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On the Equivalence of two Measures of Lower Bound to Reliability

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The following two coefficients of reliability were defined, among others, within the Guttman image theory of measurement: τ and ρ. Coefficient τ is defined by Momirovic in 1975 as the ratio of the maximized variance of a linear combination of item true scores and the maximized variance of a linear combination of standardized scores of items of some measuring instrument. Coefficients ρ is derived by Momirović and Dobrić in 1977 through the extremisation of the ratio between the variance of a linear combination of true scores and the variance of a linear combination of standardized scores of the items of some test. We have demonstrated the equivalence of these two measures of lower bound to relibility when the image scores and standardized scores on items are rescaled onto the universal metrics, i.e. when they are projected into the Harris space.

Key words: image theory / universal metrics / theory of measurement / reliability.

1. Introduction

Let cov (.,.) denotes the covariance between two variables. Let z_j and z_k are the scores on any two items of a composite measuring instrument. Let t_j and t_k are the true scores on these two items, and let e_j and e_k are the error scores on these two items, respectively. Let

$$zj = tj + ej$$

and

$$z_k = t_k + e_k$$

Then the classical theory of measurement is based on the following five postulates:

$$\begin{array}{ll} (1) & \text{cov}\,(e_j, e_k) = 0, \\ (2) & \text{cov}\,(t_j, e_j) = 0, \\ (3) & \text{cov}\,(t_k, e_k) = 0, \\ (4) & \text{cov}\,(t_j, ek) = 0, \\ (5) & \text{cov}\,(z_j, e_k) = 0. \end{array}$$

This model of measurement was considered in the majority of psychometric texts (Cronbach, Glaser, Nanda and Rajaratnam, 1972; Cronbach, Rajaratnam and Glaser, 1963; Guilford, 1954; Krkovic, 1966; Lord and Novick, 1968; Pfanzagl, 1968; etc.). Two types of solutions for estimating coefficients of reliability were proposed within the framerwork of this model. The first one consists in the maximization of theoretical reliability measure, leading to coefficient of reliability proposed by Kaiser and Caffrey (1965). The second, generally accepted aproach, results in an abundancy of different computation formulas (Cronbach, Spearman - Brown, Kuder - Richardson, Rulon, etc.), based on the additional postulates that all items have the same error of measurement, and the equal intercorrelations; these solutions, however, could be completely reduced to each other since they are only different variation of the same equation (Momirović and Gredelj, 1980).

Whitin a theory of measurement, based on Guttman image theory (Guttman, 1953), the postulates (1) and (4) are no loger valid. Namely, model of measurement based on image theory provides the possibility to compute the values of true scores and error scores on each item of a test. This possibility is created by the definition of true scores as image variables, and error scores as antiimage variables derived from the set of items. This model, therefore, allows for non-zero covariances of error scores and non-zero covariances of true scores and error scores for different items (Momirović, 1975; Nicewander, 1975; Momirović and Dobrić, 1977; Zakrajšek, Momirović and Dobrić, 1977; Momirović and Gredelj, 1980).

Within this model of measurement several measures of reliability have been defined. Among them two are particularly important for our discussion: τ - the measure of reliability defined by Momirović in 1975, and ρ, the measure derived by Momirović and Dobrić in 1977. Coefficient τ is defined as the ratio of maximized variance of a linear combination of true item scores and maximized variance of a linear combination of standardized item scores. Coefficent ρ is derived through a direct extremization of the ratio between the variance of a linear combination of true scores and variance of a linear combination of standardized item scores.

The aim of this paper is to prove the equivalence of the two measures when true scores and item scores are rescaled onto Harris universal metrics.

2. Definitions

Let be $E = (e_i;\ i = 1,...,n)$ a random sample from a homogeneous population of individuals. Let be $V = (v_j;\ j = 1,...,m)$ a set of items of a measuring instrument T. Supose $m \ll n$. Then

$$\mathbf{B} = (b_{ij}) = E \otimes V \qquad \begin{matrix} i = 1,...,n \\ j = 1,...,m \end{matrix}$$

will be a data matrix, obtained by the description of individuals from E on the set of items of test T.

Let be

$$\mathbf{e}^t = (1\ 1\ ...1)$$

a sumation vector of order (1,n) and define

$$\mathbf{P} = \mathbf{e}\ (\mathbf{e}^t\mathbf{e})^{-1}\,\mathbf{e}^t$$

as a centroid projector in R^n space defined by E.

The covariance matrix of items from V will be

$$\mathbf{C} = (\mathbf{B}^t\mathbf{B} - \mathbf{B}^t\mathbf{P}\ \mathbf{B})\ n^{-1}$$

so that the diagonal matrix of variances of items will be

$$\mathbf{S}^2 = \text{diag}\ \mathbf{C}.$$

Define now

$$\mathbf{Z} = (\mathbf{B} - \mathbf{PB})\,\mathbf{S}^{-1}$$

as the matrix of standardized results of individuals from *E* on the items of test *T*.

Then, the correlation matrix of items on the set *E* will be

$$\mathbf{R} = \mathbf{Z}^t\mathbf{Z}\ \mathrm{n}^{-1}$$

so that the matrix of unique variances of test items will be

$$\mathbf{U}^2 = (\mathrm{diag}\ \mathbf{R}^{-1})^{-1}$$

(Guttman, 1953; Kaiser, 1963; McDonald, 1975).

Now define the image scores on items

$$\mathbf{G} = \mathbf{Z}\ (\mathbf{I} - \mathbf{R}^{-1}\mathbf{U}^2)$$

as the true scores on items from *T* (Nicewander, 1975; Momirovic, 1975; Momirovic and Dobric, 1977; Zakrajsek, Momirovic and Dobric, 1977; Momirovic and Gredelj, 1980), with covariance matrix

$$\begin{aligned}\mathbf{H} = \mathbf{G}^t\mathbf{G}\ \mathrm{n}^{-1} &= (\mathbf{I} - \mathbf{U}^2\mathbf{R}^{-1})\ \mathbf{R}\ \ (\mathbf{I} - \mathbf{R}^{-1}\mathbf{U}^2)\\ &= \mathbf{R} + \mathbf{U}^2\mathbf{R}^{-1}\mathbf{U}^2 - 2\,\mathbf{U}^2.\end{aligned}$$

3. Reliability measures

The first coefficient of reliability which will be considered here is τ - a measure of lower bound to reliability defined by Momirović in 1975. The coefficient of reliability τ was obtained through the extremization of differences among entities on true scores - i.e. trough the maximization of variance of a linear combination of true scores and by the extremization of differrences among entities on a linear combination of standardized item scores, i.e. the maximization of variance of a linear combination of standardized results on items of a test.

According to the formal definition of any reliability coefficient

$$\alpha \;=\; \sigma^2{}_t \,/\, \sigma^2$$

with $\sigma^2{}_t$ the variance of true scores and σ^2 the variance of item scores, the reliability coefficient proposed by Momirovic is defined as the solution of the problem

$$\tau \;=\; (\mathbf{x}^t\mathbf{H}\,\mathbf{x}) \;/\; (\mathbf{y}^t\mathbf{R}\;\mathbf{y}) \;=\; \delta^2 / \lambda^2$$

under the conditions

$$\delta^2 = \mathbf{x}^2\mathbf{H}\;\;\mathbf{x} \;=\; \max \;\Big|\; \mathbf{x}^t\mathbf{x} \;=\; 1$$

and

$$\lambda^2 = \mathbf{y}^t\mathbf{R}\,\mathbf{y} \;=\; \max \;\Big|\; \mathbf{y}^t\mathbf{y} \;=\; 1.$$

Both conditions are actually the simple eigenvalues problems so that τ is ratio of maximal eigenvalues of image covariance matrix **H** and intercorrelation matrix **R** of items of test *T*.

The second measure of lower bound to reliability is coefficient ρ which was derived by Momirović and Dobrić in 1977. This coefficient was obtained through the direct maximization of the ratio of the variance of a linear combination of true scores and the variance of a linear combination of standardized item scores. Actually, ρ is obtained as the solution of the problem

$$\rho \;=\; (\mathbf{w}^t\mathbf{H}\,\mathbf{w}) \;/\; (\mathbf{w}^t\mathbf{R}\,\mathbf{w}) \;=\; \max \;\Big|\; \mathbf{w}^t\mathbf{R}\;\;\mathbf{w} \;=\; 1.$$

The function which should be extremized is

$$f \;=\; (\mathbf{w}, \phi) \;=\; (\mathbf{w}^t\mathbf{H}\,\mathbf{w}) \;/\; (\mathbf{w}^t\mathbf{R}\,\mathbf{w}) - \phi\;(\mathbf{w}^t\mathbf{R}\;\;\mathbf{w} - 1)$$

so that the derivation of this function with respect to **w** gives

$$
\begin{aligned}
\partial\, f(\mathbf{w}, \phi) \;/\; \partial \mathbf{w} = 2\,\mathbf{H}\,\mathbf{w} - 2\,\phi\,\mathbf{R}\,\mathbf{w} \\
&\Rightarrow \mathbf{H}\,\mathbf{w} = \phi\,\mathbf{R}\,\mathbf{w} \\
&\Rightarrow (\mathbf{H} - \phi\,\mathbf{R})\,\mathbf{w} = 0 \\
&\Rightarrow (\mathbf{R}^{-1}\mathbf{H} - \phi\,\mathbf{I})\,\mathbf{w} = 0 \\
&= ((\mathbf{I} - \mathbf{R}^{-1}\mathbf{U}^{2})^{2} - \phi\,\mathbf{I})\,\mathbf{w} = 0 \\
&\Rightarrow \phi = \mathbf{w}^{t}\,(\mathbf{I} - \mathbf{U}\;\mathbf{R}^{-1}\mathbf{U})\,\mathbf{w} \\
&\Rightarrow \phi = \rho.
\end{aligned}
$$

Obviously the derivation of f (ϕ, $\mathbf{w}$) with respect to $\mathbf{w}$ poses the eigenvalue problem of the matrix $(\mathbf{I} - \mathbf{U}\;\mathbf{R}^{-1}\mathbf{U})^{2}$. According to problem of maximization of reliability measure ρ we need the largest eigenvalue of this matrix. But,

$$
\begin{aligned}
&\phi = \max \;\Leftrightarrow\; \mathbf{w}^{t}\,(\mathbf{U}\;\mathbf{R}^{-1}\mathbf{U})\;\mathbf{w} = \min \\
&\mathbf{w}^{t}\,(\mathbf{U}\;\mathbf{R}^{-1}\mathbf{U})\;\mathbf{w} = \min \;\Leftrightarrow \mathbf{w}^{t}\,(\mathbf{U}^{-1}\mathbf{R}\;\mathbf{U}^{-1})\;\mathbf{w} = \max;
\end{aligned}
$$

define this eigenvalue of $\mathbf{U}^{-1}\mathbf{R}\;\mathbf{U}^{-1}$ as

$$\eta = \mathbf{w}^{t}\,(\mathbf{U}^{-1}\mathbf{R}\;\mathbf{U}^{-1})\,\mathbf{w}$$

so that so obtained lower bound to reliability is

$$\rho = (1 - \eta^{-1})^{2}.$$

Thus, the problem of maximization of the lower bound to reliability reduces to the problem of largest eigenvalue of the matrix of covariances of item scores rescaled to Harris universal metrics.

4. Equivalence of τ and ρ

It is very easy to show equivalence of τ and ρ when both true item scores, defined as image variables, and not necessarily standardized item scores were rescaled to Harris universal metrics.

The matrix of image variables projected into the universal metric space will be

$$\mathbf{Z}\;\;(\mathbf{I} - \mathbf{R}^{-1}\mathbf{U}^{2})\;\;\mathbf{U}^{-1} = \mathbf{G}\;\;\mathbf{U}^{-1} = \mathbf{G}^{*};$$

the matrix covariances of so rescaled variables is

$$\mathbf{G}^{*t}\mathbf{G}^{*}n^{-1} = \mathbf{U}^{-1}\mathbf{H}\ \mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{R}\ \mathbf{U}^{-1} + \mathbf{U}\ \mathbf{R}^{-1}\mathbf{U} - 2\ \mathbf{I} = \mathbf{H}^{*}.$$

The matrix of not necessarily standardized results on items of test *T* rescaled to universal metrics is, of course (Harris, 1962),

$$\mathbf{Z}\ \mathbf{U}^{-1} = \mathbf{Z}^{*};$$

the covariance matrix of so rescaled variables is well known Harris matrix

$$\mathbf{Z}^{*t}\mathbf{Z}^{*}n^{-1} = \mathbf{U}^{-1}\mathbf{R}\ \mathbf{U}^{-1} = \mathbf{R}^{*},$$

the same matrix whose largest eigenvalue and corresponding eigenvector have been used to obtain ρ, the second of proposed measures of lower bound to reliability.

The matrices of covariances of true item scores rescaled to universal metrics and covariances of item scores rescaled to the same metrics can be diagonallised in the same base, because

$$\mathbf{w}^{t}\mathbf{R}^{*}\mathbf{w} = \eta$$

and

$$\mathbf{w}^{t}\mathbf{H}^{*}\mathbf{w} = \eta + \eta^{-1} - 2.$$

Therefore, in Harris space,

$$\begin{aligned}\tau &= (\mathbf{w}^{t}\mathbf{H}^{*}\mathbf{w}) \ / \ (\mathbf{w}^{t}\mathbf{R}^{*}\mathbf{w}) \\ &= (1 - \eta^{-1})^{2} \\ &= \rho.\end{aligned}$$

Of course, in the Harris space,

$$\begin{aligned}\rho &= (\mathbf{w}^{t}\mathbf{H}^{*}\mathbf{w}) \ / \ (\mathbf{w}^{t}\mathbf{R}^{*}\mathbf{w}) \\ &= \mathbf{w}^{t}\,(\mathbf{I} + \mathbf{U}\ \mathbf{R}^{-1}\mathbf{U}\ \mathbf{U}\ \mathbf{R}^{-1}\mathbf{U} - 2\ \mathbf{U}\ \mathbf{R}^{-1}\mathbf{U})\,\mathbf{w} \\ &= 1 + \eta^{-2} - 2\ \eta^{-1} \\ &= (1 - \eta^{-1})^{2} \\ &= \tau\end{aligned}$$

so that

$$\tau = \rho$$

which means that these two measures of lower bound to reliability are equivalent in Harris univeral metrics space.

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O ekvivalentnosti dve mere donje granice pouzdanosti

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Definisana su dva koeficijenta pouzdanosti, između ostalih, u okviru Gutmanove imaž teorije merenja: τ i ρ. Koeficijent τ definisao je Momirović 1975 kao odnos maksimizirane varijanse linearne kombinacije pravih skorova ajtema i maksimizirane varijanse linearne kombinacije standardizovanih skorova ajtema nekog mernog instrumenta. Koeficijent ρ su izveli Momirović i Dobrić 1977 pomoću ekstremizacije odnosa između varijanse linearne kombinacije pravih skorova i varijanse linearne kombinacije standardizovanih skorova ajtema nekog testa. Pokazali smo ekvivalentnost ove dve mere donje granice pouzdanosti kada su imaž skorovi i standardizovani skorovi ajtema reskalirani na univerzalnu metriku, tj. kada su projektovani u Harisov prostor.

Ključne reči: imaž teorija, univerzalna metrika, teorija merenja, pouzdanost.