Karakteristike matematičkih metoda i specijalizovanih softvera za dinamičku analizu liftova i izvoznih postrojenja

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U radu je prikazana problematika definisanja dinamičkih modela liftova i izvoznih postrojenja u rudnicima i metode za rešavanja formiranih matematičkih modela u zavisnosti od osnovnih parametara postrojenja. U eksploataciji se sreću različiti tipovi ovih uređaja od rudničkih i svremenih putničkih liftova sa ekstremnim visinama i brzinama dizanja do klasičnih liftova sa relativno malim vrednostima ovih parametara što uslovljava korišćenje različitih dinamičkih modela i matematičkih metoda za njihovo rešavanje. Pored analize karakteristika i pogodnosti primene pojedinih matematičkih metoda u radu je dat prikaz korišćenja specijalizovanih softvera za dinamičku analizu liftova i izvoznih postrojenja.

Ključne reči: Mašine vertikalnog transporta, Mehanički modeli, Analitičke i numeričke metode

0. UVOD

U sistemima vertikalnog transporta, posebno kod savremenih putničkih liftova i izvoznih postrojenja u rudnicima (rudnički liftovi) su izražena dinamička opterećenja. Kod liftova u visokogradnji uglavnom je u primeni sistem sa pogonskom užetnjacom dok se u rudnicima koriste dva sistema za prenos snage sa pogonskog mehanizma na noseće elemente (užad) i to sa dobošem i pogonskom užetnjacom (Koepe sistem). Kod sistema sa dobošem za dizanje, noseća užad se namotavaju i skladište na doboš. Sistemi dizanja putem trenja (Koepe) se najčešće primenjuju u evropskim rudnicima.

Formiranje mehaničkih modela sistema vertikalnog transporta zavisi od velikog broja parametara. U zavisnosti od mesta ugradnje i kinematskih veličina (visina i brzina dizanja) potrebno je na različite načine pristupiti modeliranju nosećih elemenata.

Analiza sistema i mašina vertikalnog transporta podrazumeva formiranje matematičkih modela kroz sisteme diferencijalnih i algebarskih jednačina. U literaturi je dostupan veliki broj analitičko-numeričkih metoda i postupaka za rešavanje matematičkih modela. Na sl. 1 dat je algoritam [8], za rešavanje realnih problema.

Sl. 1: Dijagram toka za rešavanje matematičkih modela [8]

1. MODELI SISTEMA VERTIKALNOG TRANSPORTA I METODE ZA REŠAVANJE

Merodavan dinamički model za opisivanje dinamičkog ponašanja uređaja sa pogonskom užetnjacom prikazan je na sl. 2. Model predstavlja model liftova i izvoznih postrojenja sa „teškim“ užetom promenljive dužine.

Za liftove velikih visina dizanja (do 2000 m i više) i brzina kretanja (do 20 m/s), pored težine slobodnog dela užeta, značajno se menja i osnovni parametar dinamičkih modela – krutost EA/L . Ovakav model se mora primeniti kod savremenih liftova, kao i izvoznih postrojenja pri analizi dinamičkog ponašanja i stabilnosti kretanja sistema.

Sl. 2: Dinamički model lifta [3], [4]

Model uzima u razmatranje uticaj promene slobodne dužine užeta na nailaznoj i silaznoj strani pogonske užetnjace. Promena slobodne dužine užeta (skraćenje i produženje) direktno utiče na krutost užeta, atakođe i na dinamičko ponašanje užeta, koje je od izuzetnog značaja i sistemima sa velikim brzinama dizanja. Uticaj elastičnog klizanja užadi preko pogonske užetnjace obuhvaćen je kroz adekvatne granične uslove.

Prema [4] i sl. 2, postavljajući uslov ravnoteže elementarnih delova nailaznog i silaznog kraka užeta i ravnotežu momenata na pogonskoj užetnjači, dobija se sledeći sistem jednačina:

$$\frac{q}{g} \cdot \frac{\partial^2 u_1(x,t)}{\partial t^2} = E \cdot A \cdot \frac{\partial^2}{\partial x^2} \left[u_1(x,t) + b \cdot \frac{\partial u_1(x,t)}{\partial t} \right] + q \cdot \left(1 \pm \frac{a}{g} \right) \quad (1)$$

$$\frac{q}{g} \cdot \frac{\partial^2 u_2(x,t)}{\partial t^2} = E \cdot A \cdot \frac{\partial^2}{\partial x^2} \left[u_2(x,t) + b \cdot \frac{\partial u_2(x,t)}{\partial t} \right] + q \cdot \left(1 \pm \frac{a}{g} \right) \quad (2)$$

$$M_m = \frac{R}{i \cdot \eta} \cdot E \cdot A \cdot \frac{\partial}{\partial x} \left\{ u_1(l_1, t) - u_2(l_2, t) + \frac{\partial}{\partial t} [u_1(l_1, t) - u_2(l_2, t)] - J_r \cdot \frac{a \cdot i}{\eta} \right\} \quad (3)$$

gde je:

u_1, u_2 - elastična deformacija užeta, m

E - modul elastičnosti, Pa

A - površina poprečnog preseka užeta, m²

a - ubrzanje pogonskog mehanizma, ms⁻²

M_m - moment pogonskog elektromotora, Nm

i - prenosni odnos reduktora,

η - stepen iskorišćenja pogonskog mehanizma,

J_r - moment inercije rotacionih masa, redukovan na vratilo pogonske užetnjače, kgm²

R - poluprečnik pogonske užetnjače, m

q - težina užeta po dužnom metru, Nm⁻¹

Za rešavanje sistema parcijalnih diferencijelnih jednačina (1) i (2) neophodno je definisati odgovarajuće granične uslove koji zavise od gore naznačenih parametara postrojenja. U ovom radu je dat pregled analitičkih i numeričkih metoda i njihovih karakteristika pogodnih za rešavanje matematičkih modela, koje opisuju dinamičko ponašanje mašina vertikalnog transporta.

1.1. Analitičko rešavanje matematičkog modela

Kod postrojenja sa ekstremnim vrednostima brzina i visina dizanja, ako se na osnovu sl. 2, odnosno sistema jednačina (1), (2) i (3) posmatra samo nailazna strana užeta na bubanj ili užetnjaču može se napisati:

$$\frac{q}{g} \cdot \frac{\partial^2 u}{\partial t^2} = E \cdot A \cdot \frac{\partial^2}{\partial x^2} \left(u + b \cdot \frac{\partial u}{\partial t} \right) + q \quad (4)$$

Granični uslov na mestu nailaska užeta na užetnjaču bez klizanja, sl. 3 (tačka C), dat je u obliku:

$$u_1(l_1, t) = \int_0^t \frac{\partial u_1(l_1, t)}{\partial x} \left(\frac{dl_1}{dt} \right) \cdot dt \quad (5)$$

gde je:

l_1 - namotana dužina užeta, m

$dl_1/dt = v$ - obimna brzina, ms⁻¹

Granični uslov u tački veze užeta sa kabinom lifta (košem izvoznog postrojenja) ili protivtegom, kada se zanemari sila trenja između klizača i vođica, sl. 3, dat je u obliku:

$$u_1(x, t) = - \int_{l_1}^L K \cdot q(y) \cdot \left[\frac{\partial^2 u_1}{\partial t^2} - g - a \right] \cdot dy + \int_0^t \frac{\partial u_1(l_1, t)}{\partial x} \left[\left(\frac{dl_1}{dt} \right) - \frac{d}{dt} (\Delta l_1) \right] \cdot dt + \Delta l \cdot \frac{\partial u_1(l_1, t)}{\partial x} + E \cdot A \int_{l_1}^L K \cdot \frac{\partial^3 u_1}{\partial s^2 \partial t} \cdot ds - b \cdot E \cdot A \cdot \frac{\partial^2 u_1(l_1, t)}{\partial x \partial t} \cdot K(x, l_1, l_1) \quad (9)$$

Konačno rešenje jednačine (9) dato je u [14].

Parcijalna diferencijalna jednačina (4) se može prevesti u sistem običnih diferencijalnih jednačina primenom metode razdvajanja promenljivih. Primenjujući metod partikularnih integrala, prema [12], dobija se:

$$u(x, t) = X(x) \cdot T(t) \quad (10)$$

$$Q = E \cdot A \cdot \frac{\partial}{\partial x} \left(u_1(L_1, t) + b \cdot \frac{\partial u_1(L_1, t)}{\partial t} \right) + \left(\frac{Q}{g} \cdot \frac{\partial^2 u_1(L_1, t)}{\partial t^2} - a \right) \quad (6)$$

Uže zanemarljive težine (beztežinska nit) opterećeno u tački M silom F_i prikazano je na sl. 3. U području $y < x$ deformacija užeta (bez klizanja preko pogonske užetnjače) bi se menjala od nule do granične vrednosti $u(y)$, dok bi u delu užeta koje je ispod napadne tačke M pomeranje svih tačaka ostalo isto graničnoj vrednosti $u(y)$. Prema ovome, za jediničnu silu ($F_i=1$) i za pomeranje pojedinih tačaka užeta, dobija se tzv. funkcija uticaja, prema [9], kao:

$$K(x, y) = \begin{cases} \frac{x-l}{E \cdot A} & \text{for } y > x \\ \frac{y-l}{E \cdot A} & \text{for } y < x \end{cases}$$

a pomeranje bilo koje tačke užeta: $u(x) = K(x, y, l) \cdot F_i$.

Sl. 3: Deformacija užeta [4]

U slučaju delovanja većeg broja sila sa napadnim tačkama u $x = y_i$, pomeranje ima sledeći oblik:

$$u(x) = \sum_{i=1}^n K(x, y_i, l) \cdot F_i \quad (7)$$

Za slučaj da na uže deluje ravnomerno raspoređeno opterećenje, na osnovu gornjeg izraza može se napisati:

$$u(x) = \int_l^L K(x, y, l) \cdot q \cdot dy \quad (8)$$

Nakon primene ovog postupka na diferencijalnoj jednačini (1), zamenjujući argumente sa (y) , množeći ih sa funkcijom $K(x, y, l_1) = K$ i obavljanjem neophodnih matematičkih transformacija, prema [6] i [11], deformacija užeta dobija sledeću formu:

Prevođenjem parcijalne jednačine u sistem običnih diferencijalnih jednačina one se mogu svesti na Košijev problem odnosno tzv. Metod neodređenih koeficijenata:

$$y' = f(x, y), \quad y(x_0) = y_0$$

Rešenje ovog problema je u obliku reda:

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots \quad (11)$$

gde je:

$a_0, a_1, a_2, \dots$ - nepoznati koeficijenti.

Na osnovu početnog uslova $y(x_0) = y_0$, dobija se za prvi koeficijent $a_0 = y_0$. Ostali koeficijenti se dobijaju izjednačavanjem:

$$y'(x) = f(x, y(x)) \quad (12)$$

gde je:

$$y'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \dots$$

Iz uslova da leva i desna strana u izrazu (12) budu jednake, određuju se koeficijenti a_k .

Slično tome Košijev problem $y(x_0) = y_0$ može se prikazati u ekvivalentnom obliku tzv. Pikardov (eng. Picard) metod sukcesivnih aproksimacija:

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t)) dt \quad (13)$$

Polazeći od neke početne aproksimativne funkcije $y^{[0]}(x)$, na osnovu (13) može se sprovesti niz uzastopnih aproksimacija $\{y^{[k]}\}_{k \in N}$ pomoću formule:

$$y^{[n+1]}(x) = y_0 + \int_{x_0}^x f(t, y^{[n]}(t)) dt \quad (14)$$

Detaljnije obrazloženje Pikardovog metoda dato je u [7].

Za razliku od mašina vertikalnog transporta velikih visina dizanja i brzina kretanja, kod mašina relativno malih brzina kretanja (do 3 m/s) i malih visina dizanja (do 25÷35 m) mehanički modeli su mnogo jednostavniji i mogu se posmatrati kao modeli sa jednim (poznat kinematski uslov u tački namotavanja užeta), sl. 4a, ili dva stepena slobode kretanja, u zavisnosti od „vučnih“ i inercijalnih karakteristika pogonskog mehanizma. Modeli mašina vertikalnog transporta (liftova) sa dva stepena slobode kretanja prikazani su na sl. 4b, kod kojih je M_1 - redukovana masa pogonskog dela, M_2 - masa kabine i tereta, i P - pogonska sila. Formiranje i rešavanje ovakvih dinamičkih modela se svodi na klasične primere iz teorije oscilacija pa se ovde neće detaljnije razmatrati.

Sl. 4: Dinamički modeli sa jednim (a) i dva stepena slobode (b) i užetom predstavljeno u obliku Hukovog i Kelvinovog modela

Na osnovu ovog kratkog pregleda analitičkih metoda može se uočiti da je rešavanje dinamičkih modela ograničeno samo na specijalne slučajeve, pa je najčešće neophodno pristupiti određivanju približnog rešenja odnosno primeni numeričkih metoda.

1.2. Numeričko rešavanje matematičkog modela

Numeričko rešenje je poželjno čak i u slučajevima kada postoji rešenje u konačnom obliku, ali je vrlo komplikovano. Treba imati u vidu da se numeričko rešenje diferencijalne jednačine dobija u kvantitativnom obliku tako da analitički izraz za funkciju ostaje i dalje nepoznat.

U osnovi većine numeričkih postupaka se nalazi Ojlerov metod. Konačno rešenje se ne dobija u obliku izraza već u obliku tabele približnih vrednosti.

Da bi se pojasnio postupak traženja rešenja primenom Ojlerovog metoda posmatraće se diferencijalna jednačina (Košijev problem) čije se rešenje traži u intervalu $[a, b]$. Interval $[a, b]$ se može podeliti na n podintervala pomoću tačaka $x_0 = a$, $x_k = a + hk$, $(k = 1, \dots, n)$, gde je $h = (b - a)/n$.

Ako se pretpostavi da je funkcija y neprekidna zajedno sa svojim izvodima y' i y'' , tada se na osnovu Tejlrove (eng. Taylor) formule može reći da postoji tačka c_1 između tačaka x_0 i x ($x \in (x_0, x_1)$) takva da je:

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2}y''(c_1) \quad (15)$$

Kako je $y'(x_0) = f(x_0, y_0)$ i $h = x_1 - x_0$, odatle sledi da se za $x = x_1$ može napisati:

$$y(x_1) = y_0 + h \cdot f(x_0, y_0) + \frac{h^2}{2}y''(c_1) \quad (16)$$

Ukoliko je korak h dovoljno mali, može se zanemariti poslednji član na desnoj strani i za aproksimaciju tačne vrednosti $y(x_1)$ uzeti:

$$y_1 = y_0 + h \cdot f(x_0, y_0) \quad (17)$$

Sličnim postupkom na intervalu (x_1, x_2) dobija se aproksimacija tačne vrednosti $y(x_2)$:

$$y_2 = y_1 + h \cdot f(x_1, y_1) \quad (18)$$

Postupak se ponavlja, generišući niz vrednosti:

$$y_{k+1} = y_k + h \cdot f(x_k, y_k), \quad (k = 0, 1, \dots, n-1) \quad (19)$$

Relacija (19) predstavlja opšti oblik pojedinačnog koraka Ojlerovog metoda.

Na ovaj način se dobija niz tačaka (x_k, y_k) , $k = 0, \dots, n$, čijim spajanjem nastaje poligonalna linija koja se zove Ojlerov poligon. Ova poligonalna linija aproksimira grafik tražene funkcije $y = y(x)$ kao što je to prikazano na sl. 5. Na osnovu izraza (19) i sl. 5 vidi se da deo poligonalne linije, koji spaja tačke (x_0, y_0) i (x_1, y_1) , predstavlja deo tangente na krivoj $y(x)$ u tački (x_0, y_0) . Deo linije koji spaja (x_1, y_1) i (x_2, y_2) pripada pravoj paralelnoj tangenti u tački (x_1, y_1) itd.

Sl. 5: Ojlerov poligon

Sa sl. 5 može se zaključiti da Ojlerov metod nije naročito precizan i da zavisi od broja koraka n , tj. od veličine koraka h . Sa iste slike se vidi da će akumulirana greška Ojlerovog metoda biti značajna. U svakom koraku definisanom formulom (19) napravljena je greška jednaka izostavljenom članu $y''(c_k) \frac{h^2}{2}$. Na celom intervalu $[a, b]$ posle n koraka greška iznosi:

$$\sum_{k=1}^n y''(c_k) \frac{h^2}{2} \approx \frac{b-a}{2} y''(c) \cdot h \approx O(h) \quad (20)$$

Metodi Runge-Kuta (eng. Runge-Kutta) se zasnivaju na primeni Tejlrorovog reda, ali izbegavaju izračunavanje izvoda razmatrane diferencijalne jednačine.

Rešenje Košijevog problema u obliku Tejlrorovog reda daje:

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (21)$$

$$y(x+h) = y(x) + h \cdot y'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) + \dots \quad (22)$$

Prema metodu Runge-Kuta, da bi se izbeglo određivanje izvoda i postigla što je moguće veća tačnost koristi se sledeći izraz:

$$y(x+h) = y(x) + \omega_1 k_1 + \omega_2 k_2 \quad (23)$$

gde je:

$$k_1 = h \cdot f(x, y)$$

$$k_2 = h \cdot f(x + \alpha h, y + \beta k_1) = h \cdot f(x + \alpha h, y + \beta h f(x, y))$$

Određivanjem ω_1 , ω_2 , α i β dobija se izraz koji predstavlja metod Runge-Kuta drugog reda u obliku:

$$y(x+h) = y(x) + \frac{h}{2} f(x, y) + \frac{h}{2} f(x+h, y + hf(x, y)) \quad (24)$$

Od daleko većeg praktičnog značaja je metod Runge-Kuta četvrtog reda koji je definisan izrazom:

$$y_{n+1} = y_n + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4) \quad (25)$$

gde je:

$$k_1 = f(x, y),$$

$$k_2 = f\left(x + \frac{h}{2}, y + \frac{h}{2} k_1\right),$$

$$k_3 = f\left(x + \frac{h}{2}, y + \frac{h}{2} k_2\right),$$

$$k_4 = f(x+h, y+hk_3).$$

Veoma često se pri rešavanju dinamičkih problema koristi i Njumarkov (eng. Newmark) metod. Njumarkov metod integracije zasniva se na pretpostavci da se ubrzanje menja linearno između dva vremenska trenutka.

Koristeći Njumarkov metod, prvi izvod po vremenu se rešava kao:

$$\dot{u}_{n+1} = \dot{u}_n + \Delta t \ddot{u}_\gamma \quad (26)$$

$$\dot{u}_{n+1} = \dot{u}_n + (1-\gamma) \Delta t \ddot{u}_n + \gamma \Delta t \ddot{u}_{n+1} \quad (27)$$

gde je:

$$\ddot{u}_\gamma = (1-\gamma) \ddot{u}_n + \gamma \ddot{u}_{n+1}; \quad 0 \leq \gamma \leq 1$$

Zbog toga što se ubrzanje takođe menja sa vremenom, metod se mora proširiti na drugi izvod po vremenu da bi se dobile tačne vrednosti pomeranja. Tako da rešenje dobija oblik:

$$u_{n+1} = u_n + \Delta t \dot{u}_n + \frac{1}{2} \Delta t^2 \ddot{u}_\beta \quad (28)$$

gde je:

$$\ddot{u}_\beta = (1-2\beta) \ddot{u}_n + 2\beta \ddot{u}_{n+1}; \quad 0 \leq 2\beta \leq 1$$

Njumarkovim medodom je pokazano da je razumna vrednost za γ jednaka 0.5, zbog toga se dobija:

$$\dot{u}_{n+1} = \dot{u}_n + \frac{\Delta t}{2} (\ddot{u}_n + \ddot{u}_{n+1}) \quad (29)$$

$$u_{n+1} = u_n + \Delta t \dot{u}_n + \frac{1-2\beta}{2} \Delta t^2 \ddot{u}_n + \beta \Delta t^2 \ddot{u}_{n+1}$$

Podešavajući promenljive vrednosti za β između 0 i 0.5 može se dobiti širok dijapazon rezultata. Praktično, najpouzadaniji rezultati se dobijaju za vrednost $\beta=1/4$.

Slično kao i kod analitičkih metoda, gore navedeni numerički postupci imaju značajna ograničenja za rešavanje realnih dinamičkih problema sistema vertikalnog transporta, prvenstveno zbog potrebe prevođenja parcijalnih u obične diferencijalne jednačine sa neholonomnim graničnim uslovima. Analizom parametara matematičkog modela koji opisuje longitudinalne oscilacije mašina vertikalnog transporta, može se zaključiti da su to jednačine hiperboličnog tipa, zbog čega će u narednom tekstu najveća pažnja biti posvećena metodama i načinima rešavanja, parcijalnih diferencijalnih jednačina hiperboličnog tipa.

1.2.1. Numeričko rešavanje parcijalnih diferencijalnih jednačina

Numeričko rešenje neke parcijalne diferencijalne jednačine dobija se u obliku tabele približnih vrednosti tražene funkcije $u(x,y)$ za ekvidistantne vrednosti nezavisno promenljivih. Zbog opširnosti objašnjenja prikazaće se numeričke metode pogodne za rešavanje parcijalnih diferencijalnih jednačina hiperboličnog tipa, odnosno za dinamičku analizu liftova i izvoznih postrojenja.

Metode konačnih razlika

Metoda konačnih razlika za numeričko rešavanje parcijalnih diferencijalnih jednačina obuhvata:

- diskretizaciju oblasti (domena) nezavisno promenljivih,
- aproksimaciju parcijalnih izvoda konačnim razlikama,
- rešavanje rezultujućeg sistema algebarskih jednačina.

Pri tome unutrašnje tačke oblasti (mreže) zadovoljavaju diferencijalnu jednačinu, a tačke na granici oblasti (domena) zadovoljavaju granične uslove.

Osnovna ideja metode konačnih razlika je u zameni izvoda razmatrane funkcije njihovim približnim vrednostima. U realizaciji te ideje uvode se tačke, odnosno mreža čvorova (otuda naziv – metoda mreža), u kojima se rešenje traži.

Za primenu na specifične slučajeve, kao što su liftovi i izvozna postrojenja razmotriće se jednodimenzionalna parcijalna diferencijalna jednačina hiperboličnog tipa drugog reda:

$$\frac{\partial^2 u}{\partial t^2} - a(x,t) \frac{\partial^2 u}{\partial x^2} = f(x,t) \quad (30)$$

gde je $a(x,t) > 0$ za svako x i $t > 0$.

U graničnoj oblasti za promenljivu koordinatu x (npr. promena dužine slobodnog kraka užeta kod liftova ukupne dužine L) definišu se granični uslovi kao, sl. 6:

$$\begin{cases} u(0,t) = g_1(t) \\ u(L,t) = g_2(t) \end{cases} \quad \text{za } t > 0 \quad (31)$$

a takođe i početni uslovi kao, sl. 6:

$$\begin{aligned} u(x,0) &= u_0(x), \quad x \in \{0, L\} \\ \frac{\partial u}{\partial t} \Big|_{(x,t=0)} &= v_0(x), \quad x \in \{0, L\} \end{aligned} \quad (32)$$

gde je drugi poznat kao tzv. Košijev uslov.

Sl. 6: Mreža čvorova sa graničnim i početnim uslovima

Ako se uvede oznaka $a(x,t) = c^2$, gde c predstavlja pozitivnu konstantu, onda se opšte (analitičko) rešenje jednačine (30) daje u obliku:

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(x) dx \quad (33)$$

Radi lakšeg objašnjenja uvešće se oznake N i K , koje predstavljaju broj tačaka u prostornoj koordinati x i broj tačaka (podela) po koordinati vremena t . Onda i funkcija $u(x,t)$ dobija oblik u_n^k . Uvođenjem dodatnih oznaka:

$$-\frac{1}{2} \lambda a_n^{k+1} u_{n-1}^{k+1} + (1 + \lambda a_n^{k+1}) u_n^{k+1} - \frac{1}{2} \lambda a_{n+1}^{k+1} u_{n+1}^{k+1} = \frac{1}{2} \Delta t^2 (f_n^{k+1} + f_n^{k-1}) + 2u_n^k - \left[\frac{\lambda}{2} a_n^{k-1} u_{n-1}^{k-1} - (1 - \lambda a_n^{k-1}) u_n^{k-1} + \frac{\lambda}{2} a_n^{k-1} u_{n+1}^{k-1} \right] \quad (38)$$

Sl. 7: Ilustracija mreže prostor-vreme kod talasne jednačine

Prema jednačini (38), da bi se došlo do rešenja, takođe se moraju poznavati polje (vrednosti) u tačkama $(k-1)$. To obezbeđuju početni uslovi (36). Koristeći razvoj u Tejlorov red po t dobija se:

$$u_n^k = u_n^{k-1} + \frac{\partial u}{\partial t} \Big|_{k-1} \Delta t + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} \Big|_{k-1} \Delta t^2 \quad (39)$$

Potrebno je odrediti samo u_n^1 , gde se na osnovu jednačine (30) piše:

$$x_n = n \cdot \frac{L}{N}, \quad t_k = k \cdot \frac{T}{K} \quad (34)$$

gde je:

$$0 \leq n \leq N, \quad 0 \leq k \leq K$$

L - ukupna prostorna koorninata (dužina),

T - ukupno (krajnje) vreme integracije,

onda granični i početni uslovi postaju:

$$\begin{cases} u_0^k = g_1(t_k) \\ u_N^k = g_2(t_k) \end{cases} \quad 0 \leq k \leq K \quad (35)$$

$$\begin{cases} u_n^0 = u_0(x_n) \\ \frac{\partial u}{\partial t} \Big|_{n,0} = v_0(x_n) \end{cases} \quad 0 \leq n \leq N \quad (36)$$

Koristeći teoriju centralnih razlika, i zamenjujući odgovarajuće izvode po vremenu i prostornoj koordinati x , može se napisati, [10]:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\Delta t^2} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{1}{2} \left(\frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{\Delta x^2} + \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{\Delta x^2} \right) \end{aligned} \quad (37)$$

gde je:

$$\Delta t = T/K \quad \text{i} \quad \Delta x = L/N.$$

Uvođenjem oznake (skraćenice) $\lambda = \frac{\Delta t^2}{\Delta x^2}$, dobija

se sistem od $(N-1)$ jednačina u zamenu za parcijalnu diferencijalnu jednačinu u oblasti od $(k+1)$ tačaka po vremenu. Da bi se ovaj sistem rešio, mora se poznavati polje u $(k+1)$ -oj tački.

$$\begin{aligned} u_n^1 &= u_n^0 + v_n^0 \Delta t + \frac{1}{2} f_n^0 \Delta t^2 + \\ &+ \frac{1}{2} \lambda a_n^0 (u_{n+1}^0 - 2u_n^0 + u_{n-1}^0) \quad \text{za } (1 \leq n \leq N-1) \end{aligned} \quad (40)$$

Razmatranjem specijalnog slučaja (homogene diferencijalne jednačine) sa $f(x,t) = 0$ i korišćenjem već uvedene pozitivne konstante $a(x,t) = c^2$, dobija se jednostavan sistem algebarskih jednačina:

$$\begin{aligned} (1 + \alpha) u_n^{k+1} - \frac{1}{2} \alpha u_{n+1}^{k+1} - \frac{1}{2} \alpha u_{n-1}^{k+1} &= \\ = 2u_n^k + \frac{1}{2} \alpha u_{n+1}^{k-1} - (1 + \alpha) u_n^{k-1} + \frac{1}{2} \alpha u_{n-1}^{k-1} \end{aligned} \quad (41)$$

gde je:

$$\alpha = c^2 \frac{\Delta t^2}{\Delta x^2}$$

Jednačina (41) predstavlja pogodan oblik i podlogu za rešavanje ovakvih problema numeričkim putem u savremenim softverskim alatima, kao što su MATLAB, Mathcad, Mathematica i sl.

2. PRIMENA METODE KONAČNIH RAZLIKA NA PRIMERU IZVOZNIH POSTROJENJA

U ovom poglavlju će se dati postupak rešavanja diferencijalnih jednačina kretanja (oscilovanja) tereta sa nosećim elementom u obliku štapa („teško“ uže) konstantne dužine. Njega je moguće približno modelirati kao oscilacije sa jednim stepenom slobode kretanja, sa „teškom“ oprugom. Međutim, ovaj problem se još tačnije može modelirati kao sistem sa beskonačnim brojem stepeni slobode kretanja (longitudinalne oscilacije štapa) sa odgovarajućim graničnim uslovima.

Pri longitudinalnim oscilacijama užeta velike mase, sve čestice unutar užeta se kreću paralelno sa osom užeta. Izduženje se naizmenično ponavlja po vremenu t i po prostornoj koordinati x .

Na sl. 8 je prikazano „teško“ uže sa masom na kraju. Ovaj model generalno može da predstavlja izvozno postrojenje u rudnicima, u trenutku kada dođe do zaustavljanja koša i kada on počne slobodno da osciluje (zanemarivanjem trenja u vodicama).

Sl. 8: Model „teškog“ užeta sa okačenim teretom (kabinom, košem) na kraju

Ulazni parametri za analizu dinamičkog modela sa sl. 8 su:

$L = 500$ m - ukupna slobodna dužina užadi u trenutku zaustavljanja koša,

$M = 30$ t - redukovana oscilujuća masa,

$A = 1700$ mm² - ukupna površina metalnog poprečnog preseka užadi,

$E = 1,3 \cdot 10^5$ MPa - modul elastičnosti užeta za odabrane parametre.

Podaci (navedeni parametri) su usvojeni za konkretno izvozno postrojenje dobijeni merenjima i kasnijim određivanjem pojedinih dinamičkih parametara.

Diferencijalna jednačina kretanja je parcijalna diferencijalna jednačina hiperboličnog tipa opisana izrazom (30) za $f(x, t) = 0$ i $a(x, t) = c^2$, što je napisano izrazom:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (42)$$

gde je konstanta c jednaka:

$$c^2 = \frac{E}{\rho}$$

i predstavlja brzinu prostiranja talasa, a $\rho = 7,86$ t/m³ gustinu materijala žica u nosećim užadima.

Granični uslovi se definišu za krajeve užeta. Na gornjem kraju za $x = 0$, na osnovu izraza (31), prvi granični uslov je:

$$u(0, t) = 0 \quad (43)$$

Drugi granični uslov se dobija za donji kraj užeta, na mestu veze sa teretom. Na osnovu izvođenja datih u [12] i primenjene metode razdvajanja promenljivih, dobijeno je da je drugi granični uslov:

$$\frac{\partial u(L, t)}{\partial x} - \frac{M \cdot k^2}{A \cdot \rho \cdot c^2} u(L, t) = 0 \quad (44)$$

gde je konstanta k za ovaj slučaj jednaka, [12]:

$$k = (2m - 1) \frac{\pi}{2 \cdot L}, \quad (m = 1, 2, 3, \dots)$$

„Teško“ uže je u početnom trenutku opterećeno težinom ($M \cdot g$), pa se njegov kraj istegne za, [12]:

$$\Delta L = \frac{M \cdot g \cdot L}{E \cdot A}$$

a ostali delovi se pomeraju linearno pa je:

$$u(x, 0) = \frac{\Delta L}{L} \cdot x = \frac{M \cdot g}{E \cdot A} \cdot x \quad (45)$$

Izraz (45) predstavlja prvi početni uslov. Drugi početni uslov se definiše na osnovu činjenice da je ceo sistem u početku bio u miru, tj. da su svi delovi „teškog“ užeta imali brzinu jednaku nuli.

$$\frac{\partial u(x, 0)}{\partial t} = 0 \quad (46)$$

Rešenje diferencijalne jednačine kretanja (oscilovanja) „teškog“ užeta dobijeno je metodom konačnih razlika u softverskom paketu MATLAB, korišćenjem eksplicitne metode za rešavanje jednačina hiperboličnog tipa.

Korišćenjem navedenih graničnih i početnih uslova, i pridržavajući se metode centralnih razlika prikazane u prethodnom poglavlju, vodeći računa da je $v_n^0 = 0$, dobijeni su rezultati prikazani na sl. 9.

Sl. 9: Zavisnost izduženja od vremena t i prostorne koordinate x

3. PRIMENA SPECIJALIZOVANIH SOFTVERA I RAČUNARSKE SIMULACIJE DINAMIČKOG PONAŠANJA RUDNIČKOG LIFTA

Prethodni postupak se ne može primeniti za rešavanje sistema diferencijalnih jednačina sa promenljivim (neholonomnim) graničnim uslovima pa je rešenje moguće potražiti primenom specijalizovanih softvera za dinamičku analizu. Jedan od takvih softvera, posebno pogodan za analizu dinamičkog ponašanja sistema u obliku mehanizama je softver MSC ADAMS.

Karakteristika pogona je predstavljena putem zadavanja brzine, odnosno promene broja obrtaja pogonske užetnjače. U ovom radu, promena brzine je zadata u obliku „spline-a“ koji jednoznačno opisuje promenu broja obrtaja pogonskog elektromotora svedenog na obimnu brzinu doboša, odnosno pogonske užetnjače, dobijene pomoću inkrementalnog enkodera u okviru merenja sprovedenih na rudničkom liftu. „Spline“, korišćen u softveru, predstavlja osrednjenu liniju izmerene obimne brzine na pogonskoj užetnjači čime je odstranjen tzv. šum merenja, sl. 10.

Sl. 10: Formiranje „spline-a“ brzine kao pogonske karakteristike na užetnjači

Izgled dinamičkog modela korišćenog za simulacije u softveru MSC ADAMS prikazan je na sl. 11.

Sl. 11: Računarski model izvoznog postrojenja

Za verifikaciju dinamičkog modela prikazani su dijagrami oscilovanja, tokom celog perioda kretanja koša, kao i nakon potpunog zaustavljanja užetnjače za navedene karakteristične slučajeve, „preklopljeni“ sa dijagramima dobijenim merenjima. Kao najpovoljniji, za verifikaciju, izabran je dijagram ubrzanja prikazan na sl. 12.

Sl. 12: Dijagrami ubrzanja koša

Na osnovu prethodnog dijagrama može se konstatovati da dinamički model sa dinamičkim parametrima određenim na osnovu eksperimenta sa zadovoljavajućom tačnošću opisuje realno ponašanje rudničkog lifta, što omogućuje izvođenje računarskih simulacija radi analize realnih opterećenja izvoznog postrojenja. Prikazani postupak predstavlja nov pristup, koji omogućuje analizu dinamičkog ponašanja kod sistema za vertikalno dizanje, koji se nalaze u eksploataciji.

Upoređujući rezultate numeričke analize sa rezultatima merenja, korisno bi bilo ustanoviti neku numeričku meru za korelaciju rezultata. U ovom radu nije vršeno određivanje koeficijenta korelacije, što inače može biti jedan od predloga za korake daljih istraživanja. Prikazani deo istraživanja ukazuje mogućnosti da se odrede realne vrednosti dinamičkih parametara (modul elastičnosti, prigušenje, ...) i da se u slučaju velikih brzina dizanja analizara uticaj promenljive krutosti (EA/L) i brzine kretanja (neholonomni granični uslov u nailaznoj tački užeta na užetnjaču) na ponašanje sistema vertikalnog transporta što nije bilo moguće prikazanim analitičkim i numeričkim metodama.

Kao ilustracija mogućnosti dinamičke analize, na narednim slikama prikazani su dijagrami promene pojedinih veličina, izrađeni u softveru MSC ADAMS, koristeći realne eksperimentalne podatke.

Sl. 13: Dijagram promene položaja, brzine i ubrzanja koša

Sl. 14: Uporedni dijagrami promene izduženja, koeficijenta krutosti i parametra prigušenja noseće užadi

Sl. 15: Dijagram promene sile u užadima

4. ZAKLJUČAK

Analizom parametara konkretnog postrojenja moguće je složeni dinamički model znatno uprostiti, izostavljanjem malih veličina višeg reda i dobiti pogodan model za dinamičku analizu.

Za pogonske mehanizme koji se koriste za vertikalno dizanje tereta (liftovi i dizalice), neophodno je još u fazi projektovanja obezbediti adekvatne uslove za ispravnu dinamičku analizu ponašanja njihovih delova, a posebno užadi kao osnovnih elemenata. Zbog značajnog uticaja promene slobodne dužine užeta, njegovog klizanja preko pogonske užetnjače i njegovih mehaničkih karakteristika, nemoguće je primeniti klasični dinamički model uzdužnih oscilacija homogenog štapa konstantne dužine, posebno kod liftova velikih brzina i velikih visina dizanja (ekspres liftovi i izvozna postrojenja u rudnicima).

Sistem parcijalnih diferencijalnih jednačina hiperboličnog tipa moguće je rešiti primenom numeričkih metoda (MATLAB), ali samo za slučajeve nakon zaustavljanja (kočenja) pogonskog mehanizma, pri čemu se uzima da krutost odgovara trenutnim vrednostima slobodne dužine užeta kao konstantne veličine sa odgovarajućim graničnim uslovima. Da bi se mogla vršiti analiza dinamičkog ponašanja sistema u toku kretanja (promenljiva krutost i neholonomni granični uslovi), neophodno je primeniti specijalizovane softverske pakete sa posebno definisanim (prilagođenim) parametrima dinamičkih modela koji se mogu dobiti merenjima na realnim sistemima vertikalnog transporta.

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Characteristics of mathematical methods and specialized software systems for dynamic analysis of elevators and mining elevators

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The paper presents the issues with the elevators and the mining elevators dynamic models and methods for formed mathematical models solving according to basic parameters of exploitation facilities in mines. During the exploitation one can encounter different types of these machines, from mining and contemporary passenger elevators with extreme heights and lifting velocities, to the classic elevators with relatively small parameter values which leads to the usage of different dynamic models and mathematical methods for their solving. Besides the features analyses and the convenience of applying certain mathematical methods, the paper also provides the overview of using specialized software for dynamic analysis of elevators and export facilities.

Keywords: Machines of vertical transport, Mechanical models, Analytical and numerical methods

0. INTRODUCTION

Dynamic burdening is expressed in the vertical transport systems, especially at the contemporary passenger elevators and exploitation facilities in mines (mine elevators). Elevators in high rise constructions usually use the driving pulley system, while the mines use two systems for power transfer from the drive mechanism to the hoisting elements (ropes) with the drum and a driving pulley (Koepe system).

At systems, which have a drum for lifting, the driving ropes are winding and stored into the drum. The systems of hoisting by friction (Koepe) are most common in European mines.

Forming the mechanical models of vertical transport system depends on a great number of parameters. Depending on the place of installation and kinematic sizes (height and lifting velocity), modeling of the hoisting elements is done in different ways.

Analysis of the vertical transport systems and machines means that we must form mathematical models with systems of differential and algebraic equations. The literature provides a large number of analytical-numerical methods and ways for solving mathematical models. Fig. 1 shows algorithm [8] for solving realistic problems.

Fig. 1: Algorithm for mathematical models solving [8]

1. MODELS OF VERTICAL TRANSPORT SYSTEMS AND METHODS FOR SOLVING

The proper dynamic model for describing dynamic behavior of the device with a driving pulley is shown in Fig. 2. The model represents a model of elevators and exploitation facility in mines with a “heavy” rope of a variable length.

For the elevators with big lifting heights (to 2000 m and more) and velocity movements (to 20 m/s), apart from the weight of the rope’s free end, there is a significant change in the basic parameter of dynamic models – stiffness EA/L . Such model has to be applied with the modern elevators, and with the mining elevators in the analysis of dynamic behavior and the stability of the system’s movements.

Fig. 2: Dynamic model of elevators [3], [4]

The model takes into account the influence of the rope free length variations on both the incoming and outgoing pulley side. The variations in rope free length (shortening or lengthening) directly affect the rope stiffness, and therefore also the dynamic behavior of the rope, which is of significant importance in the systems

with high lifting velocities. The influence of elastic rope slipping over the driving pulley is considered through the adequate boundary conditions.

According to [4] and Fig. 2, on the basis of the equilibrium conditions for the elementary rope length and for the driving pulley, the following system of equations can be established:

$$\frac{q}{g} \cdot \frac{\partial^2 u_1(x,t)}{\partial t^2} = E \cdot A \cdot \frac{\partial^2}{\partial x^2} \left[u_1(x,t) + b \cdot \frac{\partial u_1(x,t)}{\partial t} \right] + q \cdot \left(1 \pm \frac{a}{g} \right) \quad (1)$$

$$\frac{q}{g} \cdot \frac{\partial^2 u_2(x,t)}{\partial t^2} = E \cdot A \cdot \frac{\partial^2}{\partial x^2} \left[u_2(x,t) + b \cdot \frac{\partial u_2(x,t)}{\partial t} \right] + q \cdot \left(1 \pm \frac{a}{g} \right) \quad (2)$$

$$M_m = \frac{R}{i \cdot \eta} \cdot E \cdot A \cdot \frac{\partial}{\partial x} \left\{ u_1(l_1, t) - u_2(l_2, t) + \frac{\partial}{\partial t} [u_1(l_1, t) - u_2(l_2, t)] - J_r \cdot \frac{a \cdot i}{\eta} \right\} \quad (3)$$

where:

u_1, u_2 - rope elastic deformations, m

E - elasticity modulus, Pa

A - rope cross-section, m²

a - driving mechanism acceleration, ms⁻²

M_m - driving motor torque, Nm

i - gear ratio,

η - driving mechanism efficiency,

J_r - moment of inertia of rotating masses, reduced to the pulley shaft, kgm²

R - driving pulley radius, m

q - rope weight pro meter, Nm⁻¹

For solving the system of partial differential equations, (1) and (2), it is necessary to define the appropriate boundary conditions which depend on the previously mentioned parameters of the facility. This paper gives an overview of analytical and numerical methods and their features suitable for solving mathematical models which describe the dynamic behavior of vertical transport machines.

1.1. Analytical solving of the mathematical model

At elevators and mining elevators with great values of lifting velocities and heights, if we observe only the rope incoming side on the drum or driving pulley, based on Fig. 2, i.e. the system of equations (1), (2) and (3), it can be written:

$$\frac{q}{g} \cdot \frac{\partial^2 u}{\partial t^2} = E \cdot A \cdot \frac{\partial^2}{\partial x^2} \left(u + b \cdot \frac{\partial u}{\partial t} \right) + q \quad (4)$$

Boundary condition for rope incoming point on the pulley without slipping, Fig. 3 (point C), is given in the form:

$$u_1(l_1, t) = \int_0^t \frac{\partial u_1(l_1, t)}{\partial x} \left(\frac{dl_1}{dt} \right) \cdot dt \quad (5)$$

where:

l_1 - wound rope length, m

$dl_1/dt = v$ - winding velocity, ms⁻¹

$$u_1(x, t) = - \int_{l_1}^L K \cdot q(y) \cdot \left[\frac{\partial^2 u_1}{\partial t^2} - g - a \right] \cdot dy + \int_0^t \frac{\partial u_1(l_1, t)}{\partial x} \left[\left(\frac{dl_1}{dt} \right) - \frac{d}{dt} (\Delta l_1) \right] \cdot dt + \Delta l_1 \cdot \frac{\partial u_1(l_1, t)}{\partial x} + E \cdot A \int_{l_1}^L K \cdot \frac{\partial^3 u_1}{\partial s^2 \cdot \partial t} \cdot ds - b \cdot E \cdot A \cdot \frac{\partial^2 u_1(L_1, t)}{\partial x \cdot \partial t} \cdot K(x, L_1, l_1) \quad (9)$$

The final solution of (9) is provided in [14].

Partial differential equation (4) can be translated into the ordinary differential equations system by the variables separation method. Applying the method of particular integrals, e.g. [12], given as:

$$u(x, t) = X(x) \cdot T(t) \quad (10)$$

Boundary condition for the connection point of the rope with the elevator car or counterweight, when the friction forces between the sliding shoes and guiding rails are neglected, Fig. 3, is given as:

$$Q = E \cdot A \cdot \frac{\partial}{\partial x} \left(u_1(L_1, t) + b \cdot \frac{\partial u_1(L_1, t)}{\partial t} \right) + \left(\frac{Q}{g} \cdot \frac{\partial^2 u_1(L_1, t)}{\partial t^2} - a \right) \quad (6)$$

A weightless string loaded at point M with the force F_i is shown in Fig. 3. The magnitude of rope point displacement (without slipping over the driving pulley) within the range $y < x$ shows linear growth from zero to the boundary value $u(y)$, while the displacement beneath the point M is of a steady value and equals to the boundary value $u(y)$. According to this, for an elementary force ($F_i=1$), deformation can be defined, e.g. [9], as:

$$K(x, y) = \begin{cases} \frac{x-l}{E \cdot A} & \text{for } y > x \\ \frac{y-l}{E \cdot A} & \text{for } y < x \end{cases}$$

and displacement as $u(x) = K(x, y, l) \cdot F_i$.

Fig. 3: Deformation of the rope [4]

In case of several forces acting in points at $x = y_i$, the displacement has the following form:

$$u(x) = \sum_{i=1}^n K(x, y_i, l) \cdot F_i \quad (7)$$

For the load case with evenly distributed rope load mass, the expression takes the form:

$$u(x) = \int_l^L K(x, y, l) \cdot q \cdot dy \quad (8)$$

After applying this procedure to the differential equation (1), replacing floating argument with (y) , multiplying it with the function $K(x, y, l_1) = K$ and performing necessary mathematical transformations, e.g. [6] and [11], the rope deformation becomes as follows:

By translating of the partial equation into the ordinary differential equations system, they can reduce on Cauchy's problem, i.e. the so-called Undetermined coefficients method:

$$y' = f(x, y), \quad y(x_0) = y_0$$

its solution is in the form of an order:

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots \quad (11)$$

where:

$a_0, a_1, a_2, \dots$ - represent unknown coefficients.

If we take the basic condition $y(x_0) = y_0$, the first coefficient is $a_0 = y_0$. The rest of them are obtained by equaling:

$$y'(x) = f(x, y(x)) \quad (12)$$

where:

$$y'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \dots$$

With the condition that the left and right side in (12) are the same, we determine coefficients a_k .

Similar to that, Cauchy's problem can be presented $y(x_0) = y_0$ in the equivalent form the so-called Picard's method of successive approximations:

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t)) dt \quad (13)$$

Starting with a basic approximative function $y^{[0]}(x)$, based on (13) we can construct a lot of successive approximations $\{y^{[k]}\}_{k \in N}$ by using the formula:

$$y^{[n+1]}(x) = y_0 + \int_{x_0}^x f(t, y^{[n]}(t)) dt \quad (14)$$

A more detailed explanation of Picard's method is provided in [7].

As opposed to the machines of vertical transport for great lifting heights and velocity movements, the machines with relatively small velocity (to 3 m/s) and small lifting heights (to 25÷35 m) have mechanical models that are much simpler and can be observed as models with one (a familiar kinematic condition at the point of winding rope), Fig. 4a, or two degrees of freedom, depending on the "pulling" features and inertial characteristics of a driving mechanism. Vertical transport machine models (elevators) with two degrees of freedom are shown in Fig. 4b, where M_1 – is a reduced mass of the driving part, M_2 – is the mass of the cabin and load, and there's P – driving force. Forming and solving of such dynamic models comes down to classic examples in the oscillation theory so they will not be discussed any further in this paper.

Fig. 4: Dynamic models with one (a) and two degrees of freedom (b) and wire ropes in the form of Hook's and Calvin's model

Considering this brief overview, it can be noted that solving dynamic models is limited only to some special cases, so most often it is necessary to determine an approximate solution, i.e. use numerical methods.

1.2. Numerical solving of a mathematical model

Numerical solution is desirable even in the cases when there is a finite solution, but it's very complicated. We should bear in mind that the numerical solution of a differential equation is obtained in a quantitative form so that the analytical expression for the function remains unknown.

The Euler's method is basically for the majority of numerical procedures. Final solution does not come in the form of an expression but in the form of the table of approximate values.

In order to explain the steps for the Euler's method we will observe the differential equation (Cauchy's problem) whose solution is searched for in the interval $[a, b]$. Interval $[a, b]$ can be divided in n subintervals with $x_0 = a, x_k = a + hk$ ($k = 1, \dots, n$), in which $h = (b - a)/n$.

If we suppose that the y function is without interruptions together with its derivatives y' and y'' , then based on Taylor's formula we can state that there is c_1 point between x_0 and x ($x \in (x_0, x_1)$) which means that:

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2}y''(c_1) \quad (15)$$

Since $y'(x_0) = f(x_0, y_0)$ and $h = x_1 - x_0$, it follows that $x = x_1$ and we can therefore write:

$$y(x_1) = y_0 + h \cdot f(x_0, y_0) + \frac{h^2}{2}y''(c_1) \quad (16)$$

If the h step is small enough, we can neglect the last member on the right side and take the following for the approximation of the exact value $y(x_1)$:

$$y_1 = y_0 + h \cdot f(x_0, y_0) \quad (17)$$

Should we apply a similar procedure on the (x_1, x_2) interval, we can obtain the approximation of the exact value $y(x_2)$:

$$y_2 = y_1 + h \cdot f(x_1, y_1) \quad (18)$$

The procedure is repeated, and it generates the following values:

$$y_{k+1} = y_k + h \cdot f(x_k, y_k), \quad (k = 0, 1, \dots, n-1) \quad (19)$$

Relation (19) represents the general form of an individual step in Euler's method.

In this way we gain a row of points (x_k, y_k) , $k = 0, \dots, n$, whose connecting produces a polygonal line called Euler's polygon. This polygonal line is approximating the graph of the desired function $y = y(x)$ just as shown in Fig. 5. By looking at (19) and Fig. 5 it can be observed that the part of the polygonal row connecting points (x_0, y_0) and (x_1, y_1) , represents the part of the tangent on the $y(x)$ curve at point. The part of the row connecting (x_1, y_1) and (x_2, y_2) belongs to the straight parallel tangent at point (x_1, y_1) , and so on.

Fig. 5: Euler's polygon

Based on Fig. 5 we can conclude that Euler's method isn't very precise and it depends on the n number of steps, i.e. the h steps' size. We can see in the same picture that the Euler's accumulated mistake will be of significance. Within every step defined by (19) there occurs a mistake which is almost equal to the omitted member $y''(c_k) \frac{h^2}{2}$. After n steps, the mistake for the whole $[a, b]$ interval would be:

$$\sum_{k=1}^n y''(c_k) \frac{h^2}{2} \approx \frac{b-a}{2} y''(c) \cdot h \approx O(h) \quad (20)$$

Runge-Kutta methods based on the application of Taylor's order, but they avoided to calculate the derivatives of the differential equation in question.

The solution of the Cauchy's problem in the form of a Taylor's order is:

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (21)$$

$$y(x+h) = y(x) + h \cdot y'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) + \dots \quad (22)$$

According to the Runge-Kutta method, so as to avoid determining the derivative, and still get the most precise result, it used the following formula:

$$y(x+h) = y(x) + \omega_1 k_1 + \omega_2 k_2 \quad (23)$$

where:

$$k_1 = h \cdot f(x, y)$$

$$k_2 = h \cdot f(x + \alpha h, y + \beta k_1) = h \cdot f(x + \alpha h, y + \beta h f(x, y))$$

By determining ω_1 , ω_2 , α and β we can obtain the expression that represents the Runge-Kutta method of order two in the form:

$$y(x+h) = y(x) + \frac{h}{2} f(x, y) + \frac{h}{2} f(x+h, y+hf(x, y)) \quad (24)$$

Much more practical is the Runge-Kutta method of order four which is defined as:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (25)$$

where:

$$\begin{aligned} k_1 &= f(x, y), \\ k_2 &= f\left(x + \frac{h}{2}, y + \frac{h}{2}k_1\right), \\ k_3 &= f\left(x + \frac{h}{2}, y + \frac{h}{2}k_2\right), \\ k_4 &= f\left(x + h, y + hk_3\right). \end{aligned}$$

At dynamic problems solving, the Newmark method is used very often. The Newmark integration method is based on the assumption that the acceleration varies linearly between two instants of time.

Using the Newmark method, the first time derivative can be solved as:

$$\dot{u}_{n+1} = \dot{u}_n + \Delta t \ddot{u}_\gamma \quad (26)$$

$$\dot{u}_{n+1} = \dot{u}_n + (1-\gamma)\Delta t\ddot{u}_n + \gamma\Delta t\ddot{u}_{n+1} \quad (27)$$

where:

$$\ddot{u}_\gamma = (1-\gamma)\ddot{u}_n + \gamma\ddot{u}_{n+1}; \quad 0 \leq \gamma \leq 1$$

Because acceleration also varies with time, however, the extended mean value theorem must also be extended to the second time derivative to obtain the correct displacement. Thus:

$$u_{n+1} = u_n + \Delta t \dot{u}_n + \frac{1}{2} \Delta t^2 \ddot{u}_\beta \quad (28)$$

where:

$$\ddot{u}_\beta = (1 - 2\beta)\ddot{u}_n + 2\beta\ddot{u}_{n+1}; \quad 0 \leq 2\beta \leq 1$$

Newmark showed that a reasonable value of γ is 0.5, therefore the update rules are:

$$\dot{u}_{n+1} = \dot{u}_n + \frac{\Delta t}{2}(\ddot{u}_n + \ddot{u}_{n+1}) \quad (29)$$

$$u_{n+1} = u_n + \Delta t \dot{u}_n + \frac{1-2\beta}{2} \Delta t^2 \ddot{u}_n + \beta \Delta t^2 \ddot{u}_{n+1}$$

Setting β to various values between 0 and 0.5 can give a wide range of results. Typically $\beta=1/4$, which yields the constant average acceleration method, is used.

Similar to the analytical methods, the previously mentioned numerical procedures have considerable limitations when it comes to solving realistic dynamic problems of vertical transport systems, especially since there is a need for transforming the partial into ordinary differential equations with nonholonomic boundary conditions. After analyzing the parameters for the mathematical model which describes longitudinal oscillations in the machines for vertical transport, it can be concluded that they are equations of a hyperbolic type, which leads to the rest of the paper focusing on the methods and manners of solving partial differential equations of the hyperbolic type.

1.2.1. Numerical solving of partial differential equations

Numerical solving of a partial differential equation is generated in the form of a table of approximate values for the requested function $u(x,y)$ for the equidistant values of independent variables. In order to explain it in the best way possible, we will show the numerical methods for solving the partial differential equations of the hyperbolic type, which are closely connected to the dynamic analyses of elevators and exploitation facilities, in mines especially.

- Finite difference methods

The finite difference method for numerical solving of partial differential equations comprises:

- Discretization of the area (domain) of the independent variables,
- Approximation of partial derivatives with finite differences,
- Solving the resulting system of algebraic equations.

That means that the inside points of the area (mesh) satisfy the differential equation, and the points on the boundary of the area (domain) satisfy the boundary conditions.

The basic idea behind the finite difference method is in the replacement of derivatives of the considered

function with their approximate values. In the realization of the idea we introduce points, i.e. mesh of nodes (hence the name – mesh method), where we look for the solution.

To consider the specific cases, such as elevators and exploitation facilities in mines, we are going to take a look at the one-dimensional partial differential equation of the hyperbolic type of the second order.

$$\frac{\partial^2 u}{\partial t^2} - a(x, t) \frac{\partial^2 u}{\partial x^2} = f(x, t) \quad (30)$$

where $a(x, t) > 0$ for every x and $t > 0$.

In the boundary area for the variable coordinate x (for instance a change in the length of the free rope end with the elevators where the total length is L) we define the boundary conditions, such as Fig. 6:

$$\begin{cases} u(0, t) = g_1(t) \\ u(L, t) = g_2(t) \end{cases} \quad \text{for } t > 0 \quad (31)$$

And also the initial conditions, Fig. 6:

$$\begin{aligned} u(x, 0) &= u_0(x), \quad x \in \{0, L\} \\ \left. \frac{\partial u}{\partial t} \right|_{(x, t=0)} &= v_0(x), \quad x \in \{0, L\} \end{aligned} \quad (32)$$

In which the second condition is known as the so-called Cauchy's condition.

Fig. 6: Mesh of nodes with boundary and initial conditions

If we introduce the $a(x, t) = c^2$, in which c represents the positive constant, then the general (analytical) solution for (30) is given in the form of:

$$-\frac{1}{2} \lambda a_n^{k+1} u_{n-1}^{k+1} + (1 + \lambda a_n^{k+1}) u_n^{k+1} - \frac{1}{2} \lambda a_{n+1}^{k+1} u_{n+1}^{k+1} = \frac{1}{2} \Delta t^2 (f_n^{k+1} + f_n^{k-1}) + 2u_n^k - \left[\frac{\lambda}{2} a_n^{k-1} u_{n-1}^{k-1} - (1 - \lambda a_n^{k-1}) u_n^{k-1} + \frac{\lambda}{2} a_n^{k-1} u_{n+1}^{k-1} \right] \quad (38)$$

Fig. 7: Illustration of the time-space mesh with a wave equation

According to (38), in order to reach the solution, we also have to know the field (values) in $(k-1)$ points. This is enabled by the initial conditions (36). Using the development in Taylor's order per t we get:

$$u(x, t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(x) dx \quad (33)$$

For the sake of a more comprehensible explanation we will introduce signs N and K , representing the number of points in the spatial coordinate x and the number of points (divisions) per time coordinate t . For that reason the function $u(x, t)$ changes into the form of u_n^k . By introducing the extra signs:

$$x_n = n \cdot \frac{L}{N}, \quad t_k = k \cdot \frac{T}{K} \quad (34)$$

where:

$$0 \leq n \leq N, \quad 0 \leq k \leq K$$

L - total spatial coordinate (length),

T - total (final) integration time,

And then the boundary and initial conditions become:

$$\begin{cases} u_0^k = g_1(t_k) \\ u_N^k = g_2(t_k) \end{cases} \quad 0 \leq k \leq K \quad (35)$$

$$\begin{cases} u_n^0 = u_0(x_n) \\ \left. \frac{\partial u}{\partial t} \right|_{n,0} = v_0(x_n) \end{cases} \quad 0 \leq n \leq N \quad (36)$$

By using the central difference theory and by replacing the suitable derivatives in time and place coordinate x , it can be written that, [10]:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\Delta t^2} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{1}{2} \left(\frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{\Delta x^2} + \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{\Delta x^2} \right) \end{aligned} \quad (37)$$

where:

$$\Delta t = T/K \quad \text{и} \quad \Delta x = L/N.$$

By introducing the sign (abbreviation) $\lambda = \frac{\Delta t^2}{\Delta x^2}$,

we get the system for $(N-1)$ equations as a replacement for the partial differential equation in the area from $(k+1)$ points per time. In order to solve the system, we have to know the field at the $(k+1)$ point.

$$u_n^k = u_n^{k-1} + \left. \frac{\partial u}{\partial t} \right|_{k-1} \Delta t + \frac{1}{2} \left. \frac{\partial^2 u}{\partial t^2} \right|_{k-1} \Delta t^2 \quad (39)$$

All that is needed is to determine the u_n^1 , in which we write the following according to (30):

$$\begin{aligned} u_n^1 &= u_n^0 + v_n^0 \Delta t + \frac{1}{2} f_n^0 \Delta t^2 + \\ &+ \frac{1}{2} \lambda a_n^0 (u_{n+1}^0 - 2u_n^0 + u_{n-1}^0) \quad \text{for } (1 \leq n \leq N-1) \end{aligned} \quad (40)$$

By taking into consideration a special case (homogenous differential equation where $f(x, t) = 0$ and using the already mentioned positive constant $a(x, t) = c^2$, we get a simple system of algebra equations:

$$\begin{aligned}
 (1+\alpha)u_n^{k+1} - \frac{1}{2}\alpha u_{n+1}^{k+1} - \frac{1}{2}\alpha u_{n-1}^{k+1} = \\
 = 2u_n^k + \frac{1}{2}\alpha u_{n+1}^{k-1} - (1+\alpha)u_n^{k-1} + \frac{1}{2}\alpha u_{n-1}^{k-1}
 \end{aligned} \quad (41)$$

where:

$$\alpha = c^2 \frac{\Delta t^2}{\Delta x^2}$$

Equation (41) represents an excellent form and basis for solving such problems in a numerical way in contemporary software tools, such as MATLAB, Mathcad, Mathematica and so on.

2. APPLICATION OF THE FINITE DIFFERENCE METHOD ON MINE ELEVATORS

This chapter is providing the steps for solving the movement differential equations (oscillations) of the load with the hoisting element (rope) which is in a stick form (the “heavy” rope) of a constant length. It is possible to be approximately determined as oscillations with one degrees of freedom, with a “heavy” spring. However, the problem can be modelled even more accurately as a system with an infinite number of degrees of freedom (longitudinal stick oscillation) with the suitable boundary conditions.

When it comes to longitudinal oscillations of the great mass ropes, all the particles within the rope are moving parallel to the rope axis. The elongation is repeated one after the other with the time t and spatial coordinate x .

Fig. 8 shows the “heavy” rope with the load at its end. This model generally can be used as a representation of the exploitation facility in mines at the moment when the basket stops and it starts to oscillate freely (if we ignore the friction in guide rails).

Fig. 8: Model of the “heavy” rope with the load (cabin, cage) at its end

The starting parameters for the dynamic model analyses in Fig. 8 are:

$L = 500$ m - the total free length of the ropes at the moment of stopping the cage,

$M = 30$ t - reduced oscillation mass,

$A = 1700$ mm² - total metal cross-section area of ropes,

$E = 1,3 \cdot 10^5$ MPa - rope elasticity modulus for the chosen parameters.

The data (suggested parameters) are acquired for a specific mine elevator obtained through measurements and later determination of certain dynamic parameters.

A movement differential equation is a hyperbolic partial differential equation described in (30) for $f(x,t) = 0$ and $a(x,t) = c^2$, which is expressed as:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (42)$$

With constant c being equal to:

$$c^2 = \sqrt{\frac{E}{\rho}}$$

And it stands for how fast the waves are spreading, whereas $\rho = 7,86$ t/m³ is the thickness of the wire materials in the driving ropes.

Boundary conditions are defined for the rope's ends. At the upper end, if $x = 0$, based on expression (31), the first boundary condition would be:

$$u(0,t) = 0 \quad (43)$$

The second boundary condition is gained for the lower part of the rope, where it is connected to the load. Based on the expressions in [12] and the applied method of separating the variables, it is confirmed that the second boundary condition is:

$$\frac{\partial u(L,t)}{\partial x} - \frac{M \cdot k^2}{A \cdot \rho \cdot c^2} u(L,t) = 0 \quad (44)$$

Where constant k in this case equals, [12]:

$$k = (2m-1) \frac{\pi}{2 \cdot L}, \quad (m = 1, 2, 3, \dots)$$

The “heavy” rope is initially burdened with the load ($M \cdot g$), so its end is stretched for [12]:

$$\Delta L = \frac{M \cdot g \cdot L}{E \cdot A}$$

And we observe the other parts in a linear manner:

$$u(x,0) = \frac{\Delta L}{L} \cdot x = \frac{M \cdot g}{E \cdot A} \cdot x \quad (45)$$

This expression represents the first initial condition. The second initial condition is obtained with the assumption that the system was still at first, i.e. all the parts of the “heavy” rope had the zero velocity.

$$\frac{\partial u(x,0)}{\partial t} = 0 \quad (46)$$

Solution for the movement differential equation (oscillation) of the “heavy” rope has been gained with the finite difference method in MATLAB software package, by using the explicit method for solving the equations of the hyperbolic type.

The results shown in Fig. 9 were obtained by using the mentioned boundary and initial conditions, with the second initial condition (46), taking that $v_n^0 = 0$.

Fig. 9: How elongation depends on the time and space coordinate

3. APPLICATION OF SPECIALIZED SOFTWARE SYSTEMS AND COMPUTER SIMULATION OF THE DYNAMIC BEHAVIOUR IN A MINING ELEVATOR

Since MATLAB software application in solving the system of differential equations with variable (nonholonomic) boundary conditions has its limits, the only solution lies in the application of specialized software systems for dynamic analysis. One of the software systems, which is especially suitable for the analysis of system's dynamic behavior in the mechanism form, is the software MSC ADAMS.

The drive characteristic is represented through setting the velocity, i.e. changing the number of rotations in a driving pulley. In this paper, the velocity is set in the form of „spline“, which is unambiguous when it comes to describing the changes in the rotations number in a driving electric motor which is reduced to the circumferential velocity on the drum, i.e. driving pulley, obtained through an incremental encoder during the measurements performed on a mine elevator. The „spline“, which was used in the software, represents the middle line of the measured the circumferential velocity on a driving pulley which therefore eliminates the so-called unwanted measuring signal, Fig. 10.

Fig. 10: Forming of the „spline“ velocity as a driving characteristic on the pulley

The dynamic model used for simulation MSC ADAMS software is shown in Fig. 11.

Fig. 11: Computer model of mining elevators

To verify the dynamic model, there are oscillation diagrams for the whole period of cage oscillation, and after the complete halt for the pulley in the characteristic cases, “overlapping” with the diagrams obtained through measurements. As the most suitable for verification we have chosen the acceleration diagram in Fig. 12.

Fig. 12: Diagrams of cage acceleration

According to the previous diagram we can draw a conclusion that the dynamic model with dynamic parameters determined by the experiment with satisfactory accuracy can describe realistic behavior of the mining elevator. That in turn enables performing computer stimulations for the analyses of realistic loads in an exploitation facility in mines. This is a new approach which enables the analysis of dynamic behavior of vertical lifting systems, which are being exploited.

While comparing the numerical analysis results with the measurement results, it would be useful to establish a numerical measure for result correlation. This paper does not set the correlation coefficients, which can be one of the suggestions for further research. The given part of the research points out the possibilities to set the realistic values of dynamic parameters (elasticity modulus, damping, ...), and to analyze the impact of variable stiffness (EA/L) and movement velocity (nonholonomic boundary condition at the point where rope comes to the pulley) on vertical transport systems at great lifting velocities, which was not possible to do using the mentioned analytical and numerical methods.

As an illustration of the possibilities of dynamic analysis, the following figures show the diagrams for changes in certain values, done in MSC ADAMS software, while using the real experimental data.

Figure 13: Diagrams of the change in cage position, velocity and acceleration

Fig. 14: Diagrams of the change in hoisting ropes elongation, stiffness coefficient and damping parameter

Fig. 15: Diagram of the force in ropes changing

4. CONCLUSION

By analyzing the parameters of a concrete exploitation facility it is possible to simplify a dynamic model to a great extent if we omit the small values of the higher order and gain the suitable model for a dynamic analysis.

For driving mechanisms used for vertical load lifting (elevators and cranes), it is necessary to provide adequate conditions for the correct dynamic analysis of their parts' behavior and especially for the rope as the basic element, already in the design phase. Due to a significant influence of rope free length changes, due to its slipping over the driving pulley and for the reason of its mechanical characteristics, it is impossible to apply a classical dynamic model of longitudinal oscillations for homogenous stick of a constant length, especially by the elevators with large velocities and large lifting heights (express and mine elevators).

The system of partial differential equations of the hyperbolic type is possible to be solved by using the numerical methods (MATLAB), but only in the cases after stopping (braking) the driving mechanism. In that case we trust the stiffness suits the current values of the rope's free length as a constant size with the suitable boundary conditions. In order to do the analysis of the dynamic behavior of the system during the movement (variable stiffness and nonholonomic boundary conditions) it is necessary to apply the specialized software packages with specifically defined (adjusted) parameters of dynamic

models that can be obtained through measuring at vertical transport real systems.

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