Applications of the expanded Darbo's fixed point theorem to fractional differential equations with finite delay

Mohamed Helal

Mustapha Stambouli University of Mascara, Science and Technology Faculty, B.P. 763, 29000, Mascara, Algeria,+
Djillali Liabes University of Sidi Bel-Abbès, Laboratory of Mathematics, B.P. 89, 22000, Sidi Bel-Abbès, Algeria,
e-mail: helalmohamed@univ-mascara.dz,
ORCID iD: https://orcid.org/0000-0002-7845-1018

https://doi.org/10.5937/vojtehg74-57904

FIELD: mathematics
ARTICLE TYPE: original scientific paper

Abstract:

Introduction/purpose: The aim of this paper is to study the existence of mild solutions to the initial value problem (IVP for short) of the Darboux problem for partial hyperbolic fractional differential equations via the Caputo derivative with finite delay in Fréchet spaces.

Methods: In our analysis, Darbo's fixed point theorem is employed together with the notion of measure of noncompactness to establish the existence of solutions by reducing the research to proving the existence and uniqueness of fixed points of appropriate operators.

Results: Under suitable conditions, existence of mild solutions to the considered fractional partial hyperbolic differential equations is proved. An illustrative example demonstrates the theoretical results applicability.

Conclusion: Throughout this paper, sufficient conditions were considered to establish the existence and uniqueness of solutions to the Darboux problem by applying Darbo's fixed point theorem on unbounded interval. The study provides a rigorous methodological framework for analyzing similar classes of problems and contributes to the broader understanding of mild solutions in fractional calculus and functional analysis. The results offer potential applications for researchers investigating fractional differential equations in infinite-dimensional spaces.

Key words: differential partial hyperbolic equation, fractional order derivative, mild solution, Fréchet space.

ACKNOWLEDGMENT: The author is grateful to the referees for their helpful remarks and suggestions, as well as to Grant PRFU: C00L03UN290120220002 from DGRSDT of Algeria.

Introduction

For many years, not only have mathematicians been interested in fractional calculus and fractional order differential equations, but physicists and engineers have been as well. In fact, there is a wide range of applications in the fields of porous media, electrochemistry, viscoelasticity, electromagnetism and rheology (Baleanu, et al.; Hilfer, 2000; Ortigueira, 2011; Tarasov, 2010). In recent years, ordinary and partial fractional differential equations have undergone significant development. For more details, see Abbas *et al.* (Abbas et al., 2012) and Podlubny (Podlubny, 1999), Abbas and Bechochra (Abbas and Benchohra, 2009, 2012), Benchohra *et al.* (Benchohra and Hellal, 2014a,b; Benchohra et al., 2017), Helal (Helal, 2020), Kilbas and Marzan (Kilbas and Marzan, 2005), Vityuk and Golushkov (Vityuk and Golushkov, 2004) and the references therein.

In 1930, Kuratowski (Kuratowski, 1930) first proposed the notion of a measure of noncompactness. This concept is a powerful tool in functional analysis, particularly in operator equation theory in Banach spaces and metric fixed point theory. The study of the existence of solutions for integral and integro-differential, ordinary and partial differential equations also makes use of this notion. We refer the reader to (Banaei, 2018; Banás et al., 2017; Banás and Mursaleen, 2014) for further information.

In this paper, we examine the existence of solutions to fractional order initial value problem (IVP for short). More precisely, we will consider the following problem:

$$({}^cD_0^r u)(t,x) = f(t,x,u_{(t,x)}), \text{ for a. a } (t,x) \in J, \tag{1}$$

$$u(t,x) = \phi(t,x), \quad if \quad (t,x) \in \tilde{J}, \tag{2}$$

$$\begin{cases} u(t,0) = \varphi(t), \\ u(0,x) = \psi(x), \quad (t,x) \in J, \\ \varphi(0) = \psi(0), \end{cases} \tag{3}$$

where $J := [0,\infty) \times [0,\infty)$, $\tilde{J} := [-\alpha,\infty) \times [-\beta,\infty) \backslash [0,\infty) \times [0,\infty)$, ${}^cD_0^r$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $f : J \times E \to E$ is a given function, $(E, ||\cdot||_E)$ is a (real or complex) Banach space, $\phi : \tilde{J} \to E$ is a given continuous function with $\phi(t,0) = \varphi(t)$, $\phi(0,x) = \psi(x)$ for each $(t,x) \in J$, φ, $\psi : [0,\infty) \to E$ are given absolutely continuous and $\varphi(t)$ and $\psi(x)$ are measurable for all $(t,x) \in J$.

We denote by $u_{(t,x)}$ the element of $C(\tilde{J}, E)$ defined by

$$u_{(t,x)}(s,\tau) = u(t+s, x+\tau); \quad (s,\tau) \in \tilde{J},$$

where $u_{(t,x)}(\cdot,\cdot)$ represents the history of the state u.

Inspired by other studies, we examine the existence of solutions for the problem (1)-(3). The existence of solutions for this problem will be found by using the expanded Darbo's fixed point theorem (Dudek, 2017; Dudek and Olszowy, 2015), and the notion of measure of noncompactness (Ayerbee Toledano et al., 1997) in Fréchet spaces. To the author's knowledge, there are few papers on fractional differential equations with delay in Fréchet spaces. This work aims to further continue this investigation.

The paper is organized as follows: we will go over some essential information in Section 2 that will be applied to the next sections. The main results are presented in Section 3, where we demonstrate that the problems (1)-(3) have mild solutions. In the last section, an example is given to justify the application of our results.

Background materials and preliminaries

In this section, we give some notations, essential definitions of fractional partial integrals and derivatives and preliminary facts to simplify the forthcoming analysis.

Let $J_0 = [0,n] \times [0,n]; \ n > 0$. A measurable function $u : J_0 \to E$ is Bochner integrable if and only if $||u||$ is Lebesgue integrable. For properties of the Bochner integral, see, for instance, Yosida (1980).

$C(J_0)$ is the Banach space of all continuous functions from J_0 into E with the norm

$$\|u\|_\infty = \sup_{(t,x)\in J_0} \|u(t,x)\|.$$

$AC(J_0)$ denotes the space of absolutely continuous functions from J_0 into E. $L^1(J_0)$ is the Banach space of measurable functions $u : J_0 \to E$ which are Bochner integrable and normed by

$$\|u\|_{L^1} = \int_0^n \int_0^n \|u(t,x)\|_E dtdx.$$

Now, we provide some basic overview of fractional calculus.

Definition 1. *(Vityuk and Golushkov, 2004) Let* $r = (r_1, r_2) \in (0,\infty) \times (0,\infty), \theta = (0,0)$ *and* $u \in L^1(J_0)$. *The left-sided mixed Riemann-Liouville integral of order* r *of* u *is defined by*

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} u(s,\tau)d\tau ds.$$

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x), \ (I_\theta^\sigma u)(t,x) = \int_0^t \int_0^x u(s,\tau)d\tau ds;$$

for almost all

$$(t,x) \in J_0,$$

where $\sigma = (1,1)$.
For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty) \times (0,\infty)$, when $u \in L^1(J_0)$. Note also that when $u \in C(J_0)$, then $(I_\theta^r u) \in C(J_0)$. Moreover

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0; \ (t,x) \in J_0.$$

Example 1. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2}, \text{ for almost all } (t,x) \in J_0.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in (0,1] \times (0,1]$. Denoted by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

Definition 2. Vityuk and Golushkov (2004) *Let* $r \in (0,1] \times (0,1]$ *and* $u \in L^1(J_0)$. *The mixed fractional Riemann-Liouville derivative of order* r *of* u *is defined by the expression*

$$D_\theta^r u(t,x) = (D_{tx}^2 I_\theta^{1-r} u)(t,x)$$

and the Caputo fractional-order derivative of order r *of* u *is defined by the expression*

$$({}^c D_\theta^r u)(t,x) = \left(I_\theta^{1-r} \frac{\partial^2}{\partial t \partial x} u \right)(t,x).$$

The case $\sigma = (1,1)$ is included, thus we have

$$(D_\theta^\sigma u)(t,x) = ({}^c D_\theta^\sigma u)(t,x) = (D_{tx}^2 u)(t,x), \text{ for almost all } (t,x) \in J_0.$$

Example 2. Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, then

$$D_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2}, \text{ for almost all } (t, x) \in J_0.$$

We recall the definition of the concept of a sequence of measures of noncompactness, as given in Dudek (2017); Dudek and Olszowy (2015), as follows:

Definition 3. *Let $\mathcal{M}_X$ be the family of all nonempty and bounded subsets of a Fréchet space X. A family of functions α where $\alpha : \mathcal{M}_X \to [0, \infty)$ is said to be a family of measures of noncompactness in the real Fréchet space X if it satisfies the following conditions for all $B, B_1, B_2 \in \mathcal{M}_X$:*

(a) *α is full, that is: $\alpha(B) = 0$ if and only if B is precompact,*

(b) *$\alpha(B_1) \leq \alpha(B_2)$ for $B_1 \subset B_2$,*

(c) *$\alpha(ConvB) = \alpha(B)$,*

(d) *If $\{B_i\}_{i=1,\cdots}$ is a sequence of closed sets from $\mathcal{M}_X$ such that $B_{i+1} \subset B_i;\ i = 1, \cdots$ and if $\lim\limits_{i\to 1} \alpha(B_i) = 0$, then the intersection set $B_\infty := \cap_{i=1}^\infty B_i$ is nonempty.*

Some Properties:

(e) *We call the family of measures of noncompactness α to be homogeneous if $\alpha(\lambda B) = |\lambda|\alpha(B)$, for $\lambda \in \mathbb{R}$.*

(f) *If the family α satisfies the condition $\alpha(B_1 \cup B_2) \leq \alpha(B_1) + \alpha(B_2)$, it is called subadditive.*

(g) *It is sublinear if both conditions (e) and (f) hold.*

(h) *We say that the family of measures α has the maximum property if $\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$.*

(i) *The family of measures of noncompactness α is said to be regular if the conditions $(a), (g)$ and (h) hold; (full sublinear and has maximum property).*

For more details on measure of noncompactness and its properties see Appell (1981).

Lemma 1. *A nonempty subset $B \subset X$ is said to be bounded if*

$$\sup_{u\in X} ||u||_n < \infty, \qquad for\ n \in \mathbb{N}.$$

Lemma 2. *Bothe (1998) If Y is a bounded subset of Fréchet space X, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\alpha(Y) \leq 2\alpha(\{y_k\}_{k=1}^{\infty}) + \epsilon.$$

Lemma 3. **?** *If $\{u_k\}_{k=1}^{\infty} \subset L^1(J_0)$ is uniformly integrable, then $\alpha(\{u_k\}_{k=1}^{\infty})$ is measurable and for each $(t, x) \in J_0$,*

$$\alpha\left(\left\{\int_0^t \int_0^x u_k(s,\tau)d\tau ds\right\}_{k=1}^{\infty}\right) \leq 2\int_0^t \int_0^x \alpha(\{u_k(s,\tau)\}_{k=1}^{\infty})d\tau ds.$$

Definition 4. *Let Ω be a nonempty subset of a Fréchet space X, and let $A : \Omega \to X$ be a continuous operator which transforms bounded subsets. One says that A satisfies the Darbo condition with constants $(k_n)_{n\in\mathbb{N}}$ with respect to a family of measures of noncompactness α, if*

$$\alpha(A(B)) \leq k_n \alpha(B),$$

for each bounded set $B \subset \Omega$ and $n \in \mathbb{N}$.

If $k_n < 1,\ n \in \mathbb{N}$ then A is called a contraction with respect to α.

The following expanded Darbo's fixed point theorem for Fréchet spaces will be utilized in the sequel.

Theorem 1. *Dudek (2017); Dudek and Olszowy (2015) Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space X and let $V : \Omega \longrightarrow \Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness α. Then V has at least one fixed point in the set Ω.*

Existence and uniqueness of mild solutions

This section is devoted to the proof of the global existence of solutions for the problem (1)-(3).

For each $n \in \mathbb{N}$ we consider the following set,

$$C_n = C([-\alpha, n] \times [-\beta, n], E)$$

and we define in $C_0 := C([-\alpha, \infty) \times [-\beta, \infty), E)$ the semi-norms by:

$$\|u\|_n = \{\sup \|u(t,x)\| : -\alpha \leq t \leq n, -\beta \leq x \leq n\}.$$

Then C_0 is a Fréchet space with the family of semi-norms $\{\|u\|_n\}$.

Let us begin by explaining what we mean by a solution of the problem (1)-(3).

Definition 5. *A function $u \in C_0$ is said to be a solution of (1)-(3) if u satisfies the equations (1) and (3) on J and the condition (2) on $\tilde{J}$.*

We shall use the following lemma for the existence of solutions for the problem (1)-(3):

Lemma 4. *A function $u \in C_0$ is a solution of problem (1)-(3) if and only if u satisfies the equation*

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,u_{(s,\tau)})d\tau ds,$$

for all $(t,x) \in J$ and the condition (2) on $\tilde{J}$, where

$$z(t,x) = \varphi(t) + \psi(x) - \varphi(0).$$

we propose the following assumptions to prove our main results:

$(H1)$ The functions $\varphi(t)$ and $\psi(x)$ are measurable and bounded for a.e. $(t,x) \in J_0$,

$(H2)$ The function $(t,x) \to f(t,x,u)$ is measurable on J_0, and the function $u \to f(t,x,u)$ is continuous on E for a. e. $(t,x) \in J_0$.

$(H3)$ There exist functions $p_1, p_2 : J_0 \to [0,\infty)$ such that for each $(t,x) \in J_0$,

$$\|f(t,x,u)\|_E \leq p_1(t,x) + p_2(t,x)\|u\|_n, \qquad \text{for all } u \in E.$$

$(H4)$ For each $(t,x) \in J_0$, and any bounded $B \subset X$,

$$\alpha(f(t,x,B)) \leq p_2(t,x)\alpha(B),$$

where α is a measure of noncompactness on the Banach space E.

$(H5)$ There exists a positive real number R_n such that

$$z^* + \frac{[p^*_{1n} + p^*_{2n}R_n]n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \leq R_n,$$

where

$$z^* = \sup_{(t,x)\in J_0} \|z(t,x)\|_E, \qquad p^*_{in} = \sup_{(t,x)\in J_0} p_i(t,x), i = 1,2.$$

Theorem 2. *Assuming that assumptions* $(H1)$-$(H5)$ *hold. If*

$$\ell_n := \frac{4p_{2n}^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1,$$

then the propsed problem (1)-(3) *has at least one mild solution.*

Proof. Define the operator $N : C_0 \to C_0$ by

$$(Nu)(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) & \\ +\dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} & \\ \times f(s,\tau,u_{(s,\tau)})d\tau ds, & (t,x) \in J. \end{cases} \tag{4}$$

It is evident that the fixed points of the operator N are solutions of our problem (1)-(3).

We consider the ball $B_{R_n} = \{u \in C_0 : \|u\|_n \leq R_n\}$, then from $(H3)$, for any $u \in B_{R_n}$, we obtain

$$\|(Nu)(t,x)\|_E \leq$$

$$\begin{aligned} &\leq \|z(t,x)\|_E + \int_0^t \int_0^x \frac{(t-s)^{r_1-1}(x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(s,\tau,u_{(s,\tau)})\|_E d\tau ds \\ &\leq \|z(t,x)\|_E + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} p_1(s,\tau) d\tau ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} p_2(s,\tau) \|u_{(s,\tau)}\| d\tau ds \\ &\leq z^*(w) + \frac{p_{1n}^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds \\ &+ \frac{p_{2n}^* R_n}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds \\ &\leq z^* + \frac{[p_{1n}^* + p_{2n}^* R_n] n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\leq R_n. \end{aligned}$$

This demonstrates that N transforms the ball B_{R_n} into itself.

We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the conditions of Theorem 1.

The proof will proceed in a sequence of steps.

Step 1: N is continuous.
Let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence such that $u_m \to u$ in C_0. Then, for each $(t,x) \in J_0$, we have

$$\begin{aligned} & \|(Nu_m)(t,x) - (Nu)(t,x)\| \\ \leq \;& \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \|f(s,\tau,u_{m(s,\tau)}) - \\ - \;& f(s,\tau,u_{(s,\tau)})\| d\tau ds. \end{aligned}$$

Since $u_m \to u$ as $m \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|Nu_m - Nu\|_n \to 0 \text{ as } m \to \infty.$$

Step 2: $N(B_{R_n})$ is bounded.
Since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded, then $N(B_{R_n})$ is bounded.
Step 3: For each bounded subset D of B_{R_n}, we have

$$\alpha(N(D)) \leq \ell_n \alpha(D).$$

From Lemmas 2 and 3, for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_m\}_{m=0}^\infty \subset D$, such that for all $(t,x) \in J_0$, we have

$$\begin{aligned} & \alpha((N(D))(t,x)) \\ &= \alpha\left(\left\{z(t,x) + \int_0^t \int_0^x \frac{(t-s)^{r_1-1}(x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,\tau,u_{(s,\tau)}) d\tau ds;\; u \in D\right\}\right) \\ &\leq \left\{\int_0^t \int_0^x \frac{(t-s)^{r_1-1}(x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,\tau,u_{m(s,\tau)}) d\tau ds\right\}_{m=1}^\infty + \epsilon \\ &\leq 4\int_0^t \int_0^x \alpha\left(\left\{\frac{(t-s)^{r_1-1}(x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,\tau,u_{m(s,\tau)})\right\}_{m=1}^\infty\right) d\tau ds + \epsilon \\ &\leq 4\int_0^t \int_0^x \frac{(t-s)^{r_1-1}(x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha\left(\{f(s,\tau,u_{m(s,\tau)})\}_{m=1}^\infty\right) d\tau ds + \epsilon \\ &\leq 4\int_0^t \int_0^x \frac{(t-s)^{r_1-1}(x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,\tau)\alpha\left(\{u_{m(s,\tau)}\}_{m=1}^\infty\right) d\tau ds + \epsilon \\ &\leq \left(4\int_0^t \int_0^x \frac{(t-s)^{r_1-1}(x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,\tau) d\tau ds\right) \alpha\left(\{u_m\}_{m=1}^\infty\right) + \epsilon \\ &\leq \left(4\int_0^t \int_0^x \frac{(t-s)^{r_1-1}(x-\tau)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,\tau) d\tau ds\right) \alpha(D) + \epsilon \end{aligned}$$

$$\leq \left(\frac{4p_{2n}^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right)\alpha(D)+\epsilon$$
$$= \ell_n\alpha(D)+\epsilon.$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha(N(D)) \leq \ell_n\alpha(D).$$

As a consequence of steps 1 to 3, together with Theorem 1, we can conclude that N has at least one fixed point in B_{R_n} which is a mild solution of problem (1)-(3).

Applications

Let us examine the following example as an application of our findings.

Example 3. *Let $E = \mathbb{R}$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$.*

$$({}^cD_0^r u)(t,x) = \frac{c_n}{e^{t+x+2}(1+|u(t-1,x-2)|)};\ \textit{if } (t,x)\in J := [0,\infty)\times[0,\infty), \tag{5}$$

$$u(t,x) = t+x^2,\ (t,x)\in \tilde{J} := [-1,\infty)\times[-2,\infty)\backslash[0,\infty)\times[0,\infty), \tag{6}$$

$$\begin{cases} u(t,0) = t, \\ u(0,x) = x^2, \end{cases} \quad (t,x)\in J, \tag{7}$$

Where

$$c_n = \frac{\Gamma(r_1+1)\Gamma(r_2+1)}{n^{r_1+r_2}};\quad n\in\mathbb{N}^*.$$

$$f(t,x,u_{(t,x)}) = \frac{c_n}{e^{t+x+2}(1+|u(t-1,x-2)|)};\qquad (t,x)\in J.$$

The function $(t,x)\mapsto f(t,x,u)$ is jointly continuous for all $u\in C(J,E)$ and hence jointly measurable for all $u\in C(J,E)$. Also the map $u\mapsto f(t,x,u)$ is continuous for all $(t,x)\in J$. Thus, the function f is Carathéodory on $J\times E$. For each $u\in E,\ (t,x)\in J$, we have

$$|f(t,x,u)| \leq 1+\frac{c_n|u|}{e^3}.$$

Hence the condition $(H3)$ is satisfied with $p_{1n}^* = 1,\ p_{2n}^* = c_ne^{-3}$.

Moreover, the condition $(H4)$ *is satisfied. We shall show that the condition* $\ell_n < 1$ *holds for each* $(r_1, r_2) \in (0;1] \times (0;1]$ *we obtain*

$$\begin{aligned}\ell_n &= \frac{4p^*_{2n} n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &= \frac{4}{e^3} \\ &< 1.\end{aligned}$$

As a result, Theorem 2 *implies that our problem* (5)-(7) *has a fixed point* u *defined on* J *which is a solution of our problem* (5)-(7).

Conclusion

Fractional calculus is a broad topic that calls for a variety of techniques and sophisticated tools. In this paper, we have contributed to the study of various classes of Darboux problems for partial hyperbolic functional fractional differential equations via the Caputo derivative with finite delay in Fréchet spaces.

Throughout most of this paper, sufficient conditions were considered in order to obtain the existence and uniqueness of solutions for our problem by reducing the research to the existence and uniqueness of fixed points of suitable operators, using the expanded Darbo's fixed point theorem associated with the notion of measure of noncompactness. There are numerous directions in which the work presented here can be extended. To find existence results, one should examine the properties of the operators and the structure of the space. The presence of weighted continuous functions in the space, their uniqueness, the composition of the solution set, and the optimality of the conditions met by the operators are just a few of the numerous other topics and questions that can be investigated.

References

Baleanu, D., Diethelm, K., Scalas, E., and Trujillo, J. J. (2012). *Fractional Calculus Models and Numerical Methods*. World Scientific Publishing, New York. Available at: https://doi.org/10.1142/9789814355212

Abbas, S. and Benchohra, M. (2009). Darboux problem for perturbed partial differential equations of fractional order with finite delay. *Nonlinear Analysis: Hybrid Systems*, 3, 597-604. Available at: https://doi.org/10.1016/j.nahs.2009.01.001

Abbas, S. and Benchohra, M. (2012). Fractional order partial hyperbolic differential equations involving Caputo's derivative. *Studia Universitatis Babeş–Bolyai Mathematica*, 57(4), 469-479.

Abbas, S., Benchohra, M., and N'Guérékata, G. M. (2012). *Topics in Fractional Differential Equations*. Springer, New York. Available at: https://doi.org/10.1007/978-1-4614-4036-0

Appell, J. (1981). Implicit functions, nonlinear integral equations, and the measure of noncompactness of the superposition operator. *Journal of Mathematical Analysis and Applications*, 83, 251-263. Available at: https://doi.org/10.1016/0022-247X(81)90160-9

Ayerbee Toledano, J. M., Dominguez Benavides, T., and López Acedo, G. (1997). *Measures of noncompactness in metric fixed point theory*. Operator Theory: Advances and Applications, 99. Birkhäuser, Basel, Boston, Berlin. Available at: https://doi.org/10.1007/978-3-0348-8875-7

Banaei, Sh. (2018). Solvability of a system of integral equations of Volterra type in the Fréchet space $L^p_{loc}(\mathbb{R}^+)$ via measure of noncompactness. *Filomat*, 32, 5255-5263. Available at: https://doi.org/10.2298/FIL1813255B

Banás, J., Jleli, M., Mursaleen, M., and Samet, B. (2017). *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*. Springer, Singapore. Available at: https://doi.org/10.1007/978-981-10-4565-8

Banás, J., and Mursaleen, M. (2014). *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*. Springer, Berlin. Available at: https://doi.org/10.1007/978-3-319-01392-4

Benchohra, M., and Hellal, M. (2014). Global uniqueness result for fractional partial hyperbolic differential equations with state-dependent delay. *Annales Polonici Mathematici*, 110(3), 259-281. Available at: https://doi.org/10.4064/ap110-3-2

Benchohra, M., and Hellal, M. (2014). Perturbed partial fractional order functional differential equations with infinite delay in Fréchet spaces. *Nonlinear Dynamics and System Theory*, 14(3), 244-257. Available at: https://doi.org/10.1007/s11071-014-1491-5

Benchohra, M., Henderson, J., and Heris, A. (2017). Fractional partial random differential equations with delay. *PanAmerican Mathematical Journal*, 27(1), 29-40. Available at: https://doi.org/10.1007/s40815-017-0034-2

Bothe, D. (1998). Multivalued perturbation of m-accretive differential inclusions. *Israel Journal of Mathematics*, 108, 109-138. Available at: https://doi.org/10.1007/BF02777780

Dudek, S. (2017). Fixed point theorems in Fréchet Algebras and Fréchet spaces and applications to nonlinear integral equations. *Applied Analysis and Discrete Mathematics*, 11, 340-357. Available at: https://doi.org/10.2298/AADM1703340D

Dudek, S., and Olszowy, L. (2015). Continuous dependence of the solutions of nonlinear integral quadratic Volterra equation on the parameter. *Journal of Function Spaces*, 2015, Article ID 471235, 9 pages. Available at: https://doi.org/10.1155/2015/471235

Helal, M. (2020). Perturbed Partial Functional Differential Equations on Unbounded Domains with Finite Delay. *Applied Mathematics E-Notes*, 20, 82-92. Available at: https://doi.org/10.2478/AMEN-2020-0010

Hilfer, R. (2000). *Applications of Fractional Calculus in Physics*. World Scientific, Singapore. Available at: https://doi.org/10.1142/9789812816223

Kilbas, A. A., and Marzan, S. A. (2005). Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions. *Differential Equations*, 41, 84-89. Available at: https://doi.org/10.1007/s10625-005-0010-4

Kuratowski, K. (1930). Sur les espaces complets. *Fundamenta Mathematicae*, 15, 301-309. Available at: https://doi.org/10.4064/fm-15-1-301-309

Ortigueira, M. D. (2011). *Fractional Calculus for Scientists and Engineers*. Lecture Notes in Electrical Engineering, 84. Springer, Dordrecht. Available at: https://doi.org/10.1007/978-94-007-0531-3

Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press, San Diego. Available at: https://doi.org/10.1016/B978-0-12-558840-9.X5000-3

Tarasov, V. E. (2010). *Fractional Dynamics. Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*. Springer, Heidelberg. Available at: https://doi.org/10.1007/978-3-642-11218-3

Vityuk, A. N., and Golushkov, A. V. (2004). Existence of solutions of systems of partial differential equations of fractional order. *Nonlinear Oscillations*, 7(3), 318-325. Available at: https://doi.org/10.1023/B:NONO.0000045252.92915.0f

Yosida, K. (1980). *Functional Analysis*, 6th edition. Springer-Verlag, Berlin. Available at: https://doi.org/10.1007/978-3-662-02410-7

Mönch, H. (1980). Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. *Nonlinear Analysis: Theory, Methods and Applications*, 4, 985-999. Available at: https://doi.org/10.1016/0362-546X(80)90076-6

Primene proširene Darboove teoreme o fiksnoj tački na frakcione diferencijalne jednačine sa konačnim kašnjenjem

Mohamed Helal

Mustapha Stambouli University of Mascara, Science and Technology Faculty, B.P. 763, 29000, Mascara, Algeria,+
Djillali Liabes University of Sidi Bel-Abbès, Laboratory of Mathematics, B.P. 89, 22000, Sidi Bel-Abbès, Algeria

OBLAST: matematika
KATEGORIJA (TIP) ČLANKA: originalni naučni rad

Sažetak:

Uvod/cilj: Cilj rada je proučavanje postojanja blagih rešenja problema početne vrednosti Darbuovog problema za parcijalne hiperboličke frakcione diferencijalne jednačine pomoću Kaputovog izvoda sa konačnim kašnjenjem u Frešeovim prostorima.

Metode: U analizi se koriste Darboova teorema o fiksnoj tački sa pojmom mere nekompaktnosti kako bi se utvrdilo postojanje rešenja svođenjem istraživanja na dokazivanje postojanja i jedinstvenosti fiksnih tačaka odgovarajućih operatora.

Rezultati: Postojanje blagih rešenja posmatranih parcijalnih hiperboličkih frakcionih diferencijalnih jednačina je dokazano pod odgovarajućim uslovima. Primenljivost teorijskih rezultata je ilustrovana primerom.

Zaključak: U radu se razmatraju dovoljni uslovi za utvrđivanje postojanja i jedinstvenosti rešenja Darbuovog problema primenom Darboove teoreme o fiksnoj tački na neograničenom intervalu. Istraživanje pruža metodološki okvir za analizu sličnih problema i doprinosi širem razumevanju blagih rešenja u frakcionom računu i funkcionalnoj analizi. Dobijeni rezultati nude potencijalne primene za istraživače koji se bave frakcionim diferencijalnim jednačinama u beskonačno-dimenzionalnim prostorima.

Ključne reči: parcijalna hiperbolička diferencijalna jednačina, frakcioni izvod, blago rešenje, Frešeov prostor.

Paper received on: 11 April 2025.
Manuscript corrections submitted on: 11 February 2025.
Paper accepted for publishing on: 16 March 2026.