Filip D. Jevtić
Mathematical Institute
SASA, Belgrade
Slobodan Vujošević
Mathematical Institute
SASA, Belgrade

Original Scientific Paper
UDC: 2-2:161/164
2-2:510.643
510 Гедел К.

LOGIC OF THE ONTOLOGICAL ARGUMENT

Abstract: *In his ontological argument Gödel says nothing about its underlying logic. The argument is modal and at least of second-order and since S5 axiom is used so it is widely accepted that the logic of the argument is the S5 second-order modal logic. However, there is a step in the proof in which Gödel applies the necessitation rule on the assumptions of the argument (see [3]). This is repeated by all of his followers (see [1] and [5]). This application of the necessitation rule can seriously harm the consequence relation of the logic of the ontological argument. It seems that the only way to preserve the modal logic S5 for the ontological argument is to assume some of its axioms in the necessitated form.*

Keywords: *Godel, ontological argument, modal logic, necessitatio*

Ontological argument

Ontological argument belongs to the family of arguments that establishes the existence of God by relying only on pure logic. Argument proceeds from the idea of God to the reality of God and was first clearly formulated by St. Anselm in his Proslogion (1077–78). Later famous versions were given by Descartes, Leibniz and others. These ontological arguments were clearly not formal, but they show a striking similarity in their form: they argue that God exists (actually, really, necessarily) if God is possible (consistent, present in our mind), and then proceed to prove that God is indeed possible. Kant use the term "ontological argument" having in mind their ontological context. Their formal and modal substance are systematized by Hartshorne (see [6]) in S5 modal propositional logic as the following theorem:

$$\Diamond Q,\ Q \rightarrow \Box Q \vdash \Box Q$$

Gödel expressed his version of the ontological argument in S5 second-order modal logic by deriving $\Diamond Q$ from the definition of the "God-like being" as having all „positive attributes", positive in both moral and aesthetic sense, independently of the accidental structure of the world (see [3]).

Gödel's reasons for his interest in the ontological argument are most clearly expressed in the following quote (see [5]): "I believe, already to be possible to perceive by pure reason (without appealing to the faith in any religion) that the theological worldview is thoroughly compatible with all known data (including the conditions that prevail on our earth). The famous philosopher and mathematician Leibniz already tried to do this 250 years ago, and this is also what I tried in my previous letters (ontological argument)."

Consequence and proof in modal logic

There is no doubt that the propositional skeleton of the logic of Gödel's argument is the modal logic S5, or something close to it. Besides the axioms of classical propositional logic, the modal axioms of the logic S5 are

$$\Box\,(A\to B)\to(\Box A\to\Box B)$$
$$\Box A\to A,$$
$$\Box A\to\Box\Box A,$$
$$\Diamond\Box A\to\Box A$$

where $\Box$ is the necessity operator and, $\Diamond$ is the possibility operator defined by $\Diamond A\leftrightarrow\neg\Box\neg A$, and the inference rules are modus ponens and necessitation: from A infer $\Box A$. The last axiom is usually called the S5 axiom. This elegant and simple axiomatization of modal logic was made possible by Gödel's introduction of the necessitation rule (see [4]).

The notions of consequence and proof in modal logic are different from those in classical logic. The relation that a sentence A is a consequence of assumptions Σ can have two meanings in modal logic: A is true at each world at which the members of Σ are true, and A is true in every model in which Σ holds. The two notions are not equivalent and to distinguish between them some authors (see [2]) are using the terms *local* assumption and *local* consequence in the first, and the terms *global* assumption and *global* consequence in the second case. We shall show how this semantical distinction is reflected in the syntax of modal logic.

Assume a sentence A globally; if A is true at an arbitrary world w in some model, then A is true at every world accessible to w (since A holds in every world), so $\Box A$ holds at w. Since w is arbitrary, $\Box A$ holds at every world of a model. This means that if A is a global assumption, the necessitation rule can be applied to A. On the other hand, if we assume A locally, so that A is known to be true at a world w of some model, there is no reason to expect that $\Box A$ is also true at w. If A is a local assumption, the necessitation rule cannot apply to it.

The distinction between global and local assumptions in formal deductions comes down to the applicability or nonapplicability of the necessitation rule. A formal proof or derivation in modal logic does not allow the use of the necessitation rule to local premises and their consequences. To insure this, some authors define modal derivations as finite sequences divided in two separate parts, global and local (see [2]). The global part comes first, containing only global premises and the necessitation rule is allowed, while the local part comes second containing local premises, but without the necessitation rule.

Necessitation in Gödel's argument

It is well known that Gödel was involved in the foundation of the modern approach to modal logic. He was among the first logicians who introduced the necesitation rule that made possible the simple and elegant modal axiom systems that are in use today. But in the early 1970s, at the time Gödel wrote his note about the ontological argument, the idea of possible world semantics was new and perhaps not well appreciated. Gödel argument is modal and is presented in at least second-order logic, however the exact logic is not specified.

According to what we have told about consequence in modal logic, to allow the unrestricted use of the necessitation rule in the logic S5, we have to assume the axioms of our theories globally. In the modal logic S5, where $\Box A \leftrightarrow \Box\Box A$, assuming A globally we assume $\Box A$. Formally, this means that all axioms of the theories in the S5 logic must come in the necessitated form, i.e. with $\Box$ prefixed.

Gödel's argument is a particular version of the general ontological argument that usually means two things: to prove that God's existence is possible and to prove that God exists necessarily if He exists. If Q is the statement that God exists, this means that in the general ontological argument we have to prove $\Diamond Q$ and $Q \rightarrow \Box Q$ (Anselm's principle). It is generally accepted that with these assumptions within S5 logic one can prove $\Box Q$: the necessitation of Anselm's principle gives $\Diamond Q \rightarrow \Diamond\Box Q$, the S5 axiom gives $\Diamond Q \rightarrow \Box Q$ and the first assumption finally gives $\Box Q$ (see [1], [3], and [5]). But the use of necessitation in this proof was not correct. It seems that the only way to overcome this incorrectness is to formulate Anselm's principle in the form $\Box(Q \rightarrow \Box Q)$: it is necessary that God exists necessary if He exists.

At some point in his note, relying on axioms that are not formulated in necessitated form, Gödel presents the theorem

$$G(x) \rightarrow \Box \exists y G(y)$$

where $G(x)$ means that "x is godlike being" (see page 403 in [3]), and without any comments proceedes in the following three steps:

$$\exists x G(x) \rightarrow \Box \exists y G(y)$$
$$\Diamond \exists x G(x) \rightarrow \Diamond \Box \exists y G(y)$$
$$\Diamond \exists x G(x) \rightarrow \Box \exists y G(y)$$

In the first step the existential quantifier is introduced, the second step comes from the necessitation rule, and the third uses of the S5 axiom. Since he was able to prove $\Diamond\exists x G(x)$, Gödel finally concludes $\Box\exists y G(y)$.

Gödel, as well as his followers and commentators in this matter, say nothing about the local or global character of the ontological argument axioms. They present these axioms in the unnecessitated form (see [2], [3], and [5]), and use the necessitation rule on them and on their consequences. Perhaps they have in mind global axioms?

Acknowledgement. We express our thanks to Kosta Došen who warned us about the problem concerning the correctness of the necessitation rule in the ontological argument.

References

[1] Fitting, M., 2002, *Types, Tableaux and Gödel's God*, Dordrecht, Kluwer.

[2] Fitting, M. and Mendelsohn, R. L., 1990, *First-Order Modal Logic*, Dordrecht, Kluwer.

[3] Gödel, K., 1995, *Ontological Proof*, In *Collected Works, Volume III, Unpublished essays and lectures*, New York, Oxford University Press, pp. 403–404.

[4] Gödel, K., 1933, *An interpretation of the intuitionistic propositional calculus*, In *Collected Works, Volume I*, New York, Oxford University Press, pp. 301.

[5] Háyek, P. 2011, *Gödel's Ontological Proof and Its Variants*, In *Kurt Gödel and the Foundations of Mathematics – Horizons of Truth*, Cambridge, Cambridge University Press, 2011, pp. 307–324.

[6] Sobel, J. H., 1987, *Gödel's Ontological Proof*, In *On Being and Saying: Essays in Honor of Richard Cartwright*, Cambridge, MA, MIT Press, pp. 24–61.