathcal{T}^n p) - \emptyset m(\mathcal{T}^{n-1}p, \mathcal{T}^n p)$$

where,

$$m(\mathcal{T}^{n-1}p, \mathcal{T}^n p)$$
$$= \max\left\{\begin{array}{c}\omega_1{}^2(\mathcal{T}^{n-1}p, \mathcal{T}^n p), \omega_1(\mathcal{T}^{n-1}p, \mathcal{T}\mathcal{T}^{n-1}p)\omega_1(\mathcal{T}^n p, \mathcal{T}\mathcal{T}^n p), \\ \omega_2(\mathcal{T}^{n-1}p, \mathcal{T}\mathcal{T}^n p)\omega_1(\mathcal{T}^n p, \mathcal{T}\mathcal{T}^{n-1}p), \\ \frac{1}{2}\left[\begin{array}{c}\omega_1(\mathcal{T}^{n-1}p, \mathcal{T}\mathcal{T}^{n-1}p)\omega_2(\mathcal{T}^{n-1}p, \mathcal{T}\mathcal{T}^n p) \\ +\omega_1(\mathcal{T}^n p, \mathcal{T}\mathcal{T}^{n-1}p)\omega_1(\mathcal{T}^n p, \mathcal{T}\mathcal{T}^n p)\end{array}\right]\end{array}\right\}$$

$$[1 + p\omega_1(\mathcal{T}^{n-1}p, \mathcal{T}^n p)]\omega_1{}^2(\mathcal{T}^n p, \mathcal{T}^{n+1}p)$$
$$\leq pmax\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\omega_1{}^2(\mathcal{T}^{n-1}p, \mathcal{T}^n p)\omega_1(\mathcal{T}^n p, \mathcal{T}^{n+1}p) + \\ \omega_1(\mathcal{T}^{n-1}p, \mathcal{T}^n p)\omega_1{}^2(\mathcal{T}^n p, \mathcal{T}^{n+1}p)\end{array}\right], \\ \omega_1(\mathcal{T}^{n-1}p, \mathcal{T}^n p)\omega_2(\mathcal{T}^{n-1}p, \mathcal{T}^{n+1}p)\omega_1(\mathcal{T}^n p, \mathcal{T}^n p), \\ \omega_2(\mathcal{T}^{n-1}p, \mathcal{T}^{n+1}p)\omega_1(\mathcal{T}^n p, \mathcal{T}^n p)\omega_1(\mathcal{T}^n p, \mathcal{T}^{n+1}p)\end{array}\right\}$$
$$+ m(\mathcal{T}^{n-1}p, \mathcal{T}^n p) - \emptyset m(\mathcal{T}^{n-1}p, \mathcal{T}^n p)$$

where,

$$m(\mathcal{T}^{n-1}p, \mathcal{T}^n p)$$
$$= \max\left\{\begin{array}{c}\omega_1{}^2(\mathcal{T}^{n-1}p, \mathcal{T}^n p), \omega_1(\mathcal{T}^{n-1}p, \mathcal{T}^n p)\omega_1(\mathcal{T}^n p, \mathcal{T}^{n+1}p), \\ \omega_2(\mathcal{T}^{n-1}p, \mathcal{T}^{n+1}p)\omega_1(\mathcal{T}^n p, \mathcal{T}^n p), \\ \frac{1}{2}[\omega_1(\mathcal{T}^{n-1}p, \mathcal{T}^n p)\omega_2(\mathcal{T}^{n-1}p, \mathcal{T}^{n+1}p) + \omega_1(\mathcal{T}^n p, \mathcal{T}^n p)\omega_1(\mathcal{T}^n p, \mathcal{T}^{n+1}p)]\end{array}\right\}$$

$$[1 + p\omega_1(\mathcal{T}^{n-1}p, p)]\omega_1{}^2(p, \mathcal{T}p)$$
$$\leq pmax\left\{\begin{array}{c}\frac{1}{2}[\omega_1{}^2(\mathcal{T}^{n-1}p, p)\omega_1(p, \mathcal{T}p) + \omega_1(\mathcal{T}^{n-1}p, p)\omega_1{}^2(p, \mathcal{T}p)], \\ \omega_1(\mathcal{T}^{n-1}p, p)\omega_2(\mathcal{T}^{n-1}p, \mathcal{T}p)\omega_1(p, p), \\ \omega_2(\mathcal{T}^{n-1}p, \mathcal{T}p)\omega_1(p, p)\omega_1(p, \mathcal{T}p)\end{array}\right\}$$
$$+ m(\mathcal{T}^{n-1}p, p) - \emptyset m(\mathcal{T}^{n-1}p, p)$$

where,

$m(\mathcal{T}^{n-1}p,p) =$
$\max\left\{\begin{matrix}\omega_1{}^2(\mathcal{T}^{n-1}p,p), \omega_1(\mathcal{T}^{n-1}p,p)\omega_1(p,\mathcal{T}p), \omega_2(\mathcal{T}^{n-1}p,\mathcal{T}p)\omega_1(p,p), \\ \frac{1}{2}[\omega_1(\mathcal{T}^{n-1}p,p)\omega_2(\mathcal{T}^{n-1}p,\mathcal{T}p) + \omega_1(p,p)\omega_1(p,\mathcal{T}p)]\end{matrix}\right\}$ =
$\omega_1{}^2(p,\mathcal{T}p)$.
If $\omega_1(\mathcal{T}^{n-1}p,p) \leq \omega_1(p,\mathcal{T}p)$ then
$\omega_1{}^2(p,\mathcal{T}p) \leq \omega_1{}^2(p,\mathcal{T}p)$ - $\emptyset\omega_1{}^2(p,\mathcal{T}p)$.
This implies that $p = \mathcal{T}p$, a contradiction.
Therefore, $p \in F(\mathcal{T})$ and $\mathcal{T}$has the property P.

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СВОЙСТВО P В МОДУЛЬНЫХ МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ

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РУБРИКА ГРНТИ: 27.01.00 Общие вопросы математики,
27.25.17 Метрическая теория функций,
27.39.27 Нелинейный функциональный анализ,
27.43.17 Математическая статистика
ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: Цель данной статьи заключается в представлении концепции обобщенного ∅-слабого сжимающего условия, включающего различные комбинации d(x,y) в модулярных метрических пространствах.

Методы: В данной статье применялись общепринятые теоретические методы функционального анализа.

Результаты: В данном исследовании представлен результат (Murthy & Vara Prasad, 2013) по однозначному отображению, соответствующему обобщенному ∅-слабому условию сокращения, включающему различные комбинации d(x,y). Оно обобщается в задании модулярных метрических пространств, а затем доказывается, что приведенное однозначное отображение отвечает свойству P. В заключении приводится пример, подтверждающий результаты.

Выводы: При соответствующих обобщениях можно сформулировать широко известные результаты классических метрических пространств для случая модулярных метрических пространств.

Ключевые слова: Фиксированная точка, ∅-слабое сжатие, модулярные метрические пространства, свойство P.

ОСОБИНА P У МОДУЛАРНИМ МЕТРИЧКИМ ПРОСТОРИМА

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ОБЛАСТ: математика
ВРСТА ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: Циљ овог рада јесте да представи концепт генерализованог ∅-слабог контрактивног услова који укључује различите комбинације d(x,y) у модуларним метричким просторима.

Методе: Конвенционалне теоријске методе функционалне анализе.

Резултати: Представљен је резултат (Murthy & Vara Prasad, 2013) за сингуларно пресликавање које задовољава уопштени ∅-слаби контрактивни услов који укључује различите комбинације d(x,y). Он је уопштен у постављању модуларних метричких простора.Такође, доказано је да ово сингуларно пресликавање задовољава својство P. На крају је наведен пример који подржава резултат.

Закључак: Уз одговарајуће генерализације могуће је формулисати добро познате резултате класичних метричких простора који се односе на случај модуларних метричких простора.

Кључне речи: фиксна тачка, ∅-слаба контракција, модуларни метрички простори, својство P.

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