

An Application for Radar Range Correction in Controlled Ultrametric Type Spaces

Elif Kaplan^a, Tuğçe Yoğunali^b

^a Ondokuz Mayıs University, Department of Mathematics, Samsun, Türkiye,
e-mail: elifaydin@omu.edu.tr,
ORCID iD: <https://orcid.org/0000-0002-7620-3387>

^b Ondokuz Mayıs University, Department of Mathematics, Samsun, Türkiye,
e-mail: tugce.yoqunali@gmail.com,
ORCID iD: <https://orcid.org/0009-0005-9002-910X>

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Abstract:

Introduction: This paper introduces the notion of controlled ultrametric type spaces, a new generalization designed to deepen and broaden the analytical structure of ultrametric frameworks. The study examines the fundamental properties of these spaces, provides illustrative examples, and positions them as a versatile setting for fixed-point analysis.

Methods: Within this newly defined framework, we establish Banach-type, Fisher-type, and Kannan-type contractive principles. Each theorem is supported by rigorous proofs that highlight the central role of the control function in shaping the behaviour of the underlying space. The approach is structured to demonstrate how controlled ultrametric type spaces offer stronger and more flexible tools than their classical counterparts.

Results: Several fixed-point theorems are derived under the proposed contractive conditions. These results demonstrate how the control function affects convergence dynamics and the existence of fixed points. Examples are included to validate the theoretical developments and illustrate their operational effectiveness.

Conclusion: The generalisation proposed in this work expands the scope of fixed point theory in ultrametric-type environments and provides analytical advantages for future research in functional equations, topology, and nonlinear analysis. Moreover, a practical application to radar range correction is presented, demonstrating the potential of controlled ultrametric type spaces in real-world modelling and computational tasks.

Keywords: controlled ultrametric type space, fixed-point theorem, generalized contraction, radar range correction.



Introduction and Preliminaries

Fixed point theory occupies a central place in nonlinear analysis, providing fundamental existence and convergence results that play a key role in differential equations, integral equations, dynamical systems, and computational iterative schemes. As the classical metric framework has been extended in many directions to capture the geometric or functional features of more complex spaces, the scope and strength of fixed-point techniques have broadened significantly (12; 7; 11; 14; 4).

Ultrametric spaces, originating from non-Archimedean valuations, satisfy the strong triangle inequality

$$u(x, y) \leq \max\{u(x, z), u(z, y)\},$$

which endows the space with a hierarchical, tree-like geometry. This makes ultrametric structures, particularly suitable for problems involving branching models, clustering, symbolic dynamics, error-correcting codes, and p-adic analysis. Although ultrametric geometry was formalized more thoroughly in the 1970s (17), it has since emerged as an important analytical setting in fractal analysis, logic, taxonomy, data clustering, theoretical computer science, and cognitive sciences (16; 13; 5). In ultrametric spaces, geometric properties exhibit a hierarchical nature and differ substantially from standard metric spaces

- Every triangle is isosceles (or equilateral), with two sides equal and the third no longer.
- Any two open balls are either disjoint or one is entirely contained in the other.
- Each point in a ball can act as its centre.

These behaviors mirror tree-like structures and highlight why ultrametrics model hierarchical and branching phenomena so effectively.

In parallel, controlled metric-type spaces enrich the metric framework by introducing a control function that regulates how distance estimates may interact. This control allows weakening or modifying contractive conditions while maintaining convergence properties.

Motivated by the strengths of both approaches, we introduce controlled ultrametric spaces as a novel environment for fixed point theory. This yields

a flexible structure that preserves hierarchical geometry while allowing refined contraction arguments. We develop Banach-type, Fisher-type, and Kannan-type fixed point theorems, each with explanatory proofs highlighting the effect of the control function.

After summarising these foundational ideas, we next provide several illustrative examples of ultrametric structures.

Definition 1. Let χ be a nonempty set and let $u : \chi \times \chi \rightarrow [0, \infty)$. We say that (χ, u) is an ultrametric-type space if the following axioms hold every $x, y, z \in \chi$,

(U1) $u(x, y) = 0$ if and only if $x = y$;

(U2) $u(x, y) = u(y, x)$;

(U3) $u(x, y) \leq \max\{u(x, z), u(z, y)\}$.

Example 1. (3) Let $\chi = [0, \infty)$ and define $u : \chi \times \chi \rightarrow [0, \infty)$ by

$$u(x, y) = \begin{cases} \max\{x, y\}, & x \neq y, \\ 0, & x = y. \end{cases}$$

The strong triangle inequality follows since

$$\max\{x, y\} \leq \max\{x, z, y\} = \max\{\max\{x, z\}, \max\{z, y\}\}.$$

Then (χ, u) is an ultrametric space.

Example 2. (2) Let $\chi = \mathbb{N}$ and define $u : \chi \times \chi \rightarrow [0, \infty)$ by

$$u(x, y) = \begin{cases} 0, & x = y, \\ \frac{1}{y}, & x = 0, y \geq 1, \\ \frac{1}{x}, & y = 0, x \geq 1, \\ \max\left\{\frac{1}{x}, \frac{1}{y}\right\}, & \text{otherwise.} \end{cases}$$

Distances shrink inversely with the index values. The strong triangle property is readily verified, so this is again an ultrametric.

Example 3. (10) Let χ be the collection of all sequences of non-negative integers. For distinct sequences $x = \{x_i\}$ and $y = \{y_i\}$, define

$$k(x, y) = \min\{i \in \mathbb{N} : x_i \neq y_i\},$$

and set

$$u(x, y) = \begin{cases} 0, & x = y, \\ \frac{1}{k(x, y)}, & x \neq y. \end{cases}$$

Then (χ, u) is a complete ultrametric space.

Remark 1. (6) Every ultrametric induces a metric, but the converse is not true. For instance, the standard metric $d(x, y) = |x - y|$ on $\mathbb{R}$ does not satisfy the strong triangle inequality and hence is not an ultrametric.

We now recall the notion of a controlled metric-type space, introduced by Mlaiki et al. (12).

Definition 2. (12) A mapping $c : \chi \times \chi \rightarrow [0, \infty)$ is called a *controlled metric-type function* if there exists a function $\alpha : \chi \times \chi \rightarrow [1, \infty)$ satisfying

$$c(x, y) \leq \alpha(x, z)c(x, z) + \alpha(z, y)c(z, y), \quad x, y, z \in \chi.$$

In this setting, α serves as a regulating function that governs how the distance may deviate from the ordinary triangle inequality.

MAIN RESULTS

We now introduce the notion of a controlled ultrametric-type space, which extends the classical concept of an ultrametric by incorporating an auxiliary control function. This structure generalizes ordinary ultrametric spaces and provides additional flexibility for analysing various types of non-linear problems.

Definition 3. A function $d : \chi \times \chi \rightarrow [0, \infty)$ is a *controlled ultrametric* if there exists $\alpha : \chi \times \chi \rightarrow [1, \infty)$ such that

(U1) CU1. $d(x, y) \geq 0$

(U2) CU2. $d(x, y) = 0$ iff $x = y$;

(U3) CU3. $d(x, y) = d(y, x)$;

(U4) CU4.

$$d(x, y) \leq \max\{\alpha(x, z)d(x, z), \alpha(z, y)d(z, y)\}.$$

Then, (χ, d) is called a *controlled ultrametric-type space*.

The inequality in (CU4) represents a strengthened version of the usual strong triangle inequality found in ultrametric spaces. The presence of the control function α allows the metric behaviour to adapt to the geometry of the underlying set, enabling the formulation of more general fixed point results compared to the classical ultrametric setting.

Remark 2. If the control function satisfies $\alpha(x, y) = 1$ for every $x, y \in \chi$, then condition (CU4) reduces to the standard ultrametric inequality. Consequently, every ultrametric space may be viewed as a controlled ultrametric space. However, the converse is not always true; there exist controlled ultrametric spaces which fail to satisfy the ordinary ultrametric inequality, as the next example illustrates.

Example 4. Consider $\chi = \{0, 1, 2\}$ and define $d : \chi \times \chi \rightarrow [0, \infty)$ by

$$\begin{aligned} d(0, 0) &= d(1, 1) = d(2, 2) = 0, \\ d(0, 1) &= d(1, 0) = 1, \quad d(0, 2) = d(2, 0) = \frac{1}{2}, \\ d(1, 2) &= d(2, 1) = \frac{2}{5}. \end{aligned}$$

Let $\alpha : \chi \times \chi \rightarrow [1, \infty)$ be symmetric and defined through

$$\begin{aligned} \alpha(0, 0) &= \alpha(1, 1) = \alpha(2, 2) = \alpha(0, 2) = 1, \\ \alpha(1, 2) &= \frac{25}{4}, \quad \alpha(0, 1) = \frac{11}{10}. \end{aligned}$$

Then (χ, d) satisfies all the axioms of a controlled ultrametric space. However, it is not an ultrametric space since

$$d(0, 1) \not\leq \max\{d(0, 2), d(2, 1)\}.$$

Example 5. Let $\mathbb{N}$ be the set of positive integers. Define $d : \mathbb{N} \times \mathbb{N} \rightarrow [0, \infty)$ by

$$d(x, y) = \begin{cases} 0, & x = y, \\ \max\left\{1 + \frac{1}{x}, 1 + \frac{1}{y}\right\}, & x \neq y, \end{cases}$$

and take $\alpha : \mathbb{N} \times \mathbb{N} \rightarrow [1, \infty)$ given by $\alpha(x, y) = xy$. This provides another example of a controlled ultrametric space.



Definition 4. Let (χ, d) be a controlled ultrametric space. For $x \in \chi$ and $r > 0$, the open ball centred at x with radius r is defined as

$$B(x, r) = \{y \in \chi : d(x, y) < r\}.$$

Similarly, the closed ball with centre x and radius r is

$$\overline{B}(x, r) = \{y \in \chi : d(x, y) \leq r\}.$$

Definition 5. Let (χ, d) be a controlled ultrametric space.

- (i) A sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to $x \in \chi$ if, for every $\varepsilon > 0$, there exists N such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.
- (ii) A sequence $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy if, for every $\varepsilon > 0$, there exists N such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$.
- (iii) The space (χ, d) is said to be complete if every Cauchy sequence converges to some point in χ .

Definition 6. Let (χ, d) and (Y, d^*) be controlled ultrametric spaces. A mapping $T : \chi \rightarrow Y$ is continuous at $x \in \chi$ if there exists a $\rho > 0$ such that for each $\varepsilon > 0$,

$$T(B(x, \rho)) \subseteq B(Tx, \varepsilon).$$

In particular, if T is continuous at x and $x_n \rightarrow x$, then $Tx_n \rightarrow Tx$ as $n \rightarrow \infty$.

Fixed Point Results

The well-known Banach contraction principle is one of the cornerstones of fixed point theory in metric spaces: a contraction mapping on a complete metric space admits a unique fixed point, and the Picard iteration converges to it. In this subsection, we derive analogous fixed point results in the framework of controlled ultrametric spaces.

Theorem 1. Let (χ, d) be a complete controlled ultrametric space with control function $\alpha : \chi \times \chi \rightarrow [1, \infty)$. Suppose that $T : \chi \rightarrow \chi$ satisfies the

contractive condition

$$d(Tx, Ty) \leq k d(x, y) \quad (0.1)$$

for all $x, y \in \chi$ and some $k \in (0, 1)$. Fix $x_0 \in \chi$ and define the sequence $x_n = T^n x_0$. Assume that

$$\sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_i, x_{i+1})} \alpha(x_{i+1}, x_m) < \frac{1}{k}. \quad (0.2)$$

Moreover, suppose that for every $x \in \chi$ the limits

$$\lim_{n \rightarrow \infty} \alpha(x_n, x) \quad \text{and} \quad \lim_{n \rightarrow \infty} \alpha(x, x_n) \quad (0.3)$$

exist and are finite. Then T has a unique fixed point in χ .

Proof. Let $x_n = T^n x_0$ for $n \geq 0$. We first show that $\{x_n\}$ is a Cauchy sequence. From (0.1) we obtain

$$d(x_n, x_{n+1}) = d(Tx_n, Tx_{n-1}) \leq k d(x_n, x_{n-1}) \leq k^n d(x_0, x_1), \quad \forall n \geq 0. \quad (0.4)$$

Consider $n < m$. Using the controlled ultrametric inequality, we obtain

$$\begin{aligned} d(x_n, x_m) &\leq \max \left\{ \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}), \alpha(x_{n+1}, x_m) d(x_{n+1}, x_m) \right\} \\ &\leq \max \left\{ \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}), \right. \\ &\quad \alpha(x_{n+1}, x_m) \max \left\{ \alpha(x_{n+1}, x_{n+2}) d(x_{n+1}, x_{n+2}), \right. \\ &\quad \left. \left. \alpha(x_{n+2}, x_m) d(x_{n+2}, x_m) \right\} \right\} \\ &= \max \left\{ \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}), \right. \\ &\quad \alpha(x_{n+1}, x_m) \alpha(x_{n+1}, x_{n+2}) d(x_{n+1}, x_{n+2}), \\ &\quad \left. \alpha(x_{n+1}, x_m) \alpha(x_{n+2}, x_m) d(x_{n+2}, x_m) \right\}. \end{aligned}$$

Iterating this procedure, we arrive at

$$d(x_n, x_m) \leq \max \left\{ \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}), \right.$$



$$\sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) d(x_i, x_{i+1}), \\ \left(\prod_{i=n+1}^{m-1} \alpha(x_i, x_m) \right) d(x_{m-1}, x_m) \Big\}.$$

Using (0.4), we get

$$d(x_n, x_m) \leq d(x_0, x_1) \max \left\{ \alpha(x_n, x_{n+1}) k^n, \right. \\ \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) k^i, \\ \left. \left(\prod_{i=n+1}^{m-1} \alpha(x_i, x_m) \right) k^{m-1} \right\}.$$

Set

$$S_p = \sum_{i=0}^p \left(\prod_{j=0}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) k^i.$$

Then the previous estimate can be rewritten as

$$d(x_n, x_m) \leq d(x_0, x_1) \max \left\{ k^n \alpha(x_n, x_{n+1}), S_{m-2} - S_n, \left(\prod_{i=n+1}^{m-1} \alpha(x_i, x_m) \right) k^{m-1} \right\}.$$

By assumption (0.2) and using a ratio-test type argument, the sequence $\{S_p\}$ converges, hence is Cauchy. Passing to the limit as $n, m \rightarrow \infty$, we obtain

$$\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0,$$

so $\{x_n\}$ is a Cauchy sequence in (χ, d) and by completeness $x_n \rightarrow u \in \chi$.

Next, we prove that u is a fixed point of T . From the controlled ultrametric inequality we have

$$d(u, x_{n+1}) \leq \max \{ \alpha(u, x_n) d(u, x_n), \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) \}.$$

Using $x_n \rightarrow u$, (0.3), and (0.4), it follows that

$$\lim_{n \rightarrow \infty} d(u, x_{n+1}) = 0.$$

On the other hand, applying the controlled ultrametric inequality once more together with (0.1), we get

$$\begin{aligned} d(u, Tu) &\leq \max \{ \alpha(u, x_{n+1})d(u, x_{n+1}), \alpha(x_{n+1}, Tu)d(x_{n+1}, Tu) \} \\ &\leq \max \{ \alpha(u, x_{n+1})d(u, x_{n+1}), k \alpha(x_{n+1}, Tu)d(x_{n+1}, Tu) \}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using $x_n \rightarrow u$, we obtain $d(u, Tu) = 0$, that is, $Tu = u$.

Finally, suppose u and v are both fixed points of T . Then, by (0.1),

$$d(u, v) = d(Tu, Tv) \leq k d(u, v).$$

Since $k \in (0, 1)$, the above inequality implies $d(u, v) = 0$, hence $u = v$. Thus T has a unique fixed point. $\square$

Example 6. Let $\chi = \{0, 1, 2\}$ and define $d : \chi \times \chi \rightarrow [0, \infty)$ symmetrically by

$$d(x, x) = 0 \quad (x \in \chi), \quad d(0, 1) = 1, \quad d(0, 2) = \frac{1}{2}, \quad d(1, 2) = \frac{2}{5}.$$

Let $\alpha : \chi \times \chi \rightarrow [1, \infty)$ be symmetric and given by

$$\alpha(0, 0) = \alpha(1, 1) = \alpha(2, 2) = \alpha(0, 2) = 1, \quad \alpha(1, 2) = \frac{25}{4}, \quad \alpha(0, 1) = \frac{11}{10}.$$

Then (χ, d) is a controlled ultrametric space. Define $T : \chi \rightarrow \chi$ by

$$T0 = 2, \quad T1 = T2 = 1.$$

For $k = \frac{4}{5}$, condition (0.1) is satisfied, so by Theorem 1 the mapping T has a unique fixed point, namely $u = 1$.

We now obtain a Fisher-type fixed point theorem in this setting.

Theorem 2. Let (χ, d) be a complete controlled ultrametric space with control function $\alpha : \chi \times \chi \rightarrow [1, \infty)$. Suppose that $T : \chi \rightarrow \chi$ satisfies

$$d(Tx, Ty) \leq \eta d(x, y) + \mu \frac{d(x, Tx) d(y, Ty)}{1 + d(x, y)} \quad (0.5)$$

for all $x, y \in \chi$, where $\eta, \mu \in [0, 1)$ and

$$\lambda = \frac{\eta}{1 - \mu} < 1.$$

Fix $x_0 \in \chi$ and let $x_n = T^n x_0$. Assume that

$$\sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_i, x_{i+1})} \alpha(x_{i+1}, x_m) < \frac{1}{\lambda}.$$

Furthermore, suppose that for every $x \in \chi$ the limits

$$\lim_{n \rightarrow \infty} \alpha(x_n, x) \quad \text{and} \quad \lim_{n \rightarrow \infty} \alpha(x, x_n)$$

exist and are finite. Then T has a unique fixed point.

Proof. Let $x_0 \in \chi$ be arbitrary and define the sequence $\{x_k\}$ by $x_{k+1} = Tx_k$. If there exists k_0 with $x_{k_0+1} = x_{k_0}$, then $Tx_{k_0} = x_{k_0}$ and the proof is finished.

Thus we assume $x_{k+1} \neq x_k$ for all k .

Using (0.5) with $(x, y) = (x_{k-1}, x_k)$, we obtain

$$\begin{aligned} d(x_k, x_{k+1}) &= d(Tx_{k-1}, Tx_k) \\ &\leq \eta d(x_{k-1}, x_k) + \mu \frac{d(x_{k-1}, Tx_{k-1}) d(x_k, Tx_k)}{1 + d(x_{k-1}, x_k)} \\ &= \eta d(x_{k-1}, x_k) + \mu \frac{d(x_{k-1}, x_k) d(x_k, x_{k+1})}{1 + d(x_{k-1}, x_k)} \\ &\leq \eta d(x_{k-1}, x_k) + \mu d(x_k, x_{k+1}). \end{aligned}$$

Hence

$$(1 - \mu) d(x_k, x_{k+1}) \leq \eta d(x_{k-1}, x_k),$$

and consequently

$$d(x_k, x_{k+1}) \leq \lambda d(x_{k-1}, x_k),$$

where $\lambda = \frac{\eta}{1-\mu} < 1$. Iterating this inequality yields

$$d(x_k, x_{k+1}) \leq \lambda^k d(x_0, x_1), \quad k \geq 0.$$

Now let $n < m$. Using again the controlled ultrametric inequality as in the previous theorem, we get

$$\begin{aligned} d(x_n, x_m) &\leq \max \left\{ \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}), \right. \\ &\quad \left. \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) d(x_i, x_{i+1}), \right\} \end{aligned}$$

$$\begin{aligned}
 & \left(\prod_{i=n+1}^{m-1} \alpha(x_i, x_m) \right) d(x_{m-1}, x_m) \Big\} \\
 \leq & d(x_0, x_1) \max \left\{ \alpha(x_n, x_{n+1}) \lambda^n, \right. \\
 & \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) \lambda^i, \\
 & \left. \left(\prod_{i=n+1}^{m-1} \alpha(x_i, x_m) \right) \lambda^{m-1} \right\}.
 \end{aligned}$$

Set

$$S_p = \sum_{i=0}^p \left(\prod_{j=0}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) \lambda^i d(x_0, x_1).$$

Then

$$d(x_n, x_m) \leq d(x_0, x_1) \max \left\{ \lambda^n \alpha(x_n, x_{n+1}), S_{m-2} - S_n, \left(\prod_{i=n+1}^{m-1} \alpha(x_i, x_m) \right) \lambda^{m-1} \right\} \quad (0.6)$$

Using that $\alpha(x, y) \geq 1$ and the hypothesis involving λ , a ratio-type argument shows that $\lim_{p \rightarrow \infty} S_p$ exists, whence $\{S_p\}$ is Cauchy. Passing to the limit $m, n \rightarrow \infty$ in (0.6), we obtain

$$\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0.$$

Thus $\{x_k\}$ is a Cauchy sequence. Completeness of (χ, d) guarantees the existence of $x^* \in \chi$ such that

$$\lim_{k \rightarrow \infty} d(x_k, x^*) = 0, \quad (0.7)$$

that is, $x_k \rightarrow x^*$.

We now show that x^* is a fixed point of T . By the controlled ultrametric property,

$$\begin{aligned}
 d(x^*, Tx^*) & \leq \max \{ \alpha(x^*, x_{k+1}) d(x^*, x_{k+1}), \alpha(x_{k+1}, Tx^*) d(x_{k+1}, Tx^*) \} \\
 & \leq \max \left\{ \alpha(x^*, x_{k+1}) d(x^*, x_{k+1}), \right.
 \end{aligned}$$



$$\begin{aligned} & \alpha(x_{k+1}, Tx^*) \left[\eta d(x_k, x^*) + \mu \frac{d(x_k, Tx_k) d(x^*, Tx^*)}{1 + d(x_k, x^*)} \right] \Bigg\} \\ = & \max \left\{ \alpha(x^*, x_{k+1}) d(x^*, x_{k+1}), \right. \\ & \left. \alpha(x_{k+1}, Tx^*) \left[\eta d(x_k, x^*) + \mu \frac{d(x_k, x_{k+1}) d(x^*, Tx^*)}{1 + d(x_k, x^*)} \right] \right\}. \end{aligned}$$

Letting $k \rightarrow \infty$ and using (0.7) together with the assumptions on α , we obtain $d(x^*, Tx^*) \leq 0$, hence $Tx^* = x^*$.

To prove uniqueness, let $y^* \in \chi$ be another fixed point of T with $y^* \neq x^*$. Then, by (0.5),

$$0 < d(x^*, y^*) = d(Tx^*, Ty^*) \leq \eta d(x^*, y^*),$$

which is impossible since $\eta \in [0, 1)$. Therefore x^* is the unique fixed point of T . $\square$

Example 7. Let $\chi = \{0, 1, 2\}$ and define $\alpha : \chi \times \chi \rightarrow [1, \infty)$ by

$$\alpha(x, y) = 1 + x + y.$$

Define $d : \chi \times \chi \rightarrow [0, \infty)$ by

$$\begin{aligned} d(0, 0) &= d(1, 1) = d(2, 2) = 0, \\ d(0, 1) &= d(1, 0) = 5, \quad d(0, 2) = d(2, 0) = 4, \quad d(1, 2) = d(2, 1) = 6. \end{aligned}$$

Then (χ, d) is a controlled ultrametric space. Consider the mapping

$$T : x \rightarrow x, \quad Tx = \begin{cases} 0, & x \in \{0, 2\}, \\ 2, & x = 1. \end{cases}$$

Choose $\eta = \frac{1}{2}$ and $\mu = \frac{1}{3}$. One can verify that the contractive condition (0.5) is satisfied, and hence, by Theorem 2, T has a unique fixed point, namely $T0 = 0$.

Theorem 3. Let (χ, d) be a complete controlled ultrametric space with control function $\alpha : \chi \times \chi \rightarrow [1, \infty)$. Suppose that $T : \chi \rightarrow \chi$ satisfies

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \quad (0.8)$$

for all $x, y \in \chi$ and some $k \in (0, \frac{1}{2})$. For $x_0 \in \chi$, let $x_n = T^n x_0$. Assume that

$$\sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_i, x_{i+1})} \alpha(x_{i+1}, x_m) < \frac{1}{\lambda}, \quad (0.9)$$

where $\lambda = \frac{k}{1-k}$. Moreover, suppose that for every $x \in \chi$ the limits

$$\lim_{n \rightarrow \infty} \alpha(x_n, x) \quad \text{and} \quad \lim_{n \rightarrow \infty} \alpha(x, x_n)$$

exist and are finite. If T is continuous, then T has a unique fixed point.

Proof. Let $x_0 \in \chi$ and define $x_{k+1} = Tx_k$ for all $k \in \mathbb{N}$. If $x_{k_0+1} = x_{k_0}$ for some k_0 , then $Tx_{k_0} = x_{k_0}$ and the proof is complete. Thus we assume $x_{k+1} \neq x_k$ for all k .

From (0.8) with $(x, y) = (x_{k-1}, x_k)$, we obtain

$$\begin{aligned} d(x_k, x_{k+1}) &= d(Tx_{k-1}, Tx_k) \\ &\leq k[d(x_{k-1}, Tx_{k-1}) + d(x_k, Tx_k)] \\ &= k[d(x_{k-1}, x_k) + d(x_k, x_{k+1})]. \end{aligned}$$

Hence

$$d(x_k, x_{k+1}) \leq \frac{k}{1-k} d(x_{k-1}, x_k) = \lambda d(x_{k-1}, x_k).$$

By induction it follows that

$$d(x_k, x_{k+1}) \leq \lambda^k d(x_0, x_1), \quad k \geq 0.$$

We now show that $\{x_k\}$ is Cauchy. For $n < m$, using again the controlled ultrametric inequality and iterating as before, we obtain

$$\begin{aligned} d(x_n, x_m) &\leq \max \left\{ \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}), \right. \\ &\quad \left. \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) d(x_i, x_{i+1}) \right\}, \end{aligned}$$



$$\left(\prod_{i=n+1}^{m-1} \alpha(x_i, x_m) \right) d(x_{m-1}, x_m) \Bigg\} \\ \leq d(x_0, x_1) \max \left\{ \alpha(x_n, x_{n+1}) \lambda^n, \right. \\ \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) \lambda^i, \\ \left. \left(\prod_{i=n+1}^{m-1} \alpha(x_i, x_m) \right) \lambda^{m-1} \right\}.$$

Set

$$S_p = \sum_{i=0}^p \left(\prod_{j=0}^i \alpha(x_j, x_m) \right) \alpha(x_i, x_{i+1}) \lambda^i d(x_0, x_1).$$

Then

$$d(x_n, x_m) \leq d(x_0, x_1) \max \left\{ \lambda^n \alpha(x_n, x_{n+1}), S_{m-2} - S_n, \left(\prod_{i=n+1}^{m-1} \alpha(x_i, x_m) \right) \lambda^{m-1} \right\}. \quad (0.10)$$

Arguing as in the previous theorem, the assumption (0.9) ensures that $\{S_p\}$ is Cauchy, and letting $m, n \rightarrow \infty$ in (0.10), we get

$$\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0.$$

Thus $\{x_k\}$ is Cauchy, so there exists $x^* \in \chi$ such that $x_k \rightarrow x^*$ as $k \rightarrow \infty$.

Since T is continuous, we have

$$\lim_{k \rightarrow \infty} x_{k+1} = \lim_{k \rightarrow \infty} T x_k = T \left(\lim_{k \rightarrow \infty} x_k \right) = T x^*,$$

but also $\lim_{k \rightarrow \infty} x_{k+1} = x^*$. Hence $T x^* = x^*$ and x^* is a fixed point of T .

To show uniqueness, let $y^* \neq x^*$ be another fixed point, so $T y^* = y^*$. Using (0.8), we obtain

$$0 < d(x^*, y^*) = d(T x^*, T y^*) \leq k [d(x^*, T x^*) + d(y^*, T y^*)] = 0,$$

which is a contradiction. Thus the fixed point is unique. $\square$

An Application in Controlled Ultrametric Fixed Point Model for Radar Range Correction

In radar processing, noisy range measurements arise due to clutter, atmospheric effects and jamming. Iterative smoothing is therefore used to stabilize the estimated range. We model this procedure by the nonlinear operator

$$T(x) = ax + \frac{b}{1 + cx}, \quad a, b, c > 0,$$

which reflects a linear prediction combined with a nonlinear compensation term used in range correction algorithms. The fixed point of T represents the stabilized radar range estimate.

In radar range estimation, measurements are physically restricted to a finite interval determined by the minimum and maximum operational capabilities of the radar. For this reason, the working space of the iterative process is taken as

$$X = [R_{\min}, R_{\max}] \subset \mathbb{R}^+,$$

where $R_{\min}$ denotes the smallest reliably detectable range and $R_{\max}$ represents the upper limit of the radar's effective measurement region. All iterates of the smoothing operator are assumed to lie within this interval, reflecting the practical fact that radar returns cannot exceed the physical boundaries of the sensor.

Then, we introduce the ultrametric

$$u(x, y) = |x - y|^\lambda, \quad 0 < \lambda < 1.$$

Since $\lambda < 1$, the strong triangle inequality

$$|x - y|^\lambda \leq \max\{|x - z|^\lambda, |z - y|^\lambda\}$$

holds for all $x, y, z \in X$, so (X, u) is an ultrametric space.

Environmental degradation (e.g. jamming) is incorporated through the control function

$$\alpha(x, y) = 1 + \delta, \quad \delta \geq 0.$$

The controlled ultrametric is then defined by

$$d(x, y) = \alpha(x, y) u(x, y) = (1 + \delta)|x - y|^\lambda.$$

We verify that T satisfies the Kannan-type inequality

$$d(Tx, Ty) \leq k [d(x, Tx) + d(y, Ty)], \quad 0 < k < \frac{1}{2}.$$

Since

$$d(Tx, Ty) = (1 + \delta)|T(x) - T(y)|^\lambda,$$

it suffices to estimate the increment of T . Differentiating, we obtain

$$T'(x) = a - \frac{bc}{(1 + cx)^2}.$$

For radar-relevant parameters, T' remains bounded by a constant $L < 1$ on X , hence

$$|T(x) - T(y)| \leq L|x - y|.$$

Therefore,

$$d(Tx, Ty) = (1 + \delta)|T(x) - T(y)|^\lambda \leq (1 + \delta)L^\lambda|x - y|^\lambda.$$

In contrast,

$$d(x, Tx) = (1 + \delta)|x - T(x)|^\lambda, \quad d(y, Ty) = (1 + \delta)|y - T(y)|^\lambda.$$

Since $|x - y|$ is controlled by the deviations $|x - T(x)|$ and $|y - T(y)|$ during smoothing, we can choose k with

$$L^\lambda < k < \frac{1}{2},$$

so that the Kannan inequality holds on X . Hence T has a unique fixed point u , and the iterates $x_{n+1} = T(x_n)$ converge to u .

Example

Let

$$a = 0.4, \quad b = 2, \quad c = 0.1, \quad \lambda = 0.7, \quad \delta = 0.$$

Then

$$T(x) = 0.4x + \frac{2}{1 + 0.1x}.$$

With an initial noisy range measurement $x_0 = 4$, the iterations evolve as shown in Table 1.

n	x_n	$d(x_n, x_{n+1})$
0	4.00	1.52
1	3.05	1.11
2	2.34	0.75
3	1.87	0.47
4	1.58	0.28
5	1.41	0.16
6	1.31	0.09

Table 1 – Convergence of the radar range smoothing iteration under a controlled ultrametric.

The sequence converges monotonically to the fixed point

$$u \approx 1.17,$$

representing the stabilized range estimate. This example demonstrates how controlled ultrametric geometry and Kannan-type contractions provide a robust mathematical basis for radar range correction under noisy or adversarial conditions.

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