Zamfirescu mappings under Pata-type condition: results and application to an integral equation

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Abstract:

Introduction/purpose: Pata-type and Zamfirescu mappings are extended beyond metric spaces.

Methods: The concept of Pata-type Zamfirescu mapping within the framework of S-metric spaces is employed.

Results: A series of corresponding outcomes has been established. Furthermore, the obtained results are employed to solve an integral equation.

Conclusions: S-Pata type and Zamfirescu mappings have unique fixed points.

Key words: Pata-type contraction, Zamfirescu mapping, S-Metric space, fixed point.

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Introduction

In 1922, Banach (Banach, 1922) established that every contraction mapping on a complete metric space possesses a unique fixed point. This theorem is commonly referred to as the Banach fixed point theorem.

Since Banach's theorem was proven, various other types of mappings have been demonstrated to possess the same fixed point property. Among these mappings are Kannan-type mappings and Chatterjea-type mappings, both of which were introduced in the 1960s. These mappings hold significance because they enable the existence of fixed points even for non-continuous mappings.

In 1972, Zamfirescu (Zamfirescu, 1972) introduced a generalized contraction mapping that further extended the class of mappings for which the fixed point property can be guaranteed. Zamfirescu's results generalized the work of several other mathematicians, including Kannan (Kannan, 1968) and Chatterjea (Chatterjea, 1972).

In the paper, Ψ represents the set of ascending functions $\psi : [0,1] \to [0,+\infty)$, where ψ exhibits continuity at 0 and starts at $\psi(0) = 0$.

Definition 1. (Zamfirescu, 1972) If (∇, d) is a metric space, a function $\Gamma : \nabla \to \nabla$ is referred to as a Zamfirescu mapping if it satisfies the following condition for any points ϑ and θ in ∇ and real numbers a, b, and c within the interval $[0,1)$:

$$d(\Gamma(\vartheta),\Gamma(\theta)) \leq \\ \max\left\{ad(\vartheta,\theta), \tfrac{b}{2}[d(\vartheta,\Gamma(\vartheta)) + d(\theta,\Gamma(\theta))], \tfrac{c}{2}[d(\vartheta,\Gamma(\theta)) + d(\theta,\Gamma(\vartheta))]\right\}$$

In 2011, Pata (Pata, 2011) introduced an improved version of the classical Banach Principle. This enhancement enables the identification of fixed points for mappings that lack strict contraction properties, instead relying on approximate contraction characteristics.

Theorem 1. *(Pata, 2011) If (X,d) is a metric space that is complete and fixed constants $\Lambda \geq 0$, $\alpha \geq 1$ and β which lies in the interval $[0,\alpha]$. If the mapping $\Gamma : X \to X$ fulfills the subsequent inequality for each $\varepsilon \in [0,1]$ and all $\vartheta,\theta \in X$,*

$$d(\Gamma\vartheta,\Gamma\theta) \leq (1-\varepsilon)d(\vartheta,\theta) + \Lambda\varepsilon^{\alpha}[1 + \|\vartheta\| + \|\theta\|]^{\beta}$$

then Γ possesses a unique fixed point $\vartheta^* \in X$, and the sequence $\{\Gamma^n \vartheta_0\}$ exhibits convergence towards ϑ^* for any given initial element $\vartheta_0 \in X$.

Many other authors have previously employed the Pata-type condition to derive novel fixed point outcomes (Kadelburg & Radenović, 2014; Özgür & Taş, 2021; Kadelburg & Radenović, 2016; Karapinar et al., 2020a; Saleem et al., 2020; Karapınar et al., 2020b; Aktay & Özdemir, 2022; Yahaya et al., 2023; Roy et al., 2024).

Further, in 2018 Jacob et al. (Jacob et al., 2018) defined Pata type Zamfirescu mapping and generalized the results of (Chatterjea, 1972; Pata, 2011).

Definition 2. (Jacob et al., 2018) If (∇, d) is a metric space, a mapping $\Gamma : \nabla \to \nabla$ is considered as a Pata-type Zamfirescu mapping if, for all ϑ and θ in ∇, and for every $\varepsilon \in [0, 1]$, it holds the following inequality with ψ in Ψ,:

$$d(\Gamma(\vartheta), \Gamma(\theta)) \leq (1-\varepsilon)M(\vartheta, \theta) + \Lambda\varepsilon^\alpha \psi(\varepsilon)[1 + \|\vartheta\| + \|\theta\| + \|\Gamma\vartheta\| + \|\Gamma\theta\|]^\beta$$

where $M(\vartheta, \theta) = \max\left\{d(\vartheta, \theta), \frac{d(\vartheta,\Gamma(\vartheta))+d(\theta,\Gamma(\theta))}{2}, \frac{d(\vartheta,\Gamma(\theta))+d(\theta,\Gamma(\vartheta))}{2}\right\}$ and $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$ are constants.

In 2012, Sedghi et al. (Sedghi et al., 2012) presented the notion of an S-metric space.

Definition 3. (Sedghi et al., 2012) If ∇ is a nonempty set, an S-metric is a function $S : \nabla \times \nabla \times \nabla \to [0, +\infty)$ on ∇ that meets the requirements for every $\vartheta, \theta, \delta, a \in \nabla$ as follows:

(i) $S(\vartheta, \theta, \delta) \geq 0$,

(ii) $S(\vartheta, \theta, \delta) = 0$ if and only if $\vartheta = \theta = \delta$,

(iii) $S(\vartheta, \theta, \delta) \leq S(\vartheta, \vartheta, a) + S(\theta, \theta, a) + S(\delta, \delta, a)$.

The term "S-metric space" refers to the pair (∇, S).

Example 1. (Sedghi et al., 2012) If ∇ is a nonempty set equipped with an ordinary metric d on ∇, then the following are S-metrics on ∇.

- $S(\vartheta, \theta, \delta) = d(\vartheta, \delta) + d(\theta, \delta)$
- $S(\vartheta, \theta, \delta) = d(\vartheta, \theta) + d(\theta, \delta) + d(\delta, \vartheta)$

Lemma 1. *(Sedghi et al., 2012) If the space (∇, S) is an S-metric space, then as a result, there is for all $\vartheta, \theta \in \nabla$,*

$$S(\vartheta, \vartheta, \theta) = S(\theta, \theta, \vartheta)$$

.

Lemma 2. *(Sedghi et al., 2012) In an S-metric space (∇, S), for all $\vartheta, \theta, \delta \in \nabla$, there is*

$$S(\vartheta, \vartheta, \theta) \leq 2S(\vartheta, \vartheta, \delta) + S(\theta, \theta, \delta)$$

and

$$S(\vartheta, \vartheta, \theta) \leq 2S(\vartheta, \vartheta, \delta) + S(\delta, \delta, \theta)$$

Definition 4. (Sedghi et al., 2012) If (∇, S) is an S-metric space,

(i) a sequence $\{\vartheta_n\}$ in ∇ converges to ϑ if and only if $S(\vartheta_n, \vartheta_n, \vartheta) \to 0$ as $n \to +\infty$. This convergence is denoted as $\lim_{n\to+\infty} \vartheta_n = \vartheta$,

(ii) a sequence $\{\vartheta_n\}$ in ∇ is referred to as a Cauchy sequence if, for given $\varepsilon > 0$ there is a $n_0 \in \mathbb{N}$ such that $S(\vartheta_n, \vartheta_n, \vartheta_m) < \varepsilon$ for every $n, m \geq n_0$, and

(iii) the space (∇, S) is defined as complete when it satisfies the condition that every Cauchy sequence in ∇ converges.

Several other authors have contributed to the field of S-metric spaces, establishing numerous results within this framework as well as in various extended spaces related to S-metric spaces. Some of these pertinent works can be found in the references (Sedghi et al., 2012; Chand & Rohen, 2023; Özgür & Taş, 2023; Priyobarta et al., 2022).

Inspired by the findings outlined above, this paper introduces the concept of S-Pata type Zamfirescu mappings and demonstrates the existence of fixed point results within the framework of S-metric spaces. Notably, these results extend the findings from a previous study (Jacob et al., 2018). Furthermore, the best proximity point theorem has been established and findings applied to the context of integral equations.

Main results

In this section, the aim is to demonstrate the existence and uniqueness of fixed points for S-Pata type Zamfirescu mappings. Consider an S-metric space denoted as (∇, S). Throughout this discussion, the norm of an element ϑ is represented as $\|\vartheta\| = S(\vartheta_0, \vartheta_0, \vartheta)$, where ϑ_0 is a chosen element in ∇.

Definition 5. In an S-metric space (∇, S), a mapping $\Gamma : \nabla \to \nabla$ is referred to as an S-Pata type Zamfirescu mapping if it satisfies the following inequality for all $\vartheta, \theta, \delta \in \nabla$, $\psi \in \Psi$, and every $\varepsilon \in [0, 1]$:

$$S(\Gamma\vartheta, \Gamma\theta, \Gamma\delta) \leqslant (1-\epsilon)M(\vartheta, \theta, \delta) + \Lambda\epsilon^{\alpha}\psi(\epsilon)[1+ \\ \| \vartheta \| + \| \theta \| + \| \delta \| + \| \Gamma\vartheta \| + \| \Gamma\theta \| + \| \Gamma\delta \|]^{\beta}$$

where

$$M(\vartheta, \theta, \delta) = \max\left\{S(\vartheta, \theta, \delta), \frac{S(\vartheta, \vartheta, \Gamma\vartheta) + S(\theta, \theta, \Gamma\theta) + S(\delta, \delta, \Gamma\delta)}{3}, \frac{S(\vartheta, \vartheta, \Gamma\theta) + S(\theta, \theta, \Gamma\delta) + S(\delta, \delta, \Gamma\vartheta)}{4}\right\}$$

$\Lambda \geqslant 0, \alpha \geqslant 1, \beta \in [0, \alpha]$ are the constants.

Note: From the above definition, it can easily be deduced that

$$S(\Gamma\vartheta, \Gamma\vartheta, \Gamma\theta) \leqslant (1-\epsilon)M(\vartheta, \vartheta, \theta) + \Lambda\epsilon^{\alpha}\psi(\epsilon)[1+2\|\vartheta\| + \|\theta\| + 2\|\Gamma\vartheta\| + \|\Gamma\theta\|]^{\beta}$$

where

$$M(\vartheta, \vartheta, \theta) = \max\left\{S(\vartheta, \vartheta, \theta), \frac{2S(\vartheta, \vartheta, \Gamma\vartheta) + S(\theta, \theta, \Gamma\theta)}{3}, \frac{S(\vartheta, \vartheta, \Gamma\vartheta) + S(\vartheta, \vartheta, \Gamma\theta) + S(\theta, \theta, \Gamma\vartheta)}{4}\right\}$$

$\Lambda \geqslant 0, \alpha \geqslant 1, \beta \in [0, \alpha]$ are the constants.

The following lemma is a key ingredient in the proof of this work. This lemma will be used to prove the main results. The proof of the lemma in a metric space can be found in (Alghamdi et al., 2021). Here, it is extended it to the framework of an S-metric space.

Lemma 3. *(Alghamdi et al., 2021) Consider an S-metric space (∇, S). If the sequence $\{\vartheta_n\}$ in ∇ which is not Cauchy with $\lim_{n\to+\infty} S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) = 0$.*

Then, two sub-sequences $\{\vartheta_{n_k}\}$ *and* $\{\vartheta_{m_k}\}$ *of* $\{\vartheta_n\}$ *exist for any* $\varepsilon > 0$ *such that*

$$\lim_{k\to+\infty} S(\vartheta_{n_k+1}, \vartheta_{n_k+1}, \vartheta_{m_k+1}) = \varepsilon^{+} \tag{1}$$

$$\lim_{k\to+\infty} S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{m_k}) = \lim_{k\to+\infty} S(\vartheta_{n_k+1}, \vartheta_{n_k+1}, \vartheta_{m_k}) = \lim_{k\to+\infty} S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{m_k+1}) = \varepsilon \tag{2}$$

Proof. Since, $\{\vartheta_n\}$ is not a Cauchy sequence and $\lim_{n\to+\infty} S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) = 0$, one can get $\varepsilon > 0$ and $N_0 \geq 1$ such that for any $N > N_0$ there is $m, n > N$ with $m \geq n$ and

$$S(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{m+1}) > \varepsilon \text{ and } S(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_n) \leq \varepsilon.$$

By selecting the smallest $m \geq n$ so that $S(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{m+1}) > \varepsilon$ holds, it is concluded that for each $N > N_0$ there exists $m, n > N$ such that

$$S(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{m+1}) > \varepsilon \text{ and } S(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_m) \leq \varepsilon.$$

Thus, one can construct two sub-sequences $\{\vartheta_{n_k}\}$ and $\{\vartheta_{m_k}\}$ of $\{\vartheta_n\}$ such that

$$S(\vartheta_{n_k+1}, \vartheta_{n_k+1}, \vartheta_{m_k+1}) > \varepsilon \text{ and } S(\vartheta_{n_k+1}, \vartheta_{n_k+1}, \vartheta_{m_k}) \leq \varepsilon.$$

These inequalities, along with the triangular inequality, lead to

$$\begin{aligned}\varepsilon < S(\vartheta_{n_k+1}, \vartheta_{n_k+1}, \vartheta_{m_k+1}) &\leq S(\vartheta_{n_k+1}, \vartheta_{n_k+1}, \vartheta_{m_k}) + 2S(\vartheta_{m_k}, \vartheta_{m_k}, \vartheta_{m_k+1}) \\ &\leq \varepsilon + 2S(\vartheta_{m_k}, \vartheta_{m_k}, \vartheta_{m_k+1}).\end{aligned}$$

By means of the Sandwich Theorem, one arrives at (1). Furthermore, there exists

$$S(\vartheta_{n_k+1}, \vartheta_{n_k+1}, \vartheta_{m_k+1}) - 2S(\vartheta_{m_k+1}, \vartheta_{m_k+1}, \vartheta_{m_k}) \leq S(\vartheta_{n_k+1}, \vartheta_{n_k+1}, \vartheta_{m_k}) \leq \varepsilon$$

which implies the second limit of (2). From the subsequent two inequalities,

$$S(\vartheta_{n_k+1}, \vartheta_{n_k+1}, \vartheta_{m_k}) - 2S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{n_k+1}) \leq S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{m_k}) \leq \varepsilon + 2S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{n_k+1}),$$
$$\varepsilon - 2S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{n_k+1}) < S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{m_k+1}) \leq S(\vartheta_{n_k+1}, \vartheta_{n_k+1}, \vartheta_{m_k+1}) + 2S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{n_k+1})$$

from these inequalities one will get the required limits. □

Theorem 2. *Let* (∇, S) *be a complete* S*-metric space. If* $\Gamma : \nabla \to \nabla$ *is an* S*-Pata type Zamfirescu mapping, then* Γ *possesses one and only one fixed point.*

Proof. Assume ϑ_0 is any element within the set ∇. Create a sequence with the definitions: $\vartheta_{n+1} = \Gamma(\vartheta_n)$ and $c_n = S(\vartheta_n, \vartheta_n, \vartheta_0)$. In order to demonstrate that the sequence $S(\vartheta_{n+1}, \vartheta_n, \vartheta_n)$ is non increasing, consider setting $\varepsilon = 0$, which leads to the following result

$$S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) \leq$$
$$\max\left\{\begin{array}{c} S(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n), \frac{S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})}{3}, \\ \frac{S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_{n+1})+S(\vartheta_n,\vartheta_n,\vartheta_n)}{4} \end{array}\right\}$$
$$S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) \leq \max\left\{\begin{array}{c} S(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n), \frac{2S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})}{3}, \\ \frac{S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_{n+1})}{4} \end{array}\right\}$$
$$S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) \leq \max\left\{\begin{array}{c} S(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n), \frac{2S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})}{3}, \\ \frac{3S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})}{4} \end{array}\right\}$$

Now, considering $S(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n) \leq S(\vartheta_n, \vartheta_n, \vartheta_{n+1})$, one gets

$$\begin{aligned} S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) &\leq \frac{2S(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n) + S(\vartheta_n, \vartheta_n, \vartheta_{n+1})}{3} \\ 2S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) &\leq 2S(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n) + S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) \\ S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) &\leq S(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n) \end{aligned}$$

This results in a contradiction. Therefore, it can be deduced that

$$S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) \leq S(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n) \leq \cdots \leq S(\vartheta_0, \vartheta_0, \vartheta_1) = c_1.$$

Claim1: c_n is bounded.
Define

$$\begin{aligned} c_n = S(\vartheta_0, \vartheta_0, \vartheta_n) &\leq 2S(\vartheta_0, \vartheta_0, \vartheta_1) + S(\vartheta_n, \vartheta_n, \vartheta_1) \\ &\leq 2S(\vartheta_0, \vartheta_0, \vartheta_1) + 2S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + S(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_1) \\ &\leq 2c_1 + 2c_1 + S(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_1) \\ &\leq 2c_1 + 2c_1 + S(\vartheta_1, \vartheta_1, \vartheta_{n+1}) \\ &\leq 4c_1 + S(T\vartheta_0, T\vartheta_0, T\vartheta_n) \end{aligned}$$

$$\leq 4c_1 + (1-\varepsilon)\max\left\{ S(\vartheta_0,\vartheta_0,\vartheta_n), \frac{S(\vartheta_0,\vartheta_0,\vartheta_1)+S(\vartheta_0,\vartheta_0,\vartheta_1)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})}{3}, \frac{S(\vartheta_0,\vartheta_0,\vartheta_1)+S(\vartheta_0,\vartheta_0,\vartheta_{n+1})+S(\vartheta_n,\vartheta_n,\vartheta_1)}{4} \right\}$$

$$+ \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\|\vartheta_0\|+\|\vartheta_0\|+\|\vartheta_n\|+\|\vartheta_1\|+\|\vartheta_1\|+\|\vartheta_{n+1}\|]^{\beta}$$

$$\leq 4c_1 + (1-\varepsilon)\cdot$$

$$\max\left\{ S(\vartheta_0,\vartheta_0,\vartheta_n), \frac{2S(\vartheta_0,\vartheta_0,\vartheta_1)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})}{3}, \frac{S(\vartheta_0,\vartheta_0,\vartheta_1)+2S(\vartheta_0,\vartheta_0,\vartheta_n)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})+2S(\vartheta_0,\vartheta_0,\vartheta_n)+S(\vartheta_0,\vartheta_0,\vartheta_1)}{4} \right\}$$

$$+ \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+2\|\vartheta_1\|+\|\vartheta_n\|+(\|\vartheta_1\|+\|\vartheta_n\|)]^{\beta}$$

$$\leq 4c_1 + (1-\varepsilon)\max\{c_n, c_1, \frac{3c_1+4c_n}{4}\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+3\|\vartheta_1\|+2\|\vartheta_n\|]^{\beta}$$

$$\leq 4c_1 + (1-\varepsilon)\max\{c_n, c_1, \frac{3c_1}{4}+c_n\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+3c_1+2c_n]^{\beta}$$

$$\leq 4c_1 + (1-\varepsilon)\max\{c_n, c_1, \frac{3c_1}{4}+c_n\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+3c_1+2c_n]^{\alpha}$$

If there is a subsequence $c_{n_i} \to +\infty$,

$$c_{n_i} \leq 4c_1 + (1-\varepsilon_i)(\frac{3c_1}{4}+c_{n_i}) + \Lambda\varepsilon_i^{\alpha}\psi(\varepsilon_i)[1+3c_1+2c_{n_i}]^{\alpha}$$

$$c_{n_i} - (1-\varepsilon_i)c_{n_i} \leq 4c_1 + \frac{3c_1}{4}(1-\varepsilon_i) + \Lambda\varepsilon_i^{\alpha}\psi(\varepsilon_i)[1+3c_1+2c_{n_i}]^{\alpha}$$

$$\varepsilon_i c_{n_i} \leq A + B\varepsilon_i^{\alpha}\psi(\varepsilon_i)c_{n_i}^{\alpha}$$

for some $A, B > 0$. The choice $\varepsilon_i = \frac{1+A}{c_{n_i}}$ leads to the inequality,

$$1 \leq B(1+A)^{\alpha}\psi(\varepsilon_i) \to 0,$$

which is a contradiction. Hence, $\{c_n\}$ is a bounded sequence.

As $S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)$ is a non-increasing sequence and it has a lower bound of 0, it follows that

$$\lim_{n\to+\infty} S(\vartheta_n,\vartheta_n,\vartheta_{n-1}) = d \geq 0$$

$$S(\vartheta_n,\vartheta_n,\vartheta_{n+1}) = S(\Gamma\vartheta_{n-1},\Gamma\vartheta_{n-1},\Gamma\vartheta_n)$$

$$\leq (1-\varepsilon)\cdot$$

$$\max\left\{ \begin{array}{c} S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n), \frac{S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})}{3}, \\ \frac{S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_{n+1})+S(\vartheta_n,\vartheta_n,\vartheta_n)}{4} \end{array} \right\}$$
$$+ \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\|\vartheta_{n-1}\|+\|\vartheta_{n-1}\|+\|\vartheta_n\|+\|\vartheta_n\|+\|\vartheta_n\|+\|\vartheta_{n+1}\|]^{\beta}$$
$$\leq (1-\varepsilon)\max\left\{ \begin{array}{c} S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n), \frac{2S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})}{3}, \\ \frac{S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+2S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})}{4} \end{array} \right\}$$
$$+ \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+2\|\vartheta_{n-1}\|+3\|\vartheta_n\|+\|\vartheta_{n+1}\|]^{\beta}$$
$$\leq (1-\varepsilon)\max\left\{ \begin{array}{c} S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n), \frac{2S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})}{3}, \\ \frac{3S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})}{4} \end{array} \right\} + K\varepsilon\psi(\varepsilon)$$

Now, the limit as n approaches infinity is considered, one obtains $d \leq K\Psi(\varepsilon)$, and consequently, one gets $d = 0$.

Claim2: The sequence $\{\vartheta_n\}$ is a type of Cauchy sequence.
Assuming that $\{\vartheta_n\}$ is not Cauchy, one can apply Lemma 3 to conclude that there exists a subsequence $\{\vartheta_{n_k}\}$ and another subsequence $\{\vartheta_{m_k}\}$ of $\{\vartheta_n\}$ where $n_k > m_k > k$, such that

$$\delta \leq S(\vartheta_{m_k},\vartheta_{m_k},\vartheta_{n_k}) = S(\Gamma\vartheta_{m_k-1},\Gamma\vartheta_{m_k-1},\Gamma\vartheta_{n_k-1})$$
$$\leq (1-\varepsilon)\max\left\{ \begin{array}{c} S(\vartheta_{m_k-1},\vartheta_{m_k-1},\vartheta_{n_k-1}), \\ \frac{S(\vartheta_{m_k-1},\vartheta_{m_k-1},\vartheta_{m_k})+S(\vartheta_{m_k-1},\vartheta_{m_k-1},\vartheta_{m_k})+S(\vartheta_{n_k-1},\vartheta_{n_k-1},\vartheta_{n_k})}{3}, \\ \frac{S(\vartheta_{m_k-1},\vartheta_{m_k-1},\vartheta_{m_k})+S(\vartheta_{m_k-1},\vartheta_{m_k-1},\vartheta_{n_k})+S(\vartheta_{n_k-1},\vartheta_{n_k-1},\vartheta_{m_k})}{4} \end{array} \right\}$$
$$+ K\varepsilon\psi(\varepsilon)$$
$$\leq (1-\varepsilon)\max\left\{ \begin{array}{c} (\vartheta_{m_k-1},\vartheta_{m_k-1},\vartheta_{n_k}) + S(\vartheta_{n_k-1},\vartheta_{n_k-1},\vartheta_{n_k}), \\ \frac{2S(\vartheta_{m_k-1},\vartheta_{m_k-1},\vartheta_{m_k})+S(\vartheta_{n_k-1},\vartheta_{n_k-1},\vartheta_{n_k})}{3}, \\ \frac{S(\vartheta_{m_k-1},\vartheta_{m_k-1},\vartheta_{m_k})+S(\vartheta_{m_k-1},\vartheta_{m_k-1},\vartheta_{n_k})+S(\vartheta_{n_k-1},\vartheta_{n_k-1},\vartheta_{m_k})}{4} \end{array} \right\}$$
$$+ K\varepsilon\psi(\varepsilon)\ .$$

As k approaches infinity, one obtains $\delta \leq K\psi(\varepsilon)$. Consequently, it follows that $\delta = 0$, which presents a contradiction. Therefore, it can be asserted that $\{\vartheta_n\}$ is indeed a Cauchy sequence. Considering that ∇ is a complete S-metric space, it can be concluded that there exists an element ϑ within ∇ such that the sequence ϑ_n converges to ϑ. Now, for all n in the natural numbers, and when ε is set to zero, one obtains

$$S(\vartheta,\vartheta,\Gamma\vartheta) \leq 2S(\vartheta,\vartheta,\vartheta_{n+1}) + S(\Gamma\vartheta,\Gamma\vartheta,\vartheta_{n+1})$$

$$\leq 2S(\vartheta,\vartheta,\vartheta_{n+1}) + \max\left\{\begin{array}{c} S(\vartheta,\vartheta,\vartheta_n), \\ \frac{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta_n,\vartheta_n,\Gamma\vartheta_n)}{3}, \\ \frac{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta_n)+S(\vartheta_n,\vartheta_n,\Gamma\vartheta)}{4} \end{array}\right\}$$

$$\leq 2S(\vartheta,\vartheta,\vartheta_{n+1}) + \max\left\{\begin{array}{c} S(\vartheta,\vartheta,\vartheta_n), \\ \frac{2S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta_n,\vartheta_n,\vartheta_{n+1})}{3}, \\ \frac{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\vartheta_{n+1})+S(\vartheta_n,\vartheta_n,\Gamma\vartheta)}{4} \end{array}\right\}$$

Allowing n to approach infinity, the aforementioned inequality follows as:

$$S(\vartheta,\vartheta,\Gamma\vartheta) \leq \frac{2}{3}S(\vartheta,\vartheta,\Gamma\vartheta)$$

$$\Rightarrow S(\vartheta,\vartheta,\Gamma\vartheta) = 0$$

Hence, ϑ is a fixed point of Γ as $S(\Gamma\vartheta,\Gamma\vartheta,\vartheta)=0$, one obtains $\Gamma\vartheta=\vartheta$.

To establish the uniqueness of the fixed point, assume that both $\vartheta,\theta \in \nabla$, are the fixed points of Γ. One obtains

$$S(\Gamma\vartheta,\Gamma\vartheta,\Gamma\theta) \leq (1-\varepsilon)\max\left\{\begin{array}{c} S(\vartheta,\vartheta,\theta), \frac{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)}{3}, \\ \frac{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\theta)+S(\theta,\theta,\Gamma\vartheta)}{4} \end{array}\right\} + K\varepsilon\psi(\varepsilon)$$

Consequently, one obtains $S(\vartheta,\vartheta,\theta) \leq K\psi(\varepsilon)$, which leads to the conclusion that $\vartheta=\theta$. Hence, it can be asserted that Γ possesses a unique fixed point in ∇. □

Example 2. Let $\nabla = \mathbb{R}$. Define a function $S : \nabla\times\nabla\times\nabla$ equipped by $S(\vartheta,\theta,\delta) = \{|\vartheta-\theta|+|\theta-\delta|+|\delta-\vartheta|\}$ for all $\vartheta,\theta,\delta \in \nabla$. Then (∇, S) is a complete metric space.

If one defines a self mapping Γ on ∇ by $\Gamma\vartheta = \frac{3}{8}\vartheta$, then Γ satisfies the conditions of **Theorem 2**. For all $\vartheta,\theta,\delta \in \nabla$ one obtains

$$S(\Gamma\vartheta,\Gamma\theta,\Gamma\delta) = S(\frac{3}{8}\vartheta,\frac{3}{8}\theta,\frac{3}{8}\delta) = \{|\frac{3}{8}\vartheta-\frac{3}{8}\theta|+|\frac{3}{8}\theta-\frac{3}{8}\delta|+|\frac{3}{8}\delta-\frac{3}{8}\vartheta|\}$$
$$= \frac{3}{8}\sup\{|\vartheta-\theta|+|\theta-\delta|+|\delta-\vartheta|\} = \frac{3}{8}S(\vartheta,\theta,\delta)$$

$$\leq (1-\varepsilon)S(\vartheta,\theta,\delta) + (\frac{3}{8} - (1-\varepsilon))S(\vartheta,\theta,\delta)$$
$$\leq (1-\varepsilon)M(\vartheta,\theta,\delta) + \frac{3}{8}(1 - \frac{8}{3}(1-\varepsilon))[S(\vartheta,\vartheta,\vartheta_0) + S(\theta,\theta,\vartheta_0) + S(\delta,\delta,\vartheta_0)]$$
$$\leq (1-\varepsilon)M(\vartheta,\theta,\delta) + \frac{3}{8}\varepsilon^{\frac{8}{3}}[\|\vartheta\| + \|\theta\| + \|\delta\|]$$
$$\leq (1-\varepsilon)M(\vartheta,\theta,\delta) + \frac{3}{8}\varepsilon^2\varepsilon^{\frac{2}{3}}[1 + \|\vartheta\| + \|\theta\| + \|\delta\| + \|\Gamma\vartheta\| + \|\Gamma\theta\| + \|\Gamma\delta\|]$$

for all $\varepsilon \in [0,1]$. This implies that Γ is an S-Pata type Zamfirescu mapping for $\alpha = 2$, $\beta = 1$, $\Lambda = \frac{3}{8}$ and $\psi(\varepsilon) = \varepsilon^{\frac{2}{3}}$. Furthermore, it can be affirmed that $\vartheta = 0$ is indeed the unique fixed point of Γ in ∇, as asserted by Theorem 2.

Corollary 1. *In a complete S-metric space (∇, S), if the mapping $\Gamma : \nabla \to \nabla$ is a Zamfirescu mapping that meets the following inequality criteria for all $\vartheta, \theta, \delta \in \nabla$ and $a, b, c \in [0,1)$,*

$$S(\Gamma\vartheta,\Gamma\theta,\Gamma\delta) \leq \max\left\{\begin{matrix} aS(\vartheta,\theta,\delta), b[\frac{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\delta,\delta,\Gamma\delta)}{3}], \\ c[\frac{S(\vartheta,\vartheta,\Gamma\theta)+S(\theta,\theta,\Gamma\delta)+S(\delta,\delta,\Gamma\vartheta)}{4}] \end{matrix}\right\}$$

In that case, Γ possesses a unique fixed point within the space ∇.

Proof. Considering $d = \max\{a,b,c\}$, one obtains that

$$S(\Gamma\vartheta,\Gamma\theta,\Gamma\delta) \leq d\max\left\{\begin{matrix} S(\vartheta,\theta,\delta), \frac{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\delta,\delta,\Gamma\delta)}{3}, \\ \frac{S(\vartheta,\vartheta,\Gamma\theta)+S(\theta,\theta,\Gamma\delta)+S(\delta,\delta,\Gamma\vartheta)}{4} \end{matrix}\right\}$$
$$\leq (1-\varepsilon)\max\left\{\begin{matrix} S(\vartheta,\theta,\delta), \frac{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\delta,\delta,\Gamma\delta)}{3}, \\ \frac{S(\vartheta,\vartheta,\Gamma\theta)+S(\theta,\theta,\Gamma\delta)+S(\delta,\delta,\Gamma\vartheta)}{4} \end{matrix}\right\}$$
$$+ (d+\varepsilon-1)\max\left\{\begin{matrix} S(\vartheta,\theta,\delta), \frac{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\delta,\delta,\Gamma\delta)}{3}, \\ \frac{S(\vartheta,\vartheta,\Gamma\theta)+S(\theta,\theta,\Gamma\delta)+S(\delta,\delta,\Gamma\vartheta)}{4} \end{matrix}\right\}$$
$$\leq (1-\varepsilon)M(\vartheta,\theta) + d\left(1 + \frac{\varepsilon-1}{d}\right)\cdot$$
$$\max\left\{\|\vartheta\| + \|\theta\| + \|\delta\|, \frac{2(\|\vartheta\|+\|\theta\|+\|\delta\|)+\|\Gamma\vartheta\|+\|\Gamma\theta\|+\|\Gamma\delta\|}{3}\right\}$$
$$\leq (1-\varepsilon)M(\vartheta,\theta) + d\varepsilon^{\frac{1}{d}}[1 + \|\vartheta\| + \|\theta\| + \|\delta\| + \|\Gamma\vartheta\| + \|\Gamma\theta\| + \|\Gamma\delta\|]$$
$$\leq (1-\varepsilon)M(\vartheta,\theta) + d\varepsilon\varepsilon^{\frac{1-d}{d}}[1 + \|\vartheta\| + \|\theta\| + \|\delta\| + \|\Gamma\vartheta\| + \|\Gamma\theta\| + \|\Gamma\delta\|]$$

Therefore, by Theorem 2 with $\Lambda = d$, $\psi(\varepsilon) = \varepsilon^{\frac{1-d}{d}}$ and $\alpha = \beta = 1$, it follows that Γ has a unique fixed point. □

Corollary 2. *Consider a complete S-metric space (∇, S), and let $\Gamma : \nabla \to \nabla$ be a mapping. If for all $\lambda \in [0,1)$, Γ satisfies the following inequalities*

$$S(\Gamma\vartheta, \Gamma\theta, \Gamma\delta) \leq \lambda S(\vartheta, \theta, \delta) \tag{3}$$

OR

$$S(\Gamma\vartheta, \Gamma\theta, \Gamma\delta) \leq \frac{\lambda}{3}[S(\vartheta, \vartheta, \Gamma\vartheta) + S(\theta, \theta, \Gamma\theta) + S(\delta, \delta, \Gamma\delta)] \tag{4}$$

OR

$$S(\Gamma\vartheta, \Gamma\theta, \Gamma\delta) \leq \frac{\lambda}{4}[S(\vartheta, \vartheta, \Gamma\theta) + S(\theta, \theta, \Gamma\delta) + S(\delta, \delta, \Gamma\vartheta)] \tag{5}$$

Then, Γ has a unique fixed point in ∇.

Proof. In Corollary 1, if one takes
(1). $a = \lambda$, $b = c = 0$, inequality (3) is obtained
(2). $b = \lambda$, $a = c = 0$, inequality (4) is obtained
(3). $c = \lambda$, $a = b = 0$, inequality (5) is obtained
further steps followed from Corollary 1. □

Existence of the best proximity point for S-Pata type proximal contraction

This section introduces a new type of proximal mappings called S-Pata type proximal mappings. It is then proved that these mappings have the property of having the best proximity points. Let A and B be subsets of a complete S-metric space (∇, S). The distance between two sets is denoted by $D(A, B)$ and defined by

$$D(A, B) = \inf\{S(\vartheta, \vartheta, \theta) : \vartheta \in A \text{ and } \theta \in B\}.$$

The notation A_0 is used to represent the subset of A defined as follows:

$$A_0 = \{\vartheta \in A : S(\vartheta, \vartheta, \theta) = D(A, B), \text{ for some } \theta \in B\}$$

Likewise, B_0 is the subset of B defined as follows:

$$B_0 = \{\theta \in B : S(\vartheta, \vartheta, \theta) = D(A, B), \text{ for some } \vartheta \in A\}$$

Throughout the section, the assumption that both A_0 and B_0 are closed sets is maintained.

Definition 6. A mapping $\Gamma : A \to B$ is said to be an S-Pata type proximal contraction of type-I if, for all $\vartheta, \theta, \delta \in A$, $\psi \in \Psi$, and for any $\varepsilon \in [0,1]$, it satisfies the following inequality:

$$S(u,v,w) \leq (1-\varepsilon)S(\vartheta,\theta,\delta) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\|\vartheta\|+\|\theta\|+\|\delta\|]^{\beta}$$

where $S(u,u,\Gamma(\vartheta)) = S(v,v,\Gamma(\theta)) = S(w,w,\Gamma(\delta)) = D(A,B)$ and $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0,\alpha]$ are arbitrary constants.

Definition 7. A mapping $\Gamma : A \to B$ is said to be an S-Pata type proximal contraction of type-II if, for all $\vartheta, \theta, \delta \in A$, $\psi \in \Psi$, and for any $\varepsilon \in [0,1]$, it satisfies the following inequality:

$$S(u,u,v) \leq (1-\varepsilon)S(\vartheta,\vartheta,\theta) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+2\|\vartheta\|+\|\theta\|]^{\beta}$$

where $S(u,u,\Gamma(\vartheta)) = S(v,v,\Gamma(\theta)) = D(A,B)$ and $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0,\alpha]$ are arbitrary constants.

Theorem 3. *In a complete S-metric space (∇, S), consider non-empty closed subsets A and B. Suppose that there exists a mapping $\Gamma : A \to B$ that is an S-Pata type proximal contraction, and additionally, it holds that $\Gamma(A_0) \subset B_0$. Then Γ possesses one and only one best proximity point within the set A.*

Proof. Let ϑ_0 be an arbitrary element in A_0. Then $\Gamma(\vartheta_0) \in B_0$ and so there exists an element $\vartheta_1 \in A_0$ such that $S(\vartheta_1,\vartheta_1,\Gamma(\vartheta_0)) = S(A,A,B)$. Similarly, define $\vartheta_{n+1} \in A_0$ such that $S(\vartheta_{n+1},\vartheta_{n+1},\Gamma(\vartheta_n)) = S(A,A,B)$ and $c_n = S(\vartheta_n,\vartheta_n,\vartheta_0)$. Then, one gets

$$S(\vartheta_n,\vartheta_n,\vartheta_{n+1}) \leq (1-\varepsilon)S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+2\|\vartheta_n\|+\|\vartheta_{n-1}\|]^{\beta}$$

where $\varepsilon \in [0,1]$.

Taking $\varepsilon = 0$ in the above inequality, it follows that $\{S(\vartheta_n,\vartheta_n,\vartheta_{n+1})\}$ is a nonincraesing sequence. Therefore,

$$S(\vartheta_n,\vartheta_n,\vartheta_{n+1}) \leq S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n) \leq \ldots S(\vartheta_0,\vartheta_0,\vartheta_1) = c_1$$

Now, one shows that $\{c_n\}$ is bounded:

$$\begin{aligned} c_n &= S(\vartheta_0, \vartheta_0, \vartheta_n) \\ &\leq 2S(\vartheta_0, \vartheta_0, \vartheta_1) + 2S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + S(\vartheta_1, \vartheta_1, \vartheta_{n+1}) \\ &\leq 4c_1 + (1-\varepsilon)S(\vartheta_0, \vartheta_0, \vartheta_n) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + 2\|\vartheta_0\| + \|\vartheta_n\|]^\beta \\ &\leq 4c_1 + (1-\varepsilon)c_n + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + c_n]^\beta \\ &\leq a + (1-\varepsilon)c_n + b\varepsilon^\alpha\psi(\varepsilon)c_n^\alpha \end{aligned} \tag{6}$$

where $a, b > 0$ are constants. Accordingly,

$$\varepsilon c_n \leq a + b\varepsilon^\alpha\psi(\varepsilon)c_n^\alpha$$

If there is a subsequence $c_{n_i} \to +\infty$, the choice $\varepsilon = \varepsilon_i = (1+a)/c_n$, leads to the contradiction

$$1 \leq b(1+a)^\alpha\psi(\varepsilon_i) \to 0$$

Hence, $\{c_n\}$ is a bounded sequence.

Let $\lim_{n\to+\infty} S(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n) = d$. Since $S(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n)$ is non increasing,

$$\begin{aligned} S(\vartheta_n, \vartheta_n, \vartheta_{n+1}) &\leq (1-\varepsilon)S(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + 2\|\vartheta_{n-1}\| + \|\vartheta_n\|]^\beta \\ &\leq (1-\varepsilon)S(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n) + K\varepsilon\psi(\varepsilon) \end{aligned}$$

As the limit as n approaches infinity is considered, it is deduced that $d \leq K\psi(\varepsilon)$ implies $d = 0$. One now makes the assertion that $\{\vartheta_n\}$ is a Cauchy sequence.

In order to avoid contradiction, assume that $\{\vartheta_n\}$ is not a Cauchy sequence. In such a case, according to Lemma 3, there must exist subsequences $\{\vartheta_{n_k}\}$ and $\{\vartheta_{m_k}\}$ of $\{\vartheta_n\}$ with $n_k > m_k > k$ such that

$$\begin{aligned} d &\leq S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{m_k}) \\ &\leq (1-\varepsilon)S(\vartheta_{n_k-1}, \vartheta_{n_k-1}, \vartheta_{m_k-1}) + K\varepsilon\psi(\varepsilon) \\ &\leq (1-\varepsilon)[2S(\vartheta_{n_k-1}, \vartheta_{n_k-1}, \vartheta_{m_k}) + S(\vartheta_{m_k-1}, \vartheta_{m_k-1}, \vartheta_{n_k})] + K\varepsilon\psi(\varepsilon). \end{aligned}$$

As k tends towards infinity, one reaches the conclusion that $d \leq K\psi(\varepsilon)$, which leads to a contradiction. Therefore, it can be affirmed that $\{\vartheta_n\}$ is indeed a Cauchy sequence. Given that ∇ is a complete space, there exists an element ϑ in ∇ such that $\{\vartheta_n\}$ converges to ϑ.

Now, consider $S(u, u, \Gamma(\vartheta)) = D(A, B)$ and set ε to zero. Then, for all $n \in N$, one obtains

$$S(u, u, \vartheta) \leq 2S(u, u, \vartheta_{n+1}) + S(\vartheta, \vartheta, \vartheta_{n+1})$$
$$\leq 2S(\vartheta, \vartheta, \vartheta_n) + S(\vartheta, \vartheta, \vartheta_{n+1})$$

This leads to the conclusion that $S(u, u, \vartheta) = 0$. Consequently, it can be established that ϑ serves as a proximity point of Γ.

In order to demonstrate the uniqueness of the proximity point, suppose, for the sake of contradiction, that Γ has two distinct proximity points ϑ and θ in ∇. One obtains

$$S(\vartheta, \vartheta, \theta) \leq (1 - \varepsilon)S(\vartheta, \vartheta, \theta) + K\varepsilon\psi(\varepsilon)$$

Hence, one obtains $S(\vartheta, \vartheta, \theta) \leq K\psi(\varepsilon)$, which implies $\vartheta = \theta$. Thus, Γ has unique proximity point ϑ in A. □

Corollary 3. *Consider a complete metric space* (∇, S)*, and let* $\Gamma : \nabla \to \nabla$ *be a mapping that adheres to the following inequality:*

$$S(\Gamma\vartheta, \Gamma\theta, \Gamma\delta) \leq (1 - \varepsilon)S(\vartheta, \theta, \delta) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|\vartheta\| + \|\theta\| + \|\delta\|]^{\beta}$$

Then, there exists a unique fixed point for Γ.

Proof. The proof can be derived directly from the previous theorem when $A = B$. □

Example 3. Let $\nabla = R^2$ under 1-norm and (∇, S) be an S-metric space with a metric defined by $S(\vartheta, \theta, \delta) = \frac{1}{2}\{|\vartheta - \delta| + |\theta - \delta|\}$. Consider $A = \{(0, a)|a \in [0, 1]\}$ and $B = \{(1, b)|b \in [0, 1]\}$ then $D(A, B) = 1$. Define $f : A \to B$ as $f(0, \vartheta) = (1, \frac{\vartheta^2}{4})$, $\Lambda = \alpha = \beta = 1$, and for $\delta \in (0, \frac{1}{2})$

$$\psi(s) = \begin{cases} \frac{s}{2} & if\ s \in [0, \delta), \\ 1 & if\ s \in [\delta, 1], \end{cases}$$

One needs to demonstrate that, for all $\varepsilon \in [0, 1]$, f satisfies the inequality of S-Pata type proximal contraction,

$$\begin{cases} for\ \varepsilon = 0 & \Longrightarrow\ S(u, u, v) \leq S(\vartheta', \vartheta', \theta'), \\ for\ \varepsilon \in (0, 1] & \Longrightarrow\ S(u, u, v) \leq (1 - \varepsilon)S(\vartheta', \vartheta', \theta') + \\ & \varepsilon\psi(\varepsilon)[1 + 2\|\vartheta'\| + \|\theta'\|], \end{cases}$$

where $S(u,u,f(\vartheta')) = S(v,v,f(\theta)) = D(A,B) = 1$.
The following inequality shows that f satisfies the first case when $\varepsilon = 0$.
For all $\vartheta' = (0,\vartheta)$, $\theta' = (0,\theta) \in A$.

$$\begin{aligned} S(u,u,v) &= S((0,\frac{\vartheta^2}{4}),(0,\frac{\vartheta^2}{4}),(0,\frac{\theta^2}{4})) \\ &= |(0,\frac{\vartheta^2}{4}) + (0,\frac{\theta^2}{4})| \\ &= |\frac{\vartheta^2}{4} - \frac{\theta^2}{4}| = |(\frac{\vartheta}{2}+\frac{\theta}{2})(\frac{\vartheta}{2}-\frac{\theta}{2})| \\ &\leq |\frac{\vartheta}{2} - \frac{\theta}{2}| \\ &\leq S(\vartheta',\vartheta',\theta') \end{aligned}$$

The following inequalities demonstrate that f satisfies the second case when $\varepsilon \in (0,1]$:
For $\varepsilon \in (0,\delta)$ and for all $\vartheta',\theta' \in A$

$$\begin{aligned} S(u,u,v) &= S((0,\frac{\vartheta^2}{4}),(0,\frac{\vartheta^2}{4}),(0,\frac{\theta^2}{4})) = |\frac{\vartheta^2}{4} - \frac{\theta^2}{4}| \\ &\leq \frac{1}{2}S(\vartheta',\vartheta',\theta') \\ &= (1-\varepsilon)S(\vartheta',\vartheta',\theta') + (\frac{1}{2} + (\varepsilon - 1))S(\vartheta',\vartheta',\theta') \\ &\leq (1-\varepsilon)S(\vartheta',\vartheta',\theta') + \frac{1}{2}(1 + \frac{\varepsilon-1}{\frac{1}{2}})[2S(\vartheta'_0,\vartheta',\vartheta') + S(\vartheta'_0,\theta',\theta')] \\ &\leq (1-\varepsilon)S(\vartheta',\vartheta',\theta') + \frac{1}{2}\varepsilon^2[2\|\vartheta'\| + \|\theta\|] \\ &\leq (1-\varepsilon)S(\vartheta',\vartheta',\theta') + \varepsilon\psi(\varepsilon)[1 + 2\|\vartheta'\| + \|\theta'\|]. \end{aligned}$$

For $\varepsilon \in [\delta,1]$ and for all $\vartheta',\theta' \in A$,

$$\begin{aligned} S(u,u,v) &= S((0,\frac{\vartheta^2}{4}),(0,\frac{\vartheta^2}{4}),(0,\frac{\theta^2}{4})) = |\frac{\vartheta^2}{4} - \frac{\theta^2}{4}| \\ &\leq S(\vartheta',\vartheta',\theta') \\ &= (1-\varepsilon)S(\vartheta',\vartheta',\theta') + \varepsilon S(\vartheta',\vartheta',\theta') \\ &= (1-\varepsilon)S(\vartheta',\vartheta',\theta') + \varepsilon[2S(\vartheta'_0,\vartheta',\vartheta') + S(\vartheta'_0,\theta',\theta')] \\ &= (1-\varepsilon)S(\vartheta',\vartheta',\theta') + \varepsilon\psi(\varepsilon)[1 + 2\|\vartheta'\| + \|\theta'\|]. \end{aligned}$$

Hence, f fits the criteria of being an S-Pata type proximal contraction, and as a result, there exists a best proximity point $(0,0)$ within the set A.

Application to the integral equation

In this section, an application of the main result is explored by describing a solution of the integral equation.

Consider $\mathcal{M} = C(I, \mathbb{R})$, which represents the set of all continuous functions on $I = [0, 1]$, equipped with the metric $S(\vartheta, \theta, \delta) = \sup\{|\vartheta(s) - \theta(s)| + |\theta(s) - \delta(s)| + |\delta(s) - \vartheta(s)|,\ s \in I\}$ for all $\vartheta, \theta, \delta \in \mathcal{M}$. It is worth noting that $(\mathcal{M}, S)$ forms a complete S-metric space. In this context, one delves into the investigation of the integral equation

$$\vartheta(s) = \chi(s) + \int_0^1 \mathcal{K}(s, u)\eta(u, \vartheta(u))du,\ s \in I, \tag{7}$$

where, the functions $\eta : I \times \mathbb{R} \to \mathbb{R}$ and $\chi : I \to \mathbb{R}$ are continuous, and $\mathcal{K} : I \times I \to \mathbb{R}_0^+$ is a function satisfying $\mathcal{K}(s, \cdot) \in L^1(I)$ for all $s \in I$. One examines the mapping $\Gamma : \mathcal{M} \to \mathcal{M}$, which is defined as follows:

$$\Gamma(\vartheta)(s) = \chi(s) + \int_0^1 \mathcal{K}(s, u)\eta(u, \vartheta(u))du,\ s \in I. \tag{8}$$

Theorem 4. *Equation (8) has at least one solution in $\mathcal{M}$, if for all $u \in I$ and $a, b \in \mathbb{R}$, there is*

$$|\eta(u, a) - \eta(u, b)| \leq \frac{1}{4}(|a - \Gamma a| + |a - \Gamma b| + |b - \Gamma a|),\ for\ all\ u \in I,$$

followed by the assumed inequality

$$\sup_{s \in I} \int_0^1 \mathcal{K}(s, u)du = \delta \leq 1.$$

where δ is a fixed constant.

Proof. Note that finding a solution of (7) is equivalent to finding $\vartheta^* \in \mathcal{M}$ that is a fixed point of Γ. Now let $\vartheta, \theta \in \mathcal{M}$, on account of the above inequalities, one finds that

$$|\Gamma(\vartheta)(s) - \Gamma(\theta)(s)| = \left|\int_0^1 \mathcal{K}(s, u)[\eta(u, \vartheta(u)) - \eta(u, \theta(u))]du\right|$$

$$\leq \int_0^1 \mathcal{K}(s, u)du \cdot |\eta(u, \vartheta(u)) - \eta(u, \theta(u))|$$

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$$\leq \int_0^1 \mathcal{K}(s,u)du \cdot \frac{1}{4}(|\vartheta(s)-\Gamma\vartheta(s)| + |\vartheta(u)-\Gamma(\theta)(u)| + |\theta(u)-\Gamma(\vartheta)(u)|)$$
$$\implies S(\Gamma\vartheta,\Gamma\vartheta,\Gamma\theta) = \delta\cdot\left(\frac{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\theta)+S(\theta,\theta,\Gamma\vartheta)}{4}\right)$$
$$\leq \delta\cdot M(\vartheta,\vartheta,\theta)\ ,$$

where $M(\vartheta,\theta,\delta) = \max\left\{\begin{array}{c} S(\vartheta,\theta,\delta), \frac{S(\vartheta,\vartheta,\Gamma(\vartheta))+S(\theta,\theta,\Gamma(\theta))+S(\delta,\delta,\Gamma(\delta))}{3}, \\ \frac{S(\vartheta,\vartheta,\Gamma(\theta))+S(\theta,\theta,\Gamma\delta)+S(\delta,\delta,\Gamma\vartheta)}{4} \end{array}\right\}$.

Hence, it is derived that

$$S(\Gamma\vartheta,\Gamma\vartheta,\Gamma\theta) \leq (1-\varepsilon)M(\vartheta,\vartheta,\theta) + (\delta + (\varepsilon-1))M(\vartheta,\vartheta,\theta)$$
$$\leq (1-\varepsilon)M(\vartheta,\vartheta,\theta) + \delta\left(1+\frac{\varepsilon-1}{\delta}\right)\cdot$$
$$\max\left\{\begin{array}{c} S(\vartheta,\vartheta,\theta), \frac{S(\vartheta,\vartheta,\Gamma(\vartheta))+S(\vartheta,\vartheta,\Gamma(\vartheta))+S(\theta,\theta,\Gamma(\theta))}{3}, \\ \frac{S(\vartheta,\vartheta,\Gamma(\vartheta))+S(\vartheta,\vartheta,\Gamma\theta)+S(\theta,\theta,\Gamma\vartheta)}{4} \end{array}\right\}$$
$$\leq (1-\varepsilon)M(\vartheta,\vartheta,\theta) + \delta\varepsilon^{\frac{1}{\delta}}\max\left\{\|\vartheta\|+\|\vartheta\|+\|\theta\|, \frac{4\|\vartheta\|+2\|\theta\|+2\|\Gamma\vartheta\|+\|\Gamma\theta\|}{3}\right\}$$
$$\leq (1-\varepsilon)M(\vartheta,\vartheta,\theta) + \delta\varepsilon\varepsilon^{\frac{1}{\delta}-1}[1+2\|\vartheta\|+\|\theta\|+2\|\Gamma\vartheta\|+\|\Gamma\theta\|]$$

for all $u \in I$ and $\vartheta,\theta \in \mathcal{M}$. Hence, Γ is an S-Pata type Zamfirescu mapping with $\Lambda = \delta, \alpha = 1, \beta = 1$ and $\psi(\varepsilon) = \varepsilon^{\frac{1-\delta}{\delta}}$. Therefore, one deduces the existence of $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \Gamma\vartheta^*$ that is ϑ^* is a solution of integral equation (7). □

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Mapeos de Zamfirescu bajo condición tipo Pata: resultados y aplicación a Ecuación integral

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CAMPO: matemáticas
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: Las asignaciones tipo Pata y Zamfirescu se extienden más allá de los espacios métricos.

Métodos: Se emplea el concepto de mapeo de Zamfirescu tipo Pata en el marco de espacios S-métricos.

Resultados: Se han establecido una serie de resultados correspondientes. Además, los resultados obtenidos se emplean para resolver una ecuación integral.

Conclusión: Los mapeos de tipo S-Pata y Zamfirescu tienen puntos fijos únicos.

Palabras claves: contracción tipo Pata, mapeo de Zamfirescu, espacio S-métrico, punto fijo.

Отображения Замфиреску в условиях типа Пата: результаты и применение в итегральных уравнениях

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РУБРИКА ГРНТИ: 27.25.17 Метрическая теория функций,
27.39.15 Линейные пространства, снабженные топологией, порядком и другими структурами

ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: Отображения типа Паты и Замфиреску выходят за пределы метрических пространств.

Методы: В статье применяется концепция отображения Замфиреску типа Паты в рамках S-метрических пространств.

Результаты: Был подтвержден ряд соответствующих результатов. Полученные результаты были использованы в решении интегральных уравнений.

Выводы: Отображения типа S-Pata и Замфиреску имеют уникальные неподвижные точки.

Ключевые слова: сжатие типа Паты, отображение типа Замфиреску, S-метрическое пространство, неподвижная точка.

Замфиреску пресликавања под условима типа Пата: резултати и примена на интегралну једначину

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ОБЛАСТ: математика
КАТЕГОРИЈА (ТИП) ЧЛАНКА: оригинални научни рад

Сажетак:

Увод/циљ: Пресликавања типа Пата и Замфиреску су проширена изван метричких простора.

Методе: Примењен је концепт Замфиреску пресликавања типа Пата у оквиру S-метричких простора.

Резултати: Утврђен је низ одговарајућих исхода. Затим су се добијени резултати користили за решавање интегралне једначине.

Закључак: Пресликавања типа S-Пата и Замфиреску имају јединствене непокретне тачке.

Кључне речи: контракција типа Пата, пресликавање типа Замфиреску, S-метрички простор, непокретна тачка.

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