Fixed point approximation generalized $(\alpha, \psi)-$ contraction mappings for fixed-point analysis in Branciari multiplicative b-quasi-metric spaces

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Abstract:

Introduction/purpose: In this paper, we introduce a new class of Branciari-type multiplicative structures, termed b-quasi-metric spaces. This framework extends and unifies several existing concepts, including Branciari metric spaces, quasi-metric spaces, and b-metric spaces. By broadening these foundational structures, we aim to establish a more general setting for the analysis of fixed-point phenomena.

Methods: Within the proposed b-quasi-metric framework, we formulate a generalized class of $(\alpha, \psi)-$ contraction mappings. We then analyze these mappings using fixed point techniques adapted to the newly defined space. The study includes the identification of sufficient conditions guaranteeing the validity of the contraction principle and the development of tools for verifying these conditions.

Results: We establish key properties of the proposed $(\alpha, \psi)-$ contraction mappings, including the existence and uniqueness of fixed points under appropriate hypotheses. These results substantially generalize several earlier fixed point theorems formulated in Branciari, quasi-metric, and b-metric settings. Additionally, illustrative examples are provided to confirm the correctness and applicability of the theoretical findings.

Conclusion: The introduction of b-quasi-metric spaces and the associated class of $(\alpha, \psi)-$ contractions yields a unified and comprehensive extension of previous fixed point results. The examples and applications to specific integral equations further demonstrate the practical relevance of the framework, offering a solid foundation for future theoretical developments and applications.

Key words: fixed points, multiplicative metric space, Branciari multiplicative b- quasi-metric space, generalized multiplicative $(\alpha, \psi)-$contraction mappings.

Introduction

Fixed point (F_p) theory stands as a foundational and adaptable branch within the domain of mathematics, boasting extensive implications spanning numerous disciplines. Its central focus revolves around the notion of F_{ps}, which refer to solutions to equations or mappings that retain their identity even when subjected to specific transformations. This theory furnishes a potent framework for comprehending the presence, singularity and stability of solutions across a broad spectrum of mathematical and real-world scenarios. The origins of F_p theory can be traced back to the early 20th century, marked by influential contributions from luminaries in mathematics, including Banach, Brouwer, and Kakutani. Since its inception, this field has matured into a thriving domain replete with diverse applications in both pure and applied mathematics (Czerwik, 1993) , as well as within a multitude of scientific and engineering domains. In its essence, F_p theory seeks to address fundamental questions such as the determination of the existence of solutions, the assessment of their uniqueness and the development of methodologies for computation or approximation. These inquiries hold profound implications across a wide spectrum of fields, encompassing areas such as real analysis, functional analysis, topology, optimization, control theory, computer science, among others. The ubiquity of F_p theory within the realm of mathematics, along with its wide-ranging relevance in diverse disciplines, underscores its paramount importance. Be it in the examination of dynamical system behavior, the formulation of numerical algorithms, the modeling of economic equilibria, or the design of efficient computational processes in the domain of computer science, F_p theory emerges as an invaluable instrument for gaining insight into the multifaceted intricacies of the world around us. This introductory overview offers but a glimpse into the rich and complex tapestry of F_p theory. As we delve deeper into this captivating field, we will uncover its pivotal concepts, theorem foundations, and the profound ramifications that it holds not just within mathematics but also extending far beyond its boundaries. The foundational concept of contraction mapping, first introduced by Banach (1922) holds immense significance within the domain of F_p theory. Subse-

quently, researchers have extended and enriched the Banach contraction principle. In 2014, Lin and fellow researchers (Lin et al., 2014) introduced the notion of rectangular quasi-metric spaces and provided empirical evidence of an F_p theorem that applies to Meir-Keeler contractive mappings within this specific spatial framework.

Ozavsar & Cevikel (2012) investigated multiplicative metric mappings and examined certain topological characteristics of the corresponding multiplicative metric space. During their discussions, they made an interesting observation: the collection of positive real numbers, denoted as $\mathbb{R}^+$, forms a complete multiplicative metric spaces when equipped with the multiplicative absolute value function. Furthermore, they established the notion of contraction mappings in the framework of multiplicative calculus and offered proofs for specific F_p theorems applicable to these mappings in complete multiplicative metric spaces. Karapinar & Lakzian (2014) introduced the concept of (α, ψ)-contractive mappings in the context of rectangular quasi-metric spaces. They subsequently proved F_p theorems for these mappings. George et al. (2015) extended this concept by introducing rectangular $\boldsymbol{b}-$metric spaces, which encompassed metric spaces and Branciari spaces. This development piqued the interest of many researchers, leading to investigations into a wide range of F_p theorems within these spaces. Alharbi et al. (2018) defined (α)-contractive mappings and provided proofs for F_p theorems tailored specifically to rectangular metric spaces. More recently, Bodaghi et al. (2023) & Abagaro et al. (2022) introduced the concept of ψ-contraction and investigated F_p properties related to ψ-contraction mappings in complete rectangular quasi-metric spaces. Their research confirmed the existence and uniqueness of F_{ps} within this context. Building on the groundwork laid by Ozavsar & Cevikel (2017), as well as the contributions of well-known authors (cf. Abagaro et al., 2022), the primary objective of this paper is to establish F_p properties for multiplicative generalized (α, ψ)-contraction mappings in the framework of Branciari multiplicative $\boldsymbol{b}-$quasi-metric spaces.

The set Ψ consists of functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ that are continuous, non-decreasing, and satisfy the following conditions:

1. The infinite product $\prod_{k=1}^{\infty} \psi_k(\nu)^{s_k}$ converges for all $\nu > 0$.
2. $\psi(\nu)^k < \nu$ for all $\nu > 0$ and for all integers k.
3. $\psi(1) = 1$ if and only if $\nu = 1$.

Here, ψ^k represents the kth iteration of the function ψ, and $s \in [1, \infty)$. In order to better comprehend multiplicative metric spaces, b-metric spaces, Branciari spaces, and Branciari b-metric spaces, readers are advised to refer to Branciari et al. (2000), Aydi et al. (2012), and Karapınar (2014). It is important to emphasize that all metric spaces belong to the Branciari metric spaces class, and every Branciari metric space can be viewed as a Branciari b-metric space with coefficient $s = 1$. Branciari metric spaces are necessarily rectangular b-metric spaces, but the opposite may not always be true. To illustrate, Lin et al. (2014) demonstrated that not all Branciari b-metric spaces can be classified as Branciari metric spaces (cf. George et al., 2015). In order to understand the significance of Branciari quasi-metric spaces, readers are advised to refer to Samet et al. (2012). To illustrate that any Branciari metric space is also a Branciari quasi-metric space, but the converse is not always the case, Lin et al. (2014) provided an example to demonstrate that not every Branciari quasi-metric space on a set Θ can be considered a Branciari metric space on the same set Θ (cf. Lakzian & Samet, 2012; Lin et al, 2014; Alma-halebi et al., 2014).

Definition 1. *Consider a non-empty set Θ, a mapping $\Gamma : \Theta \to \Theta$ of Θ, and a function $\alpha : \Theta \times \Theta \to \mathbb{R}^+$. Then the mapping Γ is considered as α-admissible if, for any $\varsigma,\ \omega \in \Theta$,*

$$\alpha(\varsigma,\ \omega) \in [1, \infty) \ \textit{implies} \ \alpha(\Gamma\varsigma, \Gamma\omega) \in [1, \infty)$$

Definition 2. *Suppose we have a multiplicative rectangular quasi-metric spaces denoted as $(\Theta, \eth)$, a sequence $\{\varsigma_\jmath\}$ in Θ, and an element $\varsigma \in \Theta$. In this context, it holds that*

1. *The sequence $\{\varsigma_\jmath\}$ is said to converge to the point $\varsigma \in \Theta$ if, as the limit as $\jmath$ approaches infinity, both $\eth(\varsigma_\jmath, x)$ and $\eth(\varsigma, \varsigma_\jmath)$ become 1. This convergence is symbolized by $\lim_{\jmath\to\infty} \varsigma_\jmath = \varsigma$ or $\varsigma_\jmath \to \varsigma$ as $\jmath \to \infty$.*
2. *The sequence $\{\varsigma_\jmath\}$ is Cauchy in $(\Theta, \eth)$ if*

$$\lim_{\jmath\to\infty} \eth\left(\varsigma_\jmath, \varsigma_{\jmath+\iota}\right) = 1 = \lim_{\jmath\to\infty} \eth\left(\varsigma_{\jmath+\iota}, \varsigma_\jmath\right)$$

   for every $\iota > 0$.
3. *In the case where Cauchy sequence within Θ converges to a particular point $\varsigma \in \Theta$, the space is considered to be a complete Branciar quasi-metric space.*

Main results

The major objective of this section is threefold. First, to explain the concept of Br_{mult} $\boldsymbol{b}$-quasi-metric spaces, second, to define generalized (α, ψ) multiplicative contraction mappings, and finally, to investigate the F_p results for these mappings within the context of Br_{mult} $\boldsymbol{b}-$quasi-metric spaces.

To begin, we will introduce the concept of a Br_{mult} $\boldsymbol{b}-$ quasi-metric space as follows:

Definition 3. *Consider $\Theta \neq \emptyset$ (a non-empty set) and $s \geq 1$. Suppose that the function $\eth$ defined from $\Theta \times \Theta$ to $\mathbb{R}+$ satisfies the subsequent conditions:*

1. $\eth(\varsigma,\ \omega) = 1$ *iff* $\varsigma = \omega$;
2. $\eth(\varsigma,\ \omega) \leq (\eth(\varsigma, \eta_1).\eth(\eta_1, \eta_2).\eth(\eta_2,\ \omega))^s$ *(Br_{mult} inequality)*

for every $\varsigma,\ \omega \in \Theta$ and all distinct points $\eta_1,\ \eta_2 \in \Theta \backslash \{\varsigma,\ \omega\}$.

In such case, we refer to $\eth$ as a Br_{mult} $\boldsymbol{b}-$ quasi-met defined on the set Θ, and we denote the pair $(\Theta, \eth)$ as a Br_{mult} $\boldsymbol{b}-$ quasi-metric space.

In the section that follows, we provide an illustration of a Br_{mult} $\boldsymbol{b}-$ quasi-metric space.

Example 1. Consider the set Θ defined as the union of $\mathcal{O}_1$ and $\mathcal{O}_2$, where $\mathcal{O}_1 = \{0.50, 0.33, 0.25, 0.20\}$ and $\mathcal{O}_2 = [1, 2]$. Let us define the generalized metric $\eth$ on Θ in the following manner:

$$\eth(0.50, 0.33) = \eth(0.25, 0.20) = 0.3; \eth(0.33, 0.50) = \eth(0.20, 0.25)$$

$$=\eth(0.33, 0.25) = 0.1;$$
$$\eth(0.50, 0.25) = \eth(0.33, 0.20) = 0.6; \eth(0.25, 0.50) = \eth(0.20, 0.33) = 0.4;$$
$$\eth(0.50, 0.20) = 1.05; \eth(0.20, 0.50) = \eth(0.25, 0.33) = 0.5;$$
$$\eth(0.50, 0.50) = \eth(0.33, 0.33) = \eth(0.25, 0.25) = \eth(0.20, 0.20) = 1;$$
and $\eth(\varsigma,\ \omega) = \left|\frac{\varsigma}{\omega}\right|$ if $\varsigma,\ \omega \in \mathcal{O}_2$ or $\varsigma \in \mathcal{O}_1,\ \omega \in \mathcal{O}_2$ or $\varsigma \in \mathcal{O}_2,\ \omega \in \mathcal{O}_1$.

Here, we have $(\Theta, \eth)$ forming a Br_{mult} $\boldsymbol{b}-$ quasi-metric space, and the coefficient is $s = 1.5$, which exceeds 1. Meeting condition (1) in Definition

3 is relatively straightforward. Let us now use an example that proves that condition (2) in Definition (3) is also satisfied:

Case (a)

Given that $\varsigma,\ \omega \in \mathcal{O}_1$, we have

$\eth(\varsigma,\ \omega) = \eth\,(0.5, 0.33) = 0.3 \leq [\eth\,(0.5, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.33)]^s$
when $\eta_1, \eta_2 \in \{0.25, 0.20\}$.

$\eth(\varsigma,\ \omega) = \eth\,(0.33, 0.50) = 0.1 \leq [\eth\,(0.33, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.50)]^s$
when $\eta_1, \eta_2 \in \{0.25, 0.20\}$.

$\eth(\varsigma,\ \omega) = \eth\,(0.33, 0.25) = 0.1 \leq [\eth\,(0.33, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.25)]^s$
when $\eta_1,\ \eta_2 \in \{0.50, 0.20\}$.

$\eth(\varsigma,\ \omega) = \eth\,(0.25, 0.33) = 0.5 \leq [\eth\,(0.25, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.33)]^s$
when $\eta_1, \eta_2 \in \{0.50, 0.20\}$.

$\eth(\varsigma,\ \omega) = \eth\,(0.25, 0.20) = 0.3 \leq [\eth\,(0.25, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.25)]^s$
when $\eta_1, \eta_2 \in \{0.50, 0.33\}$.

$\eth(\varsigma,\ \omega) = \eth\,(0.20, 0.25) = 0.1 \leq [\eth\,(0.20, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.25)]^s$
when $\eta_1, \eta_2 \in \{0.50, 0.33\}$.

$\eth(\varsigma,\ \omega) = \eth\,(0.50, 0.25) = 0.6 \leq [\eth\,(0.50, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.25)]^s$
when $\eta_1, \eta_2 \in \{0.33, 0.20\}$.

$\eth(\varsigma,\ \omega) = \eth\,(0.25, 0.50) = 0.4 \leq [\eth\,(0.25, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.50)]^s$
when $\eta_1, \eta_2 \in \{0.33, 0.20\}$.

$\eth(\varsigma,\ \omega) = \eth\,(0.50, 0.20) = 1.05 \leq [\eth\,(0.50, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.20)]^s$
when $\eta_1, \eta_2 \in \{0.33, 0.25\}$.

$\eth(\varsigma,\ \omega) = \eth\,(0.20, 0.50) = 0.5 \leq [\eth\,(0.20, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.50)]^s$
when $\eta_1, \eta_2 \in \{0.33, 0.25\}$.

$\eth(\varsigma,\ \omega) = \eth\,(0.33, 0.20) = 0.6 \leq [\eth\,(0.33, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.20)]^s$
when $\eta_1, \eta_2 \in \{0.50, 0.25\}$.

$\eth(\varsigma,\ \omega) = \eth\,(0.20, 0.33) = 0.4 \leq [\eth\,(0.20, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.33)]^s$
when $\eta_1, \eta_2 \in \{0.50, 0.25\}$.

$\eth(\varsigma,\ \omega) = \eth\,(0.20, 0.33) = 0.4 \leq [\eth\,(0.20, \eta_1) \times \eth(\eta_1, \eta_2) \times \eth\,(\eta_2, 0.33)]^s$
when $u\eta_1, \eta_2 \in \{0.50, 0.25\}$.

Case (b)

If $\varsigma,\ \omega \in \mathcal{O}_2$ or $\varsigma \in \mathcal{O}_1,\ \omega \in \mathcal{O}_2$ or $\varsigma \in \mathcal{O}_2,\ \omega \in \mathcal{O}_1$, then

$\eth(\varsigma,\ \omega) = \left|\frac{\varsigma}{\omega}\right| \leq \left\{\left|\frac{\varsigma}{\eta_1}\right| \times \left|\frac{\eta_1}{\eta_2}\right| \times \left|\frac{\eta_2}{\omega}\right|\right\}^s$ for every distinct points $\eta_1,\ \eta_2 \in \Theta\backslash\{\varsigma,\ \omega\}$.

However, it is important to note that $(\Theta, \eth)$ do not satisfy the criteria for being a metric space, a rectangular quasi-metric space or a Branciari metric space. This is due to the failure of the triangle inequality, symmetry, and Br_{mult} inequality, which can be demonstrated as follows:

$$\eth\left(0.50, 0.25\right) = 0.6 \nleq 0.4 = \eth\left(0.50, 0.33\right) \times \eth\left(0.33, 0.25\right) = 0.3 \times 0.1,$$
$$\eth\left(0.50, 0.25\right) = 0.6 \neq 0.4 = \eth\left(0.25, 0.50\right), \text{ and}$$
$$\eth\left(0.50, 0.20\right) = 1.05 \nleq 0.7 = \eth\left(0.50, 0.33\right) \times \eth\left(0.33, 0.25\right) \times \eth\left(0.25, 0.20\right).$$

Next, we will provide the definitions for the convergence of a sequence in Br_{mult} $\boldsymbol{b}-$ quasi-metric spaces and the completeness of such spaces.

Definition 4. *Suppose we have a Br_{mult} $\boldsymbol{b}-$ quasi-metric space $(\Theta, \eth)$ and consider a sequence $\{\varsigma_\jmath\}$ in Θ along with an element $\varsigma \in \Theta$. Then, the following holds:*

(i) $\{\varsigma_\jmath\}$ is said to be convergent to Θ if $\lim_{\jmath\to\infty} \eth\left(\varsigma_\jmath, \varsigma\right) = 1 = \lim_{\jmath\to\infty} \eth\left(\varsigma, \varsigma_\jmath\right)$ that is $\lim_{\jmath\to\infty} \varsigma_\jmath = \varsigma$ or $\varsigma_\jmath \to \varsigma$ as $\jmath \to \infty$.

(ii) Sequence $\{\varsigma_\jmath\}$ is referred to be the Cauchy sequence in $(\Theta, \eth)$ if

$$\lim_{\jmath\to\infty} \eth\left(\varsigma_\jmath, \varsigma_{\jmath+\iota}\right) = 1 = \lim_{\jmath\to\infty} \eth\left(\varsigma_{\jmath+\iota}, \varsigma_\jmath\right)$$

for every $\iota > 0$.

(iii) If every Cauchy sequence in Θ converges to some $\varsigma \in \Theta$ then $(\Theta, \eth)$ is called a complete Br_{mult} $\boldsymbol{b}$- quasi-metric space.

Remark 1. Let $(\Theta, \eth)$ be a Br_{mult} $\boldsymbol{b}-$ quasi-metric space. For $\varsigma \in \Theta$, the open ball with center $Theta$ and radius $r > 0$ is defined by

$$B_r(\varsigma, r) = \{\omega \in \Theta : \sup\{\eth(\varsigma,\ \omega), \eth(\omega, \varsigma)\} < r\}.$$

In a broader context, it is worth noting that an open ball within a Branciari metric space may not necessarily qualify as an open set. Similarly, a Br_{mult} $\boldsymbol{b}-$ quasi-metric space may lack continuity. The convergence of a

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sequence in a Branciari multiplicative $\boldsymbol{b}$- quasi-metric space does not necessarily imply that it is a Cauchy sequence. Additionally, such a space may not adhere to the Hausdorff property, which implies that the uniqueness of limits cannot be assured.

To exemplify these ideas, here is an example in accordance with Remark 1.

Example 2. Consider $\Theta = \mathcal{O}_1 \cup \mathcal{O}_2$, where $\mathcal{O}_1 = \left\{1 - \frac{1}{\eta},\ \alpha \in \mathbb{N}\right\}$, and $\mathcal{O}_2 = \{1, 3\}$. Let a function $\eth$ define from $\Theta \times \Theta$ to $\mathbb{R}^+$ such that

$$\eth(\varsigma,\ \omega) = \begin{cases} 1 & \text{if } \varsigma = \omega \\ 4.5 & \text{if } \varsigma,\ \omega \in \mathcal{O}_1 \\ 1 + \frac{1}{\jmath} & \text{if } \varsigma \in \mathcal{O}_1,\ \omega \in \mathcal{O}_2 \\ 1 + \frac{1}{\jmath+1} & \text{if } \varsigma \in \mathcal{O}_2,\ \omega \in \mathcal{O}_1 \\ 2 & \text{if } \varsigma,\ \omega \in \mathcal{O}_2. \end{cases}$$

where $\jmath \in \mathcal{O}_1$

The function $\eth$ is a Br_{mult} $\boldsymbol{b}$- quasi-metric space with a coefficient of $s = 2$. However, $\eth$ does not meet the criteria to be considered either a Branciari quasi-met or a Branciari $\boldsymbol{b}$-metric space, due to the following factors:

$\eth(0.50, 0.33) = 4.5 \nleq 2.75 = \eth(0.50, 0) \times \eth(0, 3) \times \eth(3, 0.33)$ and
$\eth(0.33, 0) = 0.33 \neq 0.25 = \eth(0, 0.33)$.

It is also evident that

$$\lim_{\jmath\to\infty} \eth\left(1 + \frac{1}{2\jmath}, 0\right) = \lim_{\jmath\to\infty}(1 + \frac{1}{\jmath}) = 1 = \lim_{\jmath\to\infty} \eth\left(0, 1 + \frac{1}{2\jmath}\right) = \lim_{\jmath\to\infty}(1 + \frac{1}{\jmath+1})$$

and

$$\lim_{\jmath\to\infty} \eth\left(1 + \frac{1}{2\jmath}, 3\right) = \lim_{\jmath\to\infty}(1 + \frac{1}{\jmath}) = 1 = \lim_{\jmath\to\infty} \eth\left(3, 1 + \frac{1}{2\jmath}\right) = \lim_{\jmath\to\infty}(1 + \frac{1}{\jmath+1}),$$

In other words, the sequence $\left\{1 + \frac{1}{2\jmath}\right\}$ has two distinct limits, namely, 1 and 4. Moreover, the sequence $\left\{\frac{1}{2\jmath}\right\}$ demonstrates Br_{mult} quasi-$\boldsymbol{b}$-

convergence but does not meet the requirements for being a Br_{mult} quasi-$\boldsymbol{b}$-Cauchy sequence. This is because:

$$\lim_{\jmath\to\infty} \eth\left(\varsigma_\jmath, \varsigma_{\jmath+p}\right) = \lim_{\jmath\to\infty} \eth\left(1+\frac{1}{2\jmath}, 1+\frac{1}{2\jmath+p}\right) = 4.5$$

and

$$\lim_{\jmath\to\infty} \eth\left(\varsigma_{\jmath+p}, \varsigma_\jmath\right) = \lim_{\jmath\to\infty} \eth\left(1+\frac{1}{2\jmath+p}, 1+\frac{1}{2\jmath}\right) = 4.5.$$

Next, we will establish the definition of an multiplicatibe $(\alpha,\psi)-$contraction mapping within the framework of a Br_{mult} $\boldsymbol{b}-$quasi-metric space.

Definition 5. *Consider a Br_{mult} $\boldsymbol{b}-$quasi-metric space $(\Theta,\eth)$ and a given mapping $\Gamma:\Theta\longrightarrow\Theta$. We classify Γ as a generalized $(\alpha,\psi)-$ contraction mapping under the condition of two functions: $\alpha:\Theta\times\Theta\to\mathbb{R}^+$ and $\psi\in\Psi$. This classification holds when the following conditions are met:*

$$(\eth(\Gamma\varsigma,\Gamma\omega))^{\alpha(\varsigma,\,\omega)} \leq \psi(\Xi(\varsigma,\,\omega)) \;\;\forall\;\; \varsigma,\,\omega\in\Theta, \tag{1}$$

where

$$\Xi(\varsigma,\omega) = \sup\Big\{\,(\eth(\varsigma,\Gamma\varsigma)\cdot\eth(\varsigma,\Gamma\omega))^{\left(1+\eth(\varsigma,\Gamma\omega)+\eth(\omega,\Gamma\varsigma)\right)^{-1}}, \eth(\varsigma,\Gamma\varsigma), \eth(\varsigma,\omega), \eth(\omega,\Gamma\omega)\Big\}.$$

We now introduce and illustrate the subsequent F_p theorem.

Theorem 1. Consider a complete Br_{mult} $\boldsymbol{b}-$quasi-metric space $(\Theta,\eth)$ and a generalized $(\alpha,\psi)-$ contraction mapping $\Gamma:\Theta\to\Theta$. Take into account the subsequent assumption:

1. Γ is a mapping that satisfies the α-admissibility condition;
2. $\exists\;\varsigma_0\in\Theta$ such that $\alpha\left(\varsigma_0,\Gamma\varsigma_0\right)\in[1,\infty), \alpha\left(\Gamma\varsigma_0,\varsigma_0\right)\in[1,\infty), \alpha\left(\varsigma_0,\Gamma^2\varsigma_0\right)\in[1,\infty)$, and $\alpha\left(\Gamma^2\varsigma_0,\varsigma_0\right)\in[1,\infty)$;
3. mapping Γ is continuous.

In this case, mapping Γ has a F_p.

Proof. As indicated in condition (2) mentioned earlier, $\exists\;\varsigma_0$ in Θ such that $\alpha(\varsigma_0,\Gamma\varsigma_0)\in[1,\infty)$ and $\alpha(\Gamma\varsigma_0,\varsigma_0)\in[1,\infty)$. Now, we establish a sequence

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$\{\varsigma_n\}$ in Θ by defining $\varsigma_{\jmath+1} = \Gamma\varsigma_n = \Gamma^{\jmath+1}\varsigma_0$ for every $\jmath \geq 0$. Let us assume that $\varsigma_{\jmath_0} = \varsigma_{\jmath_0+1}$ for some $\jmath_0 \geq 0$. Given that $\Gamma\varsigma_{\jmath_0} = \varsigma_{\jmath_0+1}$, it follows that the point $\eta_1 = \varsigma_{\jmath_0}$ serves as a F_p for Γ. Consequently, this concludes the proof. It should be noted that we are working under the assumption that $\varsigma_n \neq \varsigma_{\jmath+1}$ for every $\jmath \in \mathbb{N} \cup \{0\}$.

As Γ is a mapping that satisfies the α-admissibility condition, it follows that

$$\alpha(\varsigma_0, \varsigma_1) = \alpha(\varsigma_0, \Gamma\varsigma_0) \in [1, \infty) \Rightarrow \alpha(\Gamma\varsigma_0, \Gamma\varsigma_1) = \alpha(\varsigma_1, \varsigma_2) \in [1, \infty).$$

By employing the given expression, we can deduce that

$$\alpha(\varsigma_\jmath, \varsigma_{\jmath+1}) \in [1, \infty), \ \forall\, \jmath = 0,\ 1, \ldots, \tag{2}$$

and $\alpha(\varsigma_1, \varsigma_0) = \alpha(\Gamma\varsigma_0, \varsigma_0) \in [1, \infty) \Rightarrow \alpha(\Gamma\varsigma_1, \Gamma\varsigma_0) = \alpha(\varsigma_2, \varsigma_1) \in [1, \infty)$. Again, by employing the given expression, we can deduce that

$$\alpha(\varsigma_{\jmath+1}, \varsigma_\jmath) \in [1, \infty), \quad \forall\, \jmath = 0,\ 1, \ldots \tag{3}$$

Using a similar approach, we ascertain that $\alpha(\varsigma_0, \varsigma_2) = \alpha\left(\varsigma_0, \Gamma^2\varsigma_0\right) \in [1, \infty)$. This further implies that

$$\alpha(\Gamma\varsigma_0, \Gamma\varsigma_2) = \alpha(\varsigma_1, \varsigma_3) \in [1, \infty).$$

Through a series of recursive steps, it becomes evident that

$$\alpha(\varsigma_\jmath, \varsigma_{\jmath+2}) \in [1, \infty), \quad \forall\, \jmath = 0,\ 1, \ldots \tag{4}$$

In a comparable manner, we can readily conclude that

$$\alpha(\varsigma_{\jmath+2}, \varsigma_\jmath) \in [1, \infty), \quad \forall\, \jmath = 0,\ 1, \ldots \tag{5}$$

Step 1: we illustrate that

$$\eth(\varsigma_\jmath, \varsigma_{\jmath+1}) \to 1 \text{ as } \jmath \to \infty, \text{ and } \eth(\varsigma_{\jmath+1}, \varsigma_\jmath) \to 1 \text{ as } \jmath \to \infty,$$

also

$$\eth(\varsigma_\jmath, \varsigma_{\jmath+2}) \to 1 \text{ as } \jmath \to \infty, \text{ and } \eth(\varsigma_{\jmath+2}, \varsigma_\jmath) \to 1 \text{ as } \jmath \to \infty.$$

With respect to equation (1), we have

$$\eth(\varsigma_\jmath, \varsigma_{\jmath+1}) = \eth(\Gamma\varsigma_{\jmath-1}, \Gamma\varsigma_\jmath)$$

which results in

$$\begin{aligned}\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right) &\leq \left(\eth\left(\Gamma\varsigma_{\jmath-1},\Gamma\varsigma_{\jmath}\right)\right)^{\alpha(\varsigma_{\jmath-1},\varsigma_{\jmath})} \\ &\leq \psi\left(\Xi\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right)\right), \quad \forall\ \jmath \geq 1, \end{aligned} \tag{6}$$

where

$$\Xi(\varsigma_{\jmath-1},\varsigma_{\jmath}) = \sup\Bigg\{ \left(\eth(\varsigma_{\jmath-1},\Gamma\varsigma_{\jmath-1})\,\eth(\varsigma_{\jmath-1},\Gamma\varsigma_{\jmath})\right)^{\left(1+\eth(\varsigma_{\jmath-1},\Gamma\varsigma_{\jmath})+\eth(\varsigma_{\jmath},\Gamma\varsigma_{\jmath-1})\right)^{-1}},$$

$$\eth(\varsigma_{\jmath-1},\Gamma\varsigma_{\jmath-1}),\ \eth(\varsigma_{\jmath-1},\varsigma_{\jmath}),\ \eth(\varsigma_{\jmath},\Gamma\varsigma_{\jmath})\Bigg\}$$

which results in

$$\Xi\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right) = \sup\Big\{\left(\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right).\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath+1}\right)\right)^{\left(1+\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath+1}\right)+\eth\left(\varsigma_{\jmath},\varsigma_{\jmath}\right)\right)^{-1}},$$

$$\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right),\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right),\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}.\right)\Big\}$$

Therefore, we obtain

$$\Xi\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right) = \max\Big\{\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right),\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\Big\}.$$

If $\Xi\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right) = \eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)$.

Subsequently, based on (6), we obtain

$$\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right) \leq \psi\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right) \leq \left(\psi\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)\right)^{s} \leq \eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right),$$

which conflicts with the earlier assertion. Hence, $\Xi\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right)$ is equal to $\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right)$. Let

$$e_{\jmath} = \eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right), l_{\jmath} = \eth\left(\varsigma_{\jmath+1},\varsigma_{\jmath}\right), e^{*}_{\jmath} = \eth\left(\varsigma_{\jmath},\varsigma_{\jmath+2}\right)$$

and $l^{*}_{\jmath} = \eth\left(\varsigma_{\jmath+2},\varsigma_{\jmath}\right)$ for every $\jmath \geq 0$. Utilizing (6), we obtain

$$\begin{aligned} e_{\jmath} &= \eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right) = \eth\left(\Gamma\varsigma_{\jmath-1},\Gamma\varsigma_{\jmath}\right) \leq \psi\left(\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right)\right) \\ &= \psi\left(\eth\left(\Gamma\varsigma_{\jmath-2},\Gamma\varsigma_{\jmath-1}\right)\right) \\ &\leq \psi^{2}\left(\eth\left(\varsigma_{\jmath-2},\varsigma_{\jmath-1}\right)\right) \end{aligned}$$

$$\vdots$$

$$\leq \psi^{\jmath}\left(\eth\left(\varsigma_0,\varsigma_1\right)\right)=\psi^{\jmath}\left(e_0\right)\longrightarrow 1 \text{ as } \jmath\longrightarrow\infty. \tag{7}$$

Furthermore,

$$\begin{aligned} l_{\jmath}=\eth\left(\varsigma_{\jmath+1},\varsigma_{\jmath}\right)&=\eth\left(\Gamma\varsigma_{\jmath},\Gamma\varsigma_{\jmath-1}\right)\\ &\leq\left(\eth\left(\Gamma\varsigma_{\jmath},\Gamma\varsigma_{\jmath-1}\right)\right)^{\alpha\left(\varsigma_{\jmath},\varsigma_{\jmath-1}\right)}\\ &\leq\psi\left(\Xi\left(\varsigma_{\jmath},\varsigma_{\jmath-1}\right)\right), \end{aligned} \tag{8}$$

$\forall\ \jmath\in[1,\infty)$ wherein

$$\Xi(\varsigma_{\jmath},\varsigma_{\jmath-1})=\sup\left\{\begin{matrix}\left(\eth(\varsigma_{\jmath},\Gamma\varsigma_{\jmath})\,\eth(\varsigma_{\jmath},\Gamma\varsigma_{\jmath-1})\right)^{\left(1+\eth(\varsigma_{\jmath},\Gamma\varsigma_{\jmath-1})+\eth(\varsigma_{\jmath-1},\Gamma\varsigma_{\jmath})\right)^{-1}},\\ \eth(\varsigma_{\jmath},\Gamma\varsigma_{\jmath}),\quad \eth(\varsigma_{\jmath},\varsigma_{\jmath-1}),\quad \eth(\varsigma_{\jmath-1},\Gamma\varsigma_{\jmath-1})\end{matrix}\right\}.$$

As a result,

$$=\sup\left\{\begin{matrix}\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right).\eth\left(\varsigma_{\jmath},\varsigma_{\jmath}\right)\right)^{\left(1+\eth\left(\varsigma_{\jmath},\varsigma_{\jmath}\right)+\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath+1}\right)\right)^{-1}},\\ \eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right),\eth\left(\varsigma_{\jmath},\varsigma_{\jmath-1}\right),\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right)\end{matrix}\right\}$$

$$\Xi\left(\varsigma_{\jmath},\varsigma_{\jmath-1}\right)=\sup\left\{\eth\left(\varsigma_{\jmath},\varsigma_{\jmath-1}\right),\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right),\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right)\right\}.$$

We analyze three distinct cases.

Case (c) Provided that $\Xi\left(\varsigma_{\jmath},\varsigma_{\jmath-1}\right)=\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right)$, according to (8), we obtain

$$\begin{aligned}\eth\left(\varsigma_{\jmath+1},\varsigma_{\jmath}\right)&\leq\psi\left(\eth\left(\varsigma_{\jmath-1},\varsigma_{\jmath}\right)\right)\\ &\leq\psi^{\jmath}\left(\eth\left(\varsigma_0,\varsigma_1\right)\right)=\psi^{\jmath}\left(e_0\right)\longrightarrow 1\quad \text{as } \jmath\longrightarrow\infty.\end{aligned}$$

Case (d) Provided that $\Xi\left(\varsigma_{\jmath},\varsigma_{\jmath-1}\right)$ is same as $\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)$, based on equation (8), we have

$$\begin{aligned}\eth\left(\varsigma_{\jmath+1},\varsigma_{\jmath}\right)&\leq\psi\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)\\ &\leq\psi^{\jmath}\left(\eth\left(\varsigma_0,\varsigma_1\right)\right)=\psi^{\jmath}\left(e_0\right)\longrightarrow 1\quad \text{as } \jmath\longrightarrow\infty.\end{aligned}$$

Case (e) Provided that $\Xi(\varsigma_\jmath, \varsigma_{\jmath-1})$ is same as $\eth(\varsigma_\jmath, \varsigma_{\jmath-1})$, following the equation (8), we have

$$\begin{aligned} l_\jmath &= \eth(\varsigma_{\jmath+1}, \varsigma_\jmath) \\ &= \eth(\Gamma\varsigma_\jmath, \Gamma\varsigma_{\jmath-1}) \leq \psi(\eth(\varsigma_\jmath, \varsigma_{\jmath-1})) \\ &= \psi(\eth(\Gamma\varsigma_{\jmath-1}, \Gamma\varsigma_{\jmath-2})) \\ &\leq \psi^2(\eth(\varsigma_{\jmath-1}, \varsigma_{\jmath-2})) \\ &\vdots \\ &\leq \psi^\jmath(\eth(\varsigma_1, \varsigma_0)) = \psi^\jmath(l_0) \longrightarrow 0 \quad \text{as } \jmath \longrightarrow \infty. \end{aligned}$$

Combining all the above-mentioned cases, we obtain

$$l_\jmath = \eth(\varsigma_{\jmath+1}, \varsigma_\jmath) \longrightarrow 1 \quad \text{as } \jmath \longrightarrow \infty. \tag{9}$$

Using equations (7) and (9), we can infer that

$$\eth(\varsigma_\jmath, \varsigma_{\jmath+1}) \to 1, \text{ as } \jmath \to \infty \text{ and } \eth(\varsigma_{\jmath+1}, \varsigma_\jmath) \to 1 \text{ as } \jmath \to \infty. \tag{10}$$

Now, to prove that $\lim_{\jmath\to\infty} \eth(\varsigma_\jmath, \varsigma_{\jmath+2}) = 1 = \lim_{\jmath\to\infty} \eth(\varsigma_{\jmath+2}, \varsigma_\jmath)$.

$$\begin{aligned} e_\jmath^* &= \eth(\varsigma_\jmath, \varsigma_{\jmath+2}) = \eth(\Gamma\varsigma_{\jmath-1}, \Gamma\varsigma_{\jmath+1}) \\ &\leq (\eth(\Gamma\varsigma_{\jmath-1}, \Gamma\varsigma_{\jmath+1}))^{\alpha(\varsigma_{\jmath-1}, \varsigma_{\jmath+1})} \\ &\leq \psi(\Xi(\varsigma_{\jmath-1}, \varsigma_{\jmath+1})), \quad \forall\, \jmath \geq 1 \end{aligned} \tag{11}$$

wherein

$$\Xi(\varsigma_{\jmath-1}, \varsigma_{\jmath+1})$$

$$= \sup \left\{ \begin{matrix} (\eth(\varsigma_{\jmath-1}, \Gamma\varsigma_{\jmath-1})\, \eth(\varsigma_{\jmath-1}, \Gamma\varsigma_{\jmath+1})^{\left(1+\eth(\varsigma_{\jmath-1}, \Gamma\varsigma_{\jmath+1})+\eth(\varsigma_{\jmath+1}, \Gamma\varsigma_{\jmath-1})\right)^{-1}}, \\ \eth(\varsigma_{\jmath-1}, \Gamma\varsigma_{\jmath-1}), \quad \eth(\varsigma_{\jmath-1}, \varsigma_{\jmath+1}), \quad \eth(\varsigma_{\jmath+1}, \Gamma\varsigma_{\jmath+1}) \end{matrix} \right\}.$$

As a result,

$$\Xi(\varsigma_{\jmath-1}, \varsigma_{\jmath+1}) = \sup \left\{ \begin{matrix} (\eth(\varsigma_{\jmath-1}, \varsigma_\jmath)\, \eth(\varsigma_{\jmath-1}, \varsigma_{\jmath+2}))^{\left(1+\eth(\varsigma_{\jmath-1}, \varsigma_{\jmath+2})+\eth(\varsigma_{\jmath+1}, \varsigma_\jmath)\right)^{-1}}, \\ \eth(\varsigma_{\jmath-1}, \varsigma_\jmath), \quad \eth(\varsigma_{\jmath-1}, \varsigma_{\jmath+1}), \quad \eth(\varsigma_{\jmath+1}, \varsigma_{\jmath+2}) \end{matrix} \right\}.$$

Therefore, we have

$$\Xi(\varsigma_{\jmath-1}, \varsigma_{\jmath+1}) = \sup \left\{ \eth(\varsigma_{\jmath-1}, \varsigma_{\jmath+1}), \eth(\varsigma_{\jmath-1}, \varsigma_\jmath), \eth(\varsigma_{\jmath+1}, \varsigma_{\jmath+2}) \right\}.$$

We analyze three distinct cases:

Case (f) If $\Xi(\varsigma_\jmath, \varsigma_{\jmath-1}) = \eth(\varsigma_{\jmath-1}, \varsigma_{\jmath+1})$, with the use of (11), we obtain

$$\begin{aligned}\eth(\varsigma_\jmath, \varsigma_{\jmath+2}) &\leq \psi(\eth(\varsigma_{\jmath-1}, \varsigma_{\jmath+1})) \\ &\leq \psi^{\jmath-1}(\eth(\varsigma_0, \varsigma_2)) = \psi^{\jmath-1}(e_0^*) \longrightarrow 1 \quad \text{as } \jmath \longrightarrow \infty.\end{aligned}$$

Case (g) If $\Xi(\varsigma_{\jmath-1}, \varsigma_{\jmath+1}) = \eth(\varsigma_{\jmath-1}, \varsigma_\jmath)$, with the use of (11), we obtain

$$\begin{aligned}\eth(\varsigma_\jmath, \varsigma_{\jmath+2}) &\leq \psi(\eth(\varsigma_{\jmath-1}, \varsigma_\jmath)) \\ &\leq \psi^{\jmath-1}(\eth(\varsigma_0, \varsigma_1)) = \psi^{\jmath-1}(e_0) \longrightarrow 0 \quad \text{as } \jmath \longrightarrow \infty.\end{aligned}$$

Case (h) If $\Xi(\varsigma_{\jmath-1}, \varsigma_{\jmath+1}) = \eth(\varsigma_{\jmath+1}, \varsigma_{\jmath+2})$, based on equation (11), we have

$$\begin{aligned}\eth(\varsigma_\jmath, \varsigma_{\jmath+2}) &\leq \psi(\eth(\varsigma_{\jmath+1}, \varsigma_{\jmath+2})) \\ &\leq \psi^{\jmath+1}(\eth(\varsigma_0, \varsigma_1)) \\ &= \psi^{\jmath+1}(e_0) \longrightarrow 1 \text{ as } \jmath \longrightarrow \infty.\end{aligned}$$

Aggregation of all the aforementioned cases results in

$$e_\jmath^* = \eth(\varsigma_\jmath, \varsigma_{\jmath+2}) \longrightarrow 1 \quad \text{as } \jmath \longrightarrow \infty \tag{12}$$

$$\begin{aligned}\eth(\varsigma_{\jmath+2}, \varsigma_\jmath) &= \eth(\Gamma\varsigma_{\jmath+1}, \Gamma\varsigma_{\jmath-1}) \\ &\leq (\eth(\Gamma\varsigma_{\jmath+1}, \Gamma\varsigma_{\jmath-1}))^{\alpha(\varsigma_{\jmath+1}, \varsigma_{\jmath-1})} \\ &\leq \psi(\Xi(\varsigma_{\jmath+1}, \varsigma_{\jmath-1})), \quad \forall\, \jmath \in [1, \infty)\end{aligned} \tag{13}$$

wherein

$$\Xi(\varsigma_{\jmath+1}, \varsigma_{\jmath-1})$$

$$= \sup \begin{Bmatrix} (\eth(\varsigma_{\jmath+1}, \Gamma\varsigma_{\jmath+1}) . \eth(\varsigma_{\jmath+1}, \Gamma\varsigma_{\jmath-1}))^{(1+\eth(\varsigma_{\jmath+1}, \Gamma\varsigma_{\jmath-1}) + \eth(\varsigma_{\jmath-1}, \Gamma\varsigma_{\jmath+1}))^{-1}}, \\ \eth(\varsigma_{\jmath+1}, \Gamma\varsigma_{\jmath+1}), \eth(\varsigma_{\jmath+1}, \varsigma_{\jmath-1}), \eth(\varsigma_{\jmath-1}, \Gamma\varsigma_{\jmath-1}) \end{Bmatrix}.$$

$$= \sup \begin{Bmatrix} (\eth(\varsigma_{\jmath+1}, \varsigma_{\jmath+2}) . \eth(\varsigma_{\jmath+1}, \varsigma_\jmath))^{(1+\eth(\varsigma_{\jmath+1}, \varsigma_\jmath) + \eth(\varsigma_{\jmath-1}, \varsigma_{\jmath+2}))^{-1}}, \\ \eth(\varsigma_{\jmath+1}, \varsigma_{\jmath-1}), \eth(\varsigma_{\jmath+1}, \varsigma_{\jmath-1}), \eth(\varsigma_{\jmath+1}, \varsigma_{\jmath+2}, \eth(\varsigma_{\jmath-1}, \varsigma_\jmath) \end{Bmatrix}$$

Now, we analyze three distinct cases:

Case (i) Given that $\Xi(\varsigma_{\jmath+1}, \varsigma_{\jmath-1}) = \eth(\varsigma_{\jmath+1}, \varsigma_{\jmath-1})$, then by (13), we get

$$\begin{aligned}\eth(\varsigma_{\jmath+2}, \varsigma_{\jmath}) &\leq \psi(\eth(\varsigma_{\jmath+1}, \varsigma_{\jmath-1})) \\ &\leq \psi^{\jmath+1}(\eth(\varsigma_2, \varsigma_0)) \\ &= \psi^{\jmath+1}(l_0^*) \longrightarrow 1 \quad \text{as } \jmath \longrightarrow \infty.\end{aligned}$$

Case (j) Given that $\Xi(\varsigma_{\jmath+1}, \varsigma_{\jmath-1}) = \eth(\varsigma_{\jmath+1}, \varsigma_{\jmath+2})$, by using equation (13), we have

$$\begin{aligned}\eth(\varsigma_{\jmath+2}, \varsigma_{\jmath}) &\leq \psi(\eth(\varsigma_{\jmath+1}, \varsigma_{\jmath+2})) \\ &\leq \psi^{\jmath+1}(\eth(\varsigma_0, \varsigma_1)) \\ &= \psi^{\jmath+1}(e_0) \longrightarrow 1 \quad \text{as } \jmath \longrightarrow \infty.\end{aligned}$$

Case (k) Provided that $\Xi(\varsigma_{\jmath+1}, \varsigma_{\jmath-1}) = \eth(\varsigma_{\jmath-1}, \varsigma_{\jmath})$, by employing equation (13), we have $\eth(\varsigma_{\jmath+2}, \varsigma_{\jmath}) \leq \psi(\eth(\varsigma_{\jmath-1}, \varsigma_{\jmath}))$ and hence

$$\eth(\varsigma_{\jmath+2}, \varsigma_{\jmath}) \leq \psi^{\jmath-1}(\eth(\varsigma_0, \varsigma_1)). \tag{14}$$

It results that $\eth(\varsigma_{\jmath+2}, \varsigma_{\jmath}) = \psi^{\jmath-1}(e_0) \longrightarrow 0 \quad \text{as } \jmath \longrightarrow \infty$. Based on equation (12) and equation (14), we infer that

$$\lim_{\jmath\to\infty} \eth(\varsigma_{\jmath}, \varsigma_{\jmath+2}) = 1 = \lim_{\jmath\to\infty} \eth(\varsigma_{\jmath+2}, \varsigma_{\jmath}).$$

Step 2: We will establish that $\{\varsigma_{\jmath}\}$ qualifies as a Br_{mult} $\boldsymbol{b}$- quasi-Cauchy sequence, signifying that

$$\lim_{\jmath\to\infty} \eth(\varsigma_{\jmath}, \varsigma_{\jmath+\iota}) = 1 = \lim_{\jmath\to\infty} \eth(\varsigma_{\jmath+\iota}, \varsigma_{\jmath}) \quad \forall\ \iota \in \mathbb{N}.$$

Case (l) Assume that, for certain natural numbers $\jmath$ and h, where $h > \jmath$, we find that $\varsigma_{\jmath} = \varsigma_h$, as indicated by equation (10).

$$\begin{aligned}\eth(\varsigma_{\jmath}, \varsigma_{\jmath+1}) &= \eth(\varsigma_{\jmath}, \Gamma\varsigma_{\jmath}) \\ &= \eth(\varsigma_h, \Gamma\varsigma_h) \\ &= \eth(\varsigma_h, \varsigma_{h+1}) \\ &\leq \psi^{h-\jmath}(\eth(\varsigma_{\jmath}, \varsigma_{\jmath+1}))\end{aligned}$$

$$\begin{aligned}&\leq (\psi(\eth(\varsigma_{\jmath},\varsigma_{\jmath+1})))\\&< (\eth(\varsigma_{\jmath},\varsigma_{\jmath+1}))^{s}.\end{aligned}$$

This statement conflicts with the previous results.

Case (m) Let us assume that there exist natural numbers $\jmath$ and h such that $\jmath > h$. We observe that $\varsigma_{\jmath} = \varsigma_{h}$ based on equation (10).

$$\begin{aligned}\eth(\varsigma_{h+1},\varsigma_{h}) &= \eth(\Gamma\varsigma_{h},\varsigma_{h}) = \eth(\Gamma\varsigma_{\jmath},\varsigma_{\jmath}) = \eth(\varsigma_{\jmath+1},\varsigma_{\jmath})\\&\leq \psi^{\jmath-h}(\eth(\varsigma_{h+1},\varsigma_{h}))^{s}\\&\leq \psi(\eth(\varsigma_{h+1},\varsigma_{h})) < \eth(\varsigma_{h+1},\varsigma_{h})\end{aligned}$$

This inconsistency implies that, considering both Case (a) and Case (b), it is clear that $\varsigma_{\jmath}$ is not equal to ς_{h} for h is not equal to $\jmath$. Now, let us move on to the cases of $\iota = 1$ and $\iota = 2$, which we have successfully proved. Next, we will discuss the case where $\iota \geq 3$, and we will do so arbitrarily by distinguishing between four distinct cases:

Case (n) Assume that $\iota = 2h$ ($h \geq 2$). Now, by employing the Br_{mult} inequality, we have $\eth(\varsigma_{\jmath},\varsigma_{\jmath+2h}) \leq (\eth(\varsigma_{\jmath},\varsigma_{\jmath+2}) \times \eth(\varsigma_{\jmath+2},\varsigma_{\jmath+3}) \times \eth(\varsigma_{\jmath+3},\varsigma_{\jmath+2h}))^{s}$. It further results that

$$\begin{aligned}\eth(\varsigma_{\jmath},\varsigma_{\jmath+2h}) &\leq (\eth(\varsigma_{\jmath},\varsigma_{\jmath+2}))^{s} \times (\eth(\varsigma_{\jmath+2},\varsigma_{\jmath+3}))^{s} \times (\eth(\varsigma_{\jmath+3},\varsigma_{\jmath+4})\\&\quad \times \eth(\varsigma_{\jmath+4},\varsigma_{\jmath+5}) \times \eth(\varsigma_{\jmath+5},\varsigma_{\jmath+2h}))^{s^{2}}\\&= (\eth(\varsigma_{\jmath},\varsigma_{\jmath+2}))^{s} \times (\eth(\varsigma_{\jmath+2},\varsigma_{\jmath+3}))^{s} \times (\eth(\varsigma_{\jmath+3},\varsigma_{\jmath+4}))^{s^{2}}\\&\quad \times (\eth(\varsigma_{\jmath+4},\varsigma_{\jmath+5}))^{s^{2}} \times (\eth(\varsigma_{\jmath+5},\varsigma_{\jmath+2h}))^{s^{2}}.\end{aligned}$$

Inductively, we have

$$\begin{aligned}\eth(\varsigma_{\jmath},\varsigma_{\jmath+2h}) \leq &(\eth(\varsigma_{\jmath},\varsigma_{\jmath+2}))^{s} \times (\eth(\varsigma_{\jmath+2},\varsigma_{\jmath+3}))^{s^{3}} \times (\eth(\varsigma_{\jmath+3},\varsigma_{\jmath+4}))^{s^{4}},\\&\times (\eth(\varsigma_{\jmath+4},\varsigma_{\jmath+5}))^{s^{5}} \times \cdots \times (\eth(\varsigma_{\jmath+2h-1},\varsigma_{\jmath+2h}))^{s^{2h}},\end{aligned} \tag{15}$$

which results in

$$\begin{aligned}\eth(\varsigma_{\jmath},\varsigma_{\jmath+2h}) &= (\eth(\varsigma_{\jmath},\varsigma_{\jmath+2}))^{s} \times \prod_{k=\jmath+2}^{\jmath+2h-1} (\eth(\varsigma_{k},\varsigma_{k+1}))^{s^{k-\jmath+1}}\\&\leq (\eth(\varsigma_{\jmath},\varsigma_{\jmath+2}))^{s} \times \prod_{k=\jmath+2}^{\jmath+2h-1} (\psi^{k}(e_{0}))^{s^{k}}\end{aligned}$$

$$\leq (\eth(\varsigma_\jmath, \varsigma_{\jmath+2}))^s \times \sum_{k=\jmath+2}^{\infty} (\psi^k(e_0))^{s^k}.$$

Now, by referring to equation (14),

$$\lim_{\jmath\to\infty} \eth(\varsigma_\jmath, \varsigma_{\jmath+2}) = 1 \text{ and } \prod_{k=\jmath+2}^{\infty} (\psi^k(e_0))^{s^k} \longrightarrow 1 \text{ as } \jmath \longrightarrow \infty.$$

Consequently, $\lim_{\jmath,h\to\infty} \eth(\varsigma_\jmath, \varsigma_{\jmath+2h}) = 1$.

Case (o) Consider ι as $2h+1$ ($h \geq 1$). By utilizing the Br_{mult} inequality, we have

$$\begin{aligned}\eth(\varsigma_\jmath, \varsigma_{\jmath+2h+1}) \leq & (\eth(\varsigma_\jmath, \varsigma_{\jmath+1}) \times \eth(\varsigma_{\jmath+1}, \varsigma_{\jmath+2}) \times \eth(\varsigma_{\jmath+2}, \varsigma_{\jmath+2h+1}))^s \\ & \leq (\eth(\varsigma_\jmath, \varsigma_{\jmath+1}))^s \times (\eth(\varsigma_{\jmath+1}, \varsigma_{\jmath+2}))^s \\ & \times (\eth(\varsigma_{\jmath+2}, \varsigma_{\jmath+3}) \times \eth(\varsigma_{\jmath+3}, \varsigma_{\jmath+4}) \times \eth(\varsigma_{\jmath+4}, \varsigma_{\jmath+2h+1}))^{s^2}.\end{aligned}$$

As a result, we have

$$\begin{aligned}\eth(\varsigma_\jmath, \varsigma_{\jmath+2h+1}) = & (\eth(\varsigma_\jmath, \varsigma_{\jmath+1}))^s \times (\eth(\varsigma_{\jmath+1}, \varsigma_{\jmath+2}))^s \times (\eth(\varsigma_{\jmath+2}, \varsigma_{\jmath+3}))^{s^2} \\ & \times (\eth(\varsigma_{\jmath+3}, \varsigma_{\jmath+4}))^{s^2} \times (\eth(\varsigma_{\jmath+4}, \varsigma_{\jmath+2h+1}))^{s^2}\end{aligned} \quad (16)$$

With the same approach, we have

$$\begin{aligned}\eth(\varsigma_\jmath, \varsigma_{\jmath+2h+1}) \leq & (\eth(\varsigma_\jmath, \varsigma_{\jmath+1}))^s \times (\eth(\varsigma_{\jmath+1}, \varsigma_{\jmath+2}))^{s^2} \times (\eth(\varsigma_{\jmath+2}, \varsigma_{\jmath+3}))^{s^3} \\ & \times (\eth(\varsigma_{\jmath+3}, \varsigma_{\jmath+4}))^{s^4} \times \cdots \times (\eth(\varsigma_{\jmath+2h}, \varsigma_{\jmath+2h+1}))^{s^{2h+1}}.\end{aligned} \quad (17)$$

Therefore, we have

$$\eth(\varsigma_\jmath, \varsigma_{\jmath+2h+1}) = \prod_{k=\jmath}^{\jmath+2h} (\eth(\varsigma_k, \varsigma_{k+1}))^{s^{k-\jmath+1}} = \prod_{k=\jmath}^{\jmath+2h} (\psi^k(e_0))^{s^{k-\jmath+1}},$$

which results in

$$\begin{aligned}\eth(\varsigma_\jmath, \varsigma_{\jmath+2h+1}) & \leq \prod_{k=\jmath}^{\jmath+2h} (\psi^k(e_0))^{s^k} \\ & \leq \prod_{k=\jmath}^{\infty} (\psi^k(e_0))^{s^k} \longrightarrow 1 \quad \text{as } \jmath \longrightarrow \infty.\end{aligned}$$

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As a result, we have $\lim_{\jmath,h\to\infty} \eth\left(\varsigma_\jmath, \varsigma_{\jmath+2h+1}\right) = 1$.

Case (p) Consider $\iota = 2h$, where $h > 2$. Applying the Br_{mult} inequality, we have

$$\begin{aligned}\eth\left(\varsigma_{\jmath+2h}, \varsigma_\jmath\right) \leq & \left(\eth\left(\varsigma_{\jmath+2h}, \varsigma_{\jmath+2h-2}\right) \times \eth\left(\varsigma_{\jmath+2h-2}, \varsigma_{\jmath+2h-3}\right) \times \eth\left(\varsigma_{\jmath+2h-3}, \varsigma_\jmath\right)\right)^s \\ \leq & \left(\eth\left(\varsigma_{\jmath+2h}, \varsigma_{\jmath+2h-2}\right)\right)^s \times \left(\eth\left(\varsigma_{\jmath+2h-2}, \varsigma_{\jmath+2h-3}\right)\right)^s \\ & \times \left(\eth\left(\varsigma_{\jmath+2h-3}, \varsigma_{\jmath+2h-4}\right) \times \eth\left(\varsigma_{\jmath+2h-4}, \varsigma_{\jmath+2h-5}\right) \times \eth\left(\varsigma_{\jmath+2h-5}, \varsigma_\jmath\right)\right)^{s^2} \\ = & \left(\eth\left(\varsigma_{\jmath+2h}, \varsigma_{\jmath+2h-2}\right)\right)^s \times \left(\eth\left(\varsigma_{\jmath+2h-2}, \varsigma_{\jmath+2h-3}\right)\right)^s \\ & \times \left(\eth\left(\varsigma_{\jmath+2h-3}, \varsigma_{\jmath+2h-4}\right)\right)^{s^2} \times \left(\eth\left(\varsigma_{\jmath+2h-4}, \varsigma_{\jmath+2h-5}\right)\right)^{s^2} \\ & \times \left(\eth\left(\varsigma_{\jmath+2h-5}, \varsigma_\jmath\right)\right)^{s^2}.\end{aligned} \tag{18}$$

Inductively, we have

$$\begin{aligned}\eth\left(\varsigma_{\jmath+2h}, \varsigma_\jmath\right) \leq & \left(\eth\left(\varsigma_{\jmath+2h}, \varsigma_{\jmath+2h-2}\right)\right)^s \times \left(\eth\left(\varsigma_{\jmath+2h-2}, \varsigma_{\jmath+2h-3}\right)\right)^{s^{\jmath+2h-2}} \\ & \times \left(\eth\left(\varsigma_{\jmath+2h-3}, \varsigma_{\jmath+2h-4}\right)\right)^{s^{\jmath+2h-3}} \times \left(\eth\left(\varsigma_{\jmath+2h-4}, \varsigma_{\jmath+2h-5}\right)\right)^{s^{\jmath+2h-4}} \\ & \times \cdots \times \left(\eth\left(\varsigma_{\jmath-1}, \varsigma_\jmath\right)\right)^{s^{\jmath-1}}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\eth\left(\varsigma_{\jmath+2h}, \varsigma_\jmath\right) &= \left(\eth\left(\varsigma_{\jmath+2h}, \varsigma_{\jmath+2h-2}\right)\right)^s \times \prod_{k=\jmath-1}^{\jmath+2h-1} \left(\eth\left(\varsigma_k, \varsigma_{k+1}\right)\right)^{s^k} \\ &\leq \left(\eth\left(\varsigma_{\jmath+2h}, \varsigma_{\jmath+2h-2}\right)\right)^s \times \prod_{k=\jmath-1}^{\jmath+2h-1} \left(\psi^k\left(l_0^*\right)\right)^{s^k}.\end{aligned}$$

It results that

$$\eth\left(\varsigma_{\jmath+2h}, \varsigma_\jmath\right) \leq \left(\eth\left(\varsigma_{\jmath+2h}, \varsigma_{\jmath+2h-2}\right)\right)^s \times \prod_{k=\jmath-1}^{\infty} \left(\psi^k\left(l_0^*\right)\right)^{s^k}.$$

As $\lim_{m,\jmath\to\infty} \eth\left(\varsigma_{\jmath+2h}, \varsigma_{\jmath+2h-2}\right) = 1$ and $\prod_{k=\jmath-1}^{\infty} \left(\psi^k\left(l_0^*\right)\right)^{s^k} \longrightarrow 1$ as $\jmath \longrightarrow \infty$, we have

$$\eth\left(\varsigma_{\jmath+2h}, \varsigma_\jmath\right) \longrightarrow 1,\ h,\ j \to \infty\ .$$

Case (q) Consider $\iota = 2h+1$, $h \geq 1$. Now, by using Br_{mult} inequality, we get

$$\eth\left(\varsigma_{\jmath+2h+1}, \varsigma_\jmath\right) \leq \left(\eth\left(\varsigma_{\jmath+2h+1}, \varsigma_{\jmath+2h}\right) \times \eth\left(\varsigma_{\jmath+2h}, \varsigma_{\jmath+2h-1}\right) \times \eth\left(\varsigma_{\jmath+2h-1}, \varsigma_\jmath\right)\right)^s$$

$$\leq(\eth(\varsigma_{\jmath+2h+1},\varsigma_{\jmath+2h}))^{s}\times(\eth(\varsigma_{\jmath+2h},\varsigma_{\jmath+2h-1}))^{s} \tag{19}$$
$$\times(\eth(\varsigma_{\jmath+2h-1},\varsigma_{\jmath+2h-2})\times\eth(\varsigma_{\jmath+2h-2},\varsigma_{\jmath+2h-3})$$
$$\times\eth(\varsigma_{\jmath+2h-3},\varsigma_{\jmath}))^{s^2}.$$

Therefore, we have

$$\eth(\varsigma_{\jmath+2h+1},\varsigma_{\jmath})=(\eth(\varsigma_{\jmath+2h+1},\varsigma_{\jmath+2h}))^{s}\times(\eth(\varsigma_{\jmath+2h},\varsigma_{\jmath+2h-1}))^{s} \tag{20}$$
$$\times(\eth(\varsigma_{\jmath+2h-1},\varsigma_{\jmath+2h-2}))^{s^2}\times(\eth(\varsigma_{\jmath+2h-2},\varsigma_{\jmath+2h-3}))^{s^2}$$
$$\times(\eth(\varsigma_{\jmath+2h-3},\varsigma_{\jmath}))^{s^2}.$$

Inductively, we have

$$\eth(\varsigma_{\jmath+2h+1},\varsigma_{\jmath})\leq(\eth(\varsigma_{\jmath+2h+1},\varsigma_{\jmath+2h}))^{s^{\jmath+2h+1}}\times(\eth(\varsigma_{\jmath+2h},\varsigma_{\jmath+2h-1}))^{s^{\jmath+2h}}$$
$$\times(\eth(\varsigma_{\jmath+2h-1},\varsigma_{\jmath+2h-2}))^{s^{\jmath+2h-1}}\times(\eth(\varsigma_{\jmath+2h-2},\varsigma_{\jmath+2h-3}))^{s^{\jmath+2h-2}}$$
$$\times\cdots\times(\eth(\varsigma_{\jmath+1},\varsigma_{\jmath}))^{s^{\jmath+1}}$$

and hence

$$\eth(\varsigma_{\jmath+2h+1},\varsigma_{\jmath})=\prod_{k=\jmath+1}^{\jmath+2h}(\eth(\varsigma_{k+1},\varsigma_{k}))^{s^{k}}=\prod_{k=\jmath+1}^{\jmath+2h}(\psi^{k}(l_0))^{s^{k-\jmath+1}}.$$

Consecutively, we have

$$\eth(\varsigma_{\jmath+2h+1},\varsigma_{\jmath})\leq\prod_{k=\jmath+1}^{\jmath+2h}(\psi^{k}(l_0))^{s^{k}}\longrightarrow 1\ \text{ as } n\longrightarrow\infty.$$

Therefore, we can deduce that as $\jmath$ and h tend towards infinity, whereas the limiting value of $\eth(\varsigma_{\jmath+2h+1},\varsigma_{\jmath})$ is 1. Additionally, by considering the cases (a) to (d), we can conclude that as $\jmath$ approaches infinity, the limits of $\eth(\varsigma_{\jmath},\varsigma_{\jmath+\iota})$ and $\eth(\varsigma_{\jmath+\iota},\varsigma_{\jmath})$ are both equal to 1 for every value of ι greater than or equal to 3. Consequently, it can be inferred that the sequence $\{\varsigma_{\jmath}\}$ is a Br_{mult} $\boldsymbol{b}$- quasi-Cauchy within the metric spaces $(\Theta,\eth)$. Since Θ is a complete Br_{mult} $\boldsymbol{b}-$ quasi-metric space, $\exists\ \eta_1$ in Θ such that

$$\lim_{\jmath\to\infty}\varsigma_{\jmath}=\eta_1,\ \text{i.e.,}\ \lim_{\jmath\to\infty}\eth(\varsigma_{\jmath},\eta_1)=1=\lim_{\jmath\to\infty}\eth(\eta_1,\varsigma_{\jmath}). \tag{21}$$

Following that, we establish that η_1 acts a F_p under the mapping of Γ. As Γ exhibits continuity, we can infer from equation (21) that η_1 =

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$\lim_{\jmath\to\infty}\varsigma_{\jmath+1}=\lim_{\jmath\to\infty}\Gamma\left(\varsigma_{\jmath}\right)=\Xi\left(\lim_{\jmath\to\infty}\varsigma_{\jmath}\right)=\eta_1$, which gives $\eta_1=\Gamma\eta_1$. Therefore, we can conclude that η_1 is indeed a F_p when the mapping Γ is applied. Now, we introduce the following F_p theorem by omitting the condition for continuity in the mapping Γ from Theorem 1.

Theorem 2. *Consider a complete* Br_{mult} $\boldsymbol{b}-$*quasi-metric space denoted as* $(\Theta,\eth)$ *and let* $\Gamma:\Theta\longrightarrow\Theta$ *be a generalized mapping with multiplicative* $(\alpha,\psi)-$*contraction properties. Assume that:*

1. Γ *is a mapping that satisfies the* α*-admissibility condition;*
2. *there is an existence of* $\varsigma_0\in\Theta$ *in a way that* $\alpha\left(\varsigma_0,\Gamma\varsigma_0\right)\in[1,\infty),\alpha\left(\Gamma\varsigma_0,\varsigma_0\right)\in[1,\infty),\alpha\left(\varsigma_0,\Gamma^2\varsigma_0\right)\in[1,\infty)$*, as well as* $\alpha\left(\Gamma^2\varsigma_0,\varsigma_0\right)\in[1,\infty)$;
3. *As* $\varsigma_{\jmath}$ *represents a sequence within* Θ *in such a way that* $\alpha\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\in[1,\infty)\ \forall\ \jmath\geq 0$*, and as* $\jmath$ *approaches infinity,* $\varsigma_{\jmath}\longrightarrow\varsigma\in\Theta$*, then* $\alpha\left(\varsigma_{\jmath},\varsigma\right)\in[1,\infty)\ \forall\ \jmath\geq 0$.

Under these conditions, a F_p *exists for the operator* Γ.

Proof. After considering the proofs of Theorem 1, we have established that the sequence $\{\varsigma_{\jmath}\}$, which is defined by the recurrence relation $\varsigma_{\jmath+1}=\Gamma\varsigma_{\jmath}$ for every $\jmath\geq 0$, exhibits Br_{mult} quasi-$\boldsymbol{b}$-convergence towards a point η_1 within the space Θ. Our aim now is to demonstrate that Γ possesses a F_p. Utilizing the Br_{mult} inequality within the context of Br_{mult} $\boldsymbol{b}-$ quasi-metric spaces, the properties of ψ, and condition (iii), we can state the following:

$$\begin{aligned}\eth(\eta_1,\Gamma\eta_1)&\leq\left(\eth\left(\eta_1,\varsigma_{\jmath}\right)\right)^s\times\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)^s\times\left(\eth\left(\varsigma_{\jmath+1},\Gamma\eta_1\right)\right)^s\\&=\left(\eth\left(\eta_1,\varsigma_{\jmath}\right)\right)^s\times\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)^s\times\left(\eth\left(\Gamma\varsigma_{\jmath},\Gamma\eta_1\right)\right)^s\\&\leq\left(\eth\left(\eta_1,\varsigma_{\jmath}\right)\right)^s\times\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)^s\times\left(\eth\left(\Gamma\varsigma_{\jmath},\Gamma\eta_1\right)\right)^{s\alpha(\varsigma_{\jmath},\eta_1)}\\&\leq\left(\eth\left(\eta_1,\varsigma_{\jmath}\right)\right)^s\times\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)^s\times\left(\psi(\Xi\left(\Gamma\varsigma_{\jmath},\Gamma\eta_1\right))\right)^s\end{aligned}\tag{22}$$

where

$$\Xi\left(\varsigma_{\jmath},\eta_1\right)=\sup\left\{\begin{matrix}\left(\eth\left(\varsigma_{\jmath},\Gamma\varsigma_{\jmath}\right).\eth\left(\varsigma_{\jmath},\Gamma\eta_1\right)\right)^{\left(1+\eth\left(\varsigma_{\jmath},\Gamma\eta_1\right)+\eth\left(\eta_1,\Gamma\varsigma_{\jmath}\right)\right)^{-1}},\\\eth\left(\varsigma_{\jmath},\Gamma\varsigma_{\jmath}\right),\eth\left(\varsigma_{\jmath},\eta_1\right),\eth\left(\eta_1,\Gamma\eta_1\right)\end{matrix}\right\}$$

and hence, we have

$$\Xi\left(\varsigma_{\jmath},\eta_{1}\right)=\sup\left\{\begin{array}{l}\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right).\eth\left(\varsigma_{\jmath},\Gamma\eta_{1}\right)\right)^{\left(1+\eth\left(\varsigma_{\jmath},\Gamma\eta_{1}\right)+\eth\left(\eta_{1},\varsigma_{\jmath+1}\right)\right)^{-1}},\\ \eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right),\eth\left(\varsigma_{\jmath},\eta_{1}\right),\eth\left(\eta_{1},\Gamma\eta_{1}\right)\end{array}\right\}$$

which results in

$$\Xi\left(\varsigma_{\jmath},\eta_{1}\right)=\sup\left\{\eth\left(\varsigma_{\jmath},\eta_{1}\right),\eth(\varsigma_{\jmath},\varsigma_{\jmath+1}),\eth\left(\eta_{1},\Gamma\eta_{1}\right)\right\}.$$

We analyze some distinct cases, which can be described as follows:

Case (r) Given that $\Xi\left(\varsigma_{\jmath},\eta_{1}\right)$ is the same as $\eth\left(\varsigma_{\jmath},\eta_{1}\right)$, then by (22), we get

$$\begin{aligned}\eth(\eta_{1},\Gamma\eta_{1})&\leq\left(\eth\left(\eta_{1},\varsigma_{\jmath}\right)\right)^{s}\times\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)^{s}\times\left(\psi\left(\eth\left(\varsigma_{\jmath},\eta_{1}\right)\right)\right)^{s}\\&<\left(\eth\left(\eta_{1},\varsigma_{\jmath}\right)\right)^{s}\times\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)^{s}\times\eth\left(\varsigma_{\jmath},\eta_{1}\right).\end{aligned}$$

By allowing $\jmath$ to approach infinity in the preceding inequality, considering equations (10) and (21), we obtain the following:

$$\eth(\eta_{1},\Gamma\eta_{1})\leq 1.$$

Case (s) Given that $\Xi\left(\varsigma_{\jmath},\eta_{1}\right)$ and $\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)$ are the same, then by (16), we get

$$\begin{aligned}\eth(\eta_{1},\Gamma\eta_{1})&\leq\left(\eth\left(\eta_{1},\varsigma_{\jmath}\right)\right)^{s}\times\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)^{s}\times\left(\psi\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)\right)^{s}\\&<\left(\eth\left(\eta_{1},\varsigma_{\jmath}\right)\right)^{s}\times\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)^{s}\times\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right).\end{aligned}$$

By allowing $\jmath\longrightarrow\infty$ in the preceding inequality, considering equations (10) and (15), we obtain the following:

$$\eth(\eta_{1},\Gamma\eta_{1})\leq 1$$

Case (t) Let $\Xi\left(\varsigma_{\jmath},\eta_{1}\right)$ and $\eth(\eta_{1},\Gamma\eta_{1})$ be same, then by (16), we get

$$\begin{aligned}\eth(\eta_{1},\Gamma\eta_{1})&\leq\left(\eth\left(\eta_{1},\varsigma_{\jmath}\right)\right)^{s}\times\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)^{s}\times\left(\psi(\eth(\eta_{1},\Gamma\eta_{1}))\right)^{s}\\&<\left(\eth\left(\eta_{1},\varsigma_{\jmath}\right)\right)^{s}\times\left(\eth\left(\varsigma_{\jmath},\varsigma_{\jmath+1}\right)\right)^{s}\times\left(\eth(\eta_{1},\Gamma\eta_{1})\right)^{s}.\end{aligned}$$

As we let $\jmath\longrightarrow\infty$ in the inequality mentioned above, and taking into account equations (10) and (15), we arrive at the trivial statement that $0\leq 0$,

which is universally true. Obviously, $\eth(\eta_1, \Gamma\eta_1)$ is greater than or equal to 0, and by examining the aforementioned cases, we can derive the following:

$$\eth(\eta_1, \Gamma\eta_1) = 1. \tag{23}$$

Also,

$$\begin{aligned}\eth(\Gamma\eta_1, \eta_1) &\leq (\eth(\Gamma\eta_1, \varsigma_\jmath))^s \times (\eth(\varsigma_\jmath, \varsigma_{\jmath+1}))^s \times (\psi(\eth(\varsigma_{\jmath+1}, \eta_1)))^s \\ &= (\eth(\Gamma\eta_1, \Gamma\varsigma_{\jmath-1}))^s \times (\eth(\varsigma_\jmath, \varsigma_{\jmath+1}))^s \times (\psi(\eth(\varsigma_{\jmath+1}, \eta_1)))^s\end{aligned}$$

which results in

$$\begin{aligned}\eth(\Gamma\eta_1, \eta_1) &\leq (\eth(\Gamma\eta_1, \Gamma\varsigma_{\jmath-1}))^{s\alpha(\eta_1, \varsigma_{\jmath-1})} \times (\eth(\varsigma_\jmath, \varsigma_{\jmath+1}))^s \times (\eth(\varsigma_{\jmath+1}, \eta_1))^s \\ &\leq (\psi(\Xi(\eta_1, \varsigma_{\jmath-1})))^s \times (\eth(\varsigma_\jmath, \varsigma_{\jmath+1}))^s \times (\eth(\varsigma_{\jmath+1}, \eta_1))^s. \end{aligned} \tag{24}$$

where

$$\Xi(\eta_1, \varsigma_{\jmath-1}) = \sup\Big\{(\eth(\eta_1, \Gamma\eta_1)\eth(\eta_1, \Gamma\varsigma_{\jmath-1}))^{(1+\eth(\eta_1, \Gamma\varsigma_{\jmath-1})+\eth(\varsigma_{\jmath-1}, \Gamma\eta_1))^{-1}},$$
$$\eth(\eta_1, \Gamma\eta_1), \eth(\eta_1, \varsigma_{\jmath-1}), \eth(\varsigma_{\jmath-1}, \Gamma\varsigma_{\jmath-1})\Big\}$$

and hence,

$$\Xi(\eta_1, \varsigma_{\jmath-1}) = \sup\Big\{(\eth(\eta_1, \Gamma\eta_1)\eth(\eta_1, \varsigma_{\jmath-1}))^{(1+\eth(\eta_1, \varsigma_\jmath)+\eth(\varsigma_{\jmath-1}, \Gamma\eta_1))^{-1}},$$
$$\eth(\eta_1, \Gamma\eta_1), \eth(\eta_1, \varsigma_{\jmath-1}), \eth(\varsigma_{\jmath-1}, \Gamma\varsigma_{\jmath-1})\Big\}.$$

Consequently,

$$\Xi(\eta_1, \varsigma_{\jmath-1}) = \sup\Big\{\eth(\eta_1, \varsigma_{\jmath-1}), \eth(\varsigma_\jmath, \varsigma_{\jmath+1}), \eth(\eta_1, \Gamma\eta_1)\Big\}.$$

As a result, we analyze the following distinct cases:

Case (u) Given that $\Xi(\eta_1, \varsigma_{\jmath-1})$ and $\eth(\eta_1, \varsigma_{\jmath-1})$ are the same, then by (24), we get $\eth(\Gamma\eta_1, \eta_1) \leq (\psi(\eth(\eta_1, \varsigma_{\jmath-1})))^s \times (\eth(\varsigma_\jmath, \varsigma_{\jmath+1}))^s \times (\eth(\varsigma_{\jmath+1}, \eta_1))^s$, which further results in

$$\eth(\Gamma\eta_1, \eta_1) < \eth(\eta_1, \varsigma_{\jmath-1}) \times (\eth(\varsigma_\jmath, \varsigma_{\jmath+1}))^s \times (\eth(\varsigma_{\jmath+1}, \eta_1))^s.$$

By letting $\jmath$ approach infinity in the previous equation, and making use of equations (10) and (15), we can deduce that $\eth(\Gamma\eta_1, \eta_1) \leq 1$.

Case (v) Let it be given that $\Xi\left(\eta_1, \varsigma_{\jmath-1}\right) = \eth\left(\varsigma_\jmath, \varsigma_{\jmath+1}\right)$, then by (18), we get

$$\begin{aligned}\eth(\Gamma\eta_1, \eta_1) &\leq \left(\psi\left(\eth\left(\varsigma_\jmath, \varsigma_{\jmath+1}\right)\right)\right)^s \times \left(\eth\left(\varsigma_\jmath, \varsigma_{\jmath+1}\right)\right)^s \times \left(\eth\left(\varsigma_{\jmath+1}, \eta_1\right)\right)^s \\ &< \left(\eth\left(\varsigma_\jmath, \varsigma_{\jmath+1}\right)\right) \times \left(\eth\left(\varsigma_\jmath, \varsigma_{\jmath+1}\right)\right)^s \times \left(\eth\left(\varsigma_{\jmath+1}, \eta_1\right)\right)^s.\end{aligned}$$

By allowing the value of $\jmath \longrightarrow \infty$ in the inequality mentioned earlier and considering equations (10) and (21), we have $\eth(\Gamma\eta_1, \eta_1) \leq 1$.

Case (w) If $\Xi\left(\eta_1, \varsigma_{\jmath-1}\right) = \eth(\eta_1, \Gamma\eta_1)$, then by (18), we get

$$\begin{aligned}\eth(\Gamma\eta_1, \eta_1) &\leq \left(\psi(\eth(\eta_1, \Gamma\eta_1))\right)^s \times \left(\eth\left(\varsigma_\jmath, \varsigma_{\jmath+1}\right)\right)^s \times \left(\eth\left(\varsigma_{\jmath+1}, \eta_1\right)\right)^s \\ &< \eth(\eta_1, \Gamma\eta_1) \times \left(\eth\left(\varsigma_\jmath, \varsigma_{\jmath+1}\right)\right)^s \times \left(\eth\left(\varsigma_{\jmath+1}, \eta_1\right)\right)^s\end{aligned}$$

By allowing the value of $\jmath \longrightarrow \infty$ in the inequality mentioned earlier and considering equations (10), (21) and (23) we have $\eth(\Gamma\eta_1, \eta_1) \leq 1$. It is evident that $\eth(\eta_1, \Gamma\eta_1) \in [1, \infty)$, and we can deduce from the analysis of cases (i) to (iii) that

$$\eth(\Gamma\eta_1, \eta_1) = 1. \tag{25}$$

By considering equations (23) and (25), we can conclude that $\eth(\eta_1, \Gamma\eta_1) = 1 = \eth(\Gamma\eta_1, \eta_1)$, which implies that $\Gamma\eta_1 = \eta_1$. This means that η_1 remains unchanged under the action of Γ i.e. η_1 is a F_p.

To ensure that there is only one F_p for Γ, we will examine the following criterion.

Property U for every ς and ω belonging to the set of F_{ps} of Γ, denoted as $\mathrm{Fix}(\Gamma)$, holds that $\alpha(\varsigma, \omega) \in [1, \infty)$ and $\alpha(\omega, \varsigma) \in [1, \infty)$.

Theorem 3. *If we add condition (U) to the assumptions made in Theorem 1 (or Theorem 2), we can guarantee the uniqueness of the F_p of Γ.*

Proof. We establish that $\mathrm{Fix}(\Gamma)$ is not an empty set by employing proof of Theorem 1 and Theorem 2 respectively. Let us assume that $\eta_1 \neq \eta_2$ are

two F_{ps} of Γ. Then Condition (U) implies that

$$\alpha(\Gamma\eta_1, \Gamma\eta_2) = \alpha(\eta_1, \eta_2) \in [1, \infty) \text{ and } \alpha(\Gamma\eta_2, \Gamma\eta_1) = \alpha(\eta_2, \eta_1) \in [1, \infty).$$

Consequently, due to the multiplicative $\alpha-$admissibility property of mapping Γ and the aforementioned inequality, we can deduce that

$$\begin{aligned}\eth(\eta_1, \eta_2) &\leq (\eth(\eta_1, \eta_2))^{\alpha(\eta_1, \eta_2)} \\ &= (\eth(\Gamma\eta_1, \Gamma\eta_2))^{\alpha(\Gamma\eta_1, \Gamma\eta_2)}\end{aligned}$$

and hence, $\eth(\eta_1, \eta_2) \leq \psi(\Xi(\eta_1, \eta_2))$, where

$$\begin{aligned}\Xi(\eta_1, \eta_2) = \sup \Big\{ &(\eth(\eta_1, \Gamma\eta_1)\eth(\eta_1, \Gamma\eta_2))(1 + \eth(\eta_1, \Gamma\eta_2) + \eth(\eta_2, \Gamma\eta_1))^{-1}, \\ &\eth(\eta_1, \Gamma\eta_1), \eth(\eta_1, \eta_2), \eth(\eta_2, \Gamma\eta_2) \\ = \sup \Big\{ &(\eth(\eta_1, \eta_1)\eth(\eta_1, \eta_2))(1 + \eth(\eta_1, \eta_2) + \eth(\eta_2, \eta_1))^{-1}, \\ &\eth(\eta_1, \eta_1), \eth(\eta_1, \eta_2), \eth(\eta_2, \eta_2) \\ = &\eth(\eta_1, \eta_2).\end{aligned}$$

Since $(\psi(\nu))^s$ is always less than Γ for every $\nu > 0$, and considering the inequality provided, we can conclude that

$$\begin{aligned}\eth(\eta_1, \eta_2) &\leq \psi(\eth(\eta_1, \eta_2)) \\ &\leq (\psi(\eth(\eta_1, \eta_2)))^s \\ &< \eth(\eta_1, \eta_2). \end{aligned} \tag{26}$$

This contradicts our previous assertion. Similarly,

$$\begin{aligned}\eth(\eta_2, \eta_1) &\leq (\eth(\eta_2, \eta_1))^{\alpha(\eta_2, \eta_1)} \\ &= (\eth(\Gamma\eta_2, \Gamma\eta_1))^{\alpha(\Gamma\eta_2, \Gamma\eta_1)} \\ &\leq \psi(\Xi(\eta_2, \eta_1)),\end{aligned}$$

wherein

$$\begin{aligned}\Xi(\eta_2, \eta_1) = \sup \Big\{ &(\eth(\eta_2, \Gamma\eta_2)\eth(\eta_2, \Gamma\eta_1))(1 + \eth(\eta_2, \Gamma\eta_1) + \eth(\eta_1, \Gamma\eta_2))^{-1}, \\ &\eth(\eta_2, \Gamma\eta_2), \eth(\eta_2, \eta_1), \eth(\eta_1, \Gamma\eta_1) \\ = \sup \Big\{ &\eth(\eta_2, \eta_1), (\eth(\eta_2, \eta_2)\eth(\eta_2, \eta_1))(1 + \eth(\eta_2, \eta_1) + \eth(\eta_1, \eta_2))^{-1}, \\ &\eth(\eta_2, \eta_2), \eth(\eta_1, \eta_1).\end{aligned}$$

Therefore, we have $\Xi(\eta_2, \eta_1) = \eth(\eta_2, \eta_1)$. Since $(\psi(\nu))^s < \nu$ for every $\nu > 0$, and considering the inequality above, we conclude that

$$\begin{aligned}\eth(\eta_2, \eta_1) &\leq \psi(\eth(\eta_2, \eta_1)) \\ &\leq (\psi(\eth(\eta_2, \eta_1)))^s.\end{aligned}$$

It results that

$$\eth(\eta_2, \eta_1) < \eth(\eta_2, \eta_1), \tag{27}$$

which leads to a contradiction. Using equations (26) and (27), we find that $\eth(\eta_1, \eta_2) = 1 = \eth(\eta_2, \eta_1)$. Hence, we can conclude that $\eta_1 = \eta_2$. Therefore, there exists only one F_p for the mapping Γ.

Following that, we present various corollaries based on our significant results.

Corollary 1. *Consider a complete Br_{mult} $\boldsymbol{b}-$ quasi-metric space denoted as $(\Theta, \eth)$ and a mapping $\Gamma : \Theta \longrightarrow \Theta$ which is an multiplicative $(\alpha, \psi)-$contraction, meaning that is*

$$(\eth(\Gamma\varsigma, \Gamma\omega))^{\alpha(\varsigma,\, \omega)} \leq \psi(\eth(\varsigma,\, \omega)) \quad \forall\, \varsigma,\, \omega \in \Theta$$

In that case, Γ possesses a F_p, which is unique.

Proof. The conclusion can be derived by substituting $\Xi(\varsigma,\, \omega) = \eth(\varsigma,\, \omega)$ for every $\varsigma,\, \omega \in \Theta$ in the proof of Theorem 1 (or Theorem 2).

This conclusion is obtained by employing $\Xi(\varsigma,\, \omega) = \eth(\varsigma,\, \omega)\ \forall\, \varsigma,\, \omega \in \Theta$ in the proof of either Theorem 1 (or Theorem 2).

Remark 2. By substituting $s = 1$ in Corollary 1, we obtain the outcomes that were discussed in the research conducted by Karapinar and Lakzian (2014). Consequently, this study extends and broadens the results studied by Karapinar and Lakzian (2014).

Corollary 2. *Consider a complete Br_{mult} $\boldsymbol{b}-$ quasi-metric space denoted as $(\Theta, \eth)$, and let $\Gamma : \Theta \longrightarrow \Theta$ be a continuous mapping, provided that there are functions ψ belonging to the set Ψ such that*

$$\eth(\Gamma\varsigma, \Gamma\omega) \leq \psi(\eth(\varsigma,\, \omega)) \quad \forall\, \varsigma,\, \omega \in \Theta.$$

Therefore, there exists a unique F_p for the mapping Γ.

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Proof. The conclusion can be derived by setting $\alpha(\varsigma,\ \omega) = 1$ and $\Xi(\varsigma,\ \omega) = \eth(\varsigma,\ \omega)$ for every pair of elements ς and ω within the Theorem 1.

Remark 3. When we substitute $s = 1$ in Corollary 2, we arrive at the outcomes discussed in the research conducted by Khuangsatung et al. and Abagaro et al. (2022). Therefore, this study builds upon and broadens the discoveries made by Khuangsatung et al. and Abagaro et al. (2022).

Corollary 3. *Consider a complete* Br_{mult} $\boldsymbol{b}-$ *quasi-metric space denoted as* $(\Theta, \eth)$*, and let* $\Gamma : \Theta \longrightarrow \Theta$ *as a continuous mapping. Suppose there is a constant* k *within the range of* $[0, 1)$*, such that*

$$\eth(\Gamma\varsigma, \Gamma\omega) \leq (\eth(\varsigma,\ \omega))^k \quad \forall\ \varsigma,\ \omega \in \Theta.$$

Proof. The conclusion can be obtained by using $\psi(\nu) = \nu^k$, where k is within the range of $[0, 1)$ and Γ is greater than or equal to 1, within the context of Corollary 2.

Application

In the following section, we prove a theorem that establishes the existence of a solution to the integral equation:

$$\varsigma(\nu) = \ln \int_0^1 (e^{\lambda(\nu, r, \varsigma(r))})^{dr} \tag{28}$$

in this context, the functions $\lambda : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}^+$ exhibit continuity.

In this section, consider the set Θ, which comprises real continuous functions on $[0, 1]$, denoted as $\Theta = C([0, 1], \mathbb{R})$. We define the Br_{mult} $\boldsymbol{b}$-quasi-met $\eth : \Theta \times \Theta \longrightarrow [0, \infty)$ as follows:

$$\eth(\varsigma,\ \omega) = \begin{cases} \left\| (\frac{\varsigma}{\omega})^2 \right\|_\infty \times \|\varsigma\|_\infty, & \text{if } \varsigma \neq \eta_2, \\ 1, & \text{if } \varsigma = \omega, \end{cases} \tag{29}$$

where $\|\eta_1\|_\infty = \sup_{r\in[0,1]} |u\eta_1(s)|$ for every $\eta_1 \in \Theta$. We know that $(\Theta, \eth)$ forms a complete Br_{mult} $\boldsymbol{b}$-quasi-metric space (with $s = 1.5$). We will now establish the subsequent conclusion.

Theorem 4. *If the aforementioned hypotheses are true:*

1. *there exist constants* k *within the range of (0,1) and a function* $g : \Theta \times \Theta \longrightarrow [0, \infty)$ *such that,* $\forall\ \varsigma,\ \omega \in \Theta$ *satisfying* $\varsigma(\nu) \leq \omega(\nu)\ \forall\ \nu$ *in the interval* $[0, 1]$*, and for any value of* r *within the interval* $[0, 1]$*, the following condition holds:*

$$\left|\frac{\lambda(\nu, r, \varsigma(r))}{\lambda(\nu, r,\ \omega(r))}\right| \geq 0 \ \text{ and } \ \left|\frac{\lambda(\nu, r, \varsigma(r))}{\lambda(\nu, r,\ \omega(r))}\right| \leq g(\nu, r)\left|\frac{\varsigma}{\omega}\right|,$$

and

$$\sup_{t\in[0,1]} \ln \int_0^1 (e^{g(\nu,r)})^{dr} = k$$

2. λ *exhibits a monotonic increase in relation to its third variable;*
3. $\exists\ \varsigma_0$ *within the set* Θ *for which, for every* $\nu \in [0, 1]$*, the following condition holds:*

$$\varsigma_0(\nu) \leq \int_0^1 \lambda(\nu, r, \varsigma_0(r))dr$$

and

$$\varsigma_0(\nu) \leq \ln \int_0^1 (e^{\lambda(\nu,r,\ln \int_0^1 (e^{\lambda(\nu,r,\varsigma_0(r))})^{dr})})^{dr}$$

The integral Equation (29) then has a solution in Θ*.*

Proof. $\forall\ \varsigma$ in the set Θ and for every Γ within the interval $[0, 1]$, we establish the mapping $\Gamma : \Theta \to \Theta$ as follows: $\Gamma\varsigma(\nu) = \ln \int_0^1 (e^{\lambda(\nu,r,\varsigma(r))})^{dr}$. Additionally, we define $\alpha : \Theta \times \Theta \longrightarrow [0, \infty)$ as follows:

$$\alpha(\varsigma,\ \omega) = \begin{cases} 1, & \text{if } \varsigma \leq \omega \\ 0, & \text{otherwise} \end{cases}$$

Let us consider $\psi(\nu) = t^k$, where k is chosen in such a way that $\psi(\nu) < (\nu)^{\frac{1}{s}}\ \forall\ \nu > 0$ (as $s = 1.5$). We define two functions, ς and ω, both belonging to the set Θ. We say that $\varsigma \leq \omega$ if and only if $\varsigma(\nu) \leq \omega(\nu)\ \forall\ \nu \in [0, 1]$. Now, let us assume that we have two functions ς and ω in the set Θ such that $\alpha(\varsigma,\ \omega) \in [1, \infty)$. Since $\alpha(\varsigma,\ \omega) \in [1, \infty)$, it follows that $\varsigma \leq \omega$, which implies that $\varsigma(\nu) \leq \omega(\nu)$ for every Γ in the interval $[0, 1]$. Consequently, based on

condition (1)

$$\begin{aligned}\left|\frac{\Gamma\varsigma(\nu)}{\Gamma\omega(\nu)}\right| &\leq \ln\int_1^0 (e^{\left|\frac{\lambda(\nu,r,\varsigma(r))}{\lambda(\nu,r,\ \omega(r))}\right|})^{dr}\\ &\leq \ln\int_1^0 (e^{(|\frac{\varsigma(r)}{\omega(r)}|))^{g(\nu,r)}})^{dr}\\ &= \ln\int_1^0 (e^{(\sqrt{(\frac{\varsigma(r)}{\omega(r)})^2})^{g(\nu,r)}})^{dr}\\ &\leq (\sqrt{\left|\left|\frac{\varsigma}{\omega}\right|\right|_\infty^2})^k.\end{aligned}$$

Further, we have

$$\begin{aligned}|\Gamma\varsigma(\nu)| &\leq \ln\int_1^0 (e^{|\lambda(\nu,r,\varsigma(r))|})^{dr}\\ &\leq \ln\int_1^0 (e^{|\varsigma(r)|^{g(\nu,r)}})^{dr}\\ &\leq (\|\varsigma\|_\infty)^k.\end{aligned}$$

We deduce that for every $\varsigma,\ \omega \in \Theta$ such that $\varsigma \leq \omega$

$$\begin{aligned}\eth(\frac{\Gamma\varsigma}{\Gamma\omega}) &= \left\|\frac{\Gamma\varsigma}{\Gamma\omega}\right\|_\infty^2 \times \|\varsigma\|_\infty\\ &\leq (\left\|\frac{\varsigma}{\omega}\right\|_\infty^2)^{k^2} \times (\|\varsigma\|_\infty)^k\\ &\leq (\eth(\varsigma,\ \omega))^k = \psi(\eth(\varsigma,\ \omega))\\ &\leq \psi(\Xi(\varsigma,\ \omega)).\end{aligned}$$

As κ is non-decreasing concerning its third variable, for any pair of functions ς and ω within the set Θ where $\varsigma \leq \omega$, it follows that $\Gamma\varsigma(\nu) \leq \Gamma\omega(\nu)\ \forall\ \Gamma$ in the interval $[0, 1]$. In other words, if $\alpha(\varsigma,\ \omega) \in [1, \infty)$, then we can conclude that $\alpha(\Gamma\varsigma, T\omega) \in [1, \infty)$. Furthermore, condition (3) implies that there exists a function ς_0 in the set Θ such that $\alpha(\varsigma_0, \Gamma\varsigma_0) \in [1, \infty)$, $\alpha(\Gamma\varsigma_0, \varsigma_0) \in [1, \infty)$, $\alpha(\varsigma_0, \Gamma^2\varsigma_0) \in [1, \infty)$, and $\alpha(\Gamma^2\varsigma_0, \Gamma\varsigma_0) \in [1, \infty)$. Consequently, all the constraints required by Theorem 2 are satisfied with $s = 1.5$, and thus, the operator Γ possesses a F_p that serves as a solution to the integral Equation (29) in the set Θ.

References

Abagaro, B.N., Tola, K.K. & Mamud. 2022. Fixed point theorems for generalized (α, ψ)-contraction mappings in rectangular quasi-b-metric spaces. *Fixed Point Theory and Algorithms for Sciences and Engineering*, 2022, pp.89–101. Available at: https://doi.org/10.1186/s13663-022-00723-w

Abbas, M., Leyewb, B.T. & Khanc, S.H. 2017. A new ϕ-generalized quasi-metric spaces with some fixed point results and applications. *Filomat*, 31(11), pp.3157–3172. Available at: http://www.jstor.org/stable/26195044

Alharbi, N., Aydi, H., Cenap & Felhi, A. 2018. α-contractive mappings on rectangular b-metric spaces and an application to integral equations. *J. Math. Anal.*, 9(3), pp.47–60. Available at: www.ilirias.com/jma

Aydi, H., Karapinar, E. & Lakzian, H. 2012. Fixed point results on a class of generalized metric spaces. *Math. Sci.*, 6(1), p.46. Available at: https://doi.org/10.1186/2251-7456-6-46

Aydi, H., Karapınar, E. & Samet, B. 2014. Fixed points for generalized (α,ψ)-contractions on generalized metric spaces. *J. Inequal. Appl.*, p.229. Available at: https://doi.org/10.1186/1029-242X-2014-229

Banach, S. 1922. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. *Fundam. Math.*, 3(1), pp.133–181. Available at: http://eudml.org/doc/213289

Bodagi, A. & Mitrovic, Z.D. 2023. The structure of multimixed quadratic cubic mappings and an application of fixed point theory. *The Journal of Analysis*, 31(1), pp.759–777. Available at: https://doi.org/10.1007/s41478-022-00475-1

Branciari, A. 2000. A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces. *Publ. Math. (Debr.)*, 57, pp.31–37. Available at: https://doi.org/10.5486/PMD.2000.2133

Czerwik, S. 1993. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostrav.*, 1(1), pp.5–11. Available at: http://eudml.org/doc/23748

Ding, H.S., Ozturk, V. & Radenović, S. 2015. On some new fixed point results in b-rectangular metric spaces. *J. Nonlinear Sci. Appl.*, 8(4), pp.378–386. Available at: https://doi.org/10.1016/j.ajmsc.2015.05.003

George, R., Radenović, S., Reshma, K.P. & Shukla, S. 2015. Rectangular b-metric spaces and contraction principles. *J. Nonlinear Sci. Appl.*, 8(6), pp.1005–1013. Available at: https://doi.org/10.22436/jnsa.008.06.11

Karapinar, E. & Lakzian, H. 2014. α-contractive mappings on generalized quasi-metric spaces. *J. Funct. Spaces*, 2014, Article ID 914398. Available at: https://doi.org/10.1155/2014/914398

Lin, I.J., Chen, C.M. & Karapinar, E. 2014. Periodic points of weaker Meir-Keeler contractive mappings on generalized quasi-metric space. *Abstr. Appl. Anal.*, SI71, pp.1–6. Available at: https://doi.org/10.1155/2014/490450

Ozavsar, M. & Cevikel, A.C. 2012. Fixed points of multiplicative contraction mappings on multiplicative metric spaces. *arXiv*, 1205.5131. Available at: https://doi.org/10.48550/arXiv.1205.5131

Samet, B., Vetro, C. & Vetro, P. 2012. Fixed point theorems for α–ψ-contractive type mappings. *Nonlinear Anal.: Theory, Methods & Applications*, 75(4), pp.2154–2165. Available at: https://doi.org/10.1016/j.na.2011.10.014

Aproksimacija fiksnih tačaka pomoću generalizovanih (α, ψ)-kontraktivnih preslikavanja za analizu fiksnih tačaka u Branciari multiplikativnim b-kvazi-metričkim prostorima

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OBLAST: matematika
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Sažetak:

Uvod/cilj: U radu uvodimo novu klasu multiplikativnih struktura tipa Branciari nazvanu b-kvazi-metrički prostori. Ovaj okvir proširuje i objedinjuje nekoliko postojećih koncepata, uključujući Branciari metričke prostore, kvazi-metričke prostore i b-metričke prostore. Proširivanjem ovih osnovnih struktura, cilj nam je da uspostavimo opštiji okvir za analizu fenomena fiksnih tačaka.

Metode: U okviru predloženog b-kvazi-metričkog okvira, formulišemo generalizovanu klasu (α, ψ)- kontraktivnih preslikavanja. Zatim analiziramo ova preslikavanja tehnikama za fiksne tačke prilagođenim novodefinisanom prostoru. Studija obuhvata identifikaciju dovoljnih uslova koji garantuju važenje principa kontrakcije, kao i razvoj metoda za njihovu proveru.

Rezultati: Utvrđujemo ključne osobine predložene klase (α, ψ)-kontraktivnih preslikavanja, uključujući postojanje i jedinstvenost fiksnih tačaka pod odgovarajućim pretpostavkama. Ovi rezultati u značajnoj meri generalizuju nekoliko ranijih teorema o fiksnim tačkama koje su formulisane u okruženju Branciari, kvazi-metričkom i b-metričkom okruženju. Pored toga, predstavljeni su ilustrativni primeri koji potvrđuju ispravnost i primenljivost dobijenih teorijskih rezultata.

Zaključak: Uvođenje b-kvazi-metričkih prostora i odgovarajuće klase (α, ψ)- kontrakcija daje objedinjeno i sveobuhvatno proširenje prethodnih rezultata o fiksnim tačkama. Primeri i primene na konkretne integralne jednačine dodatno pokazuju praktični

značaj predloženog okvira, pružajući čvrstu osnovu za budući teorijski razvoj i primene.

Ključne reči: fiksne tačke, multiplikativni metrički prostor, multiplikativni b-kvazi-metrički prostor Branciari, generalizovana multiplikativna (α, ψ)- kontraktivna preslikavanja.

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