Analiza slobodnih poprečnih oscilacija jednostepenastih Timošenkovih i Ojler-Bernulijevih štapova koršćenjem modifikovanog Hencky-evog modela kinematičkog lanca segmenata

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U ovom radu analizirane su poprečne slobodne oscilacije jednostepenastih Timošenkovih i Ojler-Bernulijevih štapova sa prikacnim oprugama i koncentrisanim masama na krajevima i sastavljenih od dva homogenog segmenta konstantnih poprečnih preseka pomoću modifikovanog Hencky-evog modela lanca štapova. Analiziran je uticaj opruga i koncentrisanih masa na krajevima na kružne frekvencije stepenastog štapa. Teorijska razmatranja su praćena numeričkim primerima kojima je pokazana tačnost modifikovanog Hencky-evog modela lanca štapova. Rad proširuje opseg primene Hencky-evog modela lanca štapova na oblast analize oscilacija složenih mehaničkih sistema.

Ključne reči: Poprečne oscilacije, Timošenkov štap, Ojler-Bernulijev štap, Hencky-ev model lanca štapova, stepenasti štap, kružne frekvencije

1. UVOD

Stepenasti štapovi predstavljaju važan tip elemenata struktura u raznim inženjerskim primenama. Često se štapovi kontinualno promenljivog poprečnog preseka približno predstavljaju stepenastim štapovima (videti [1]).

Za određivanje kružnih frekvencija, u [2-8] je korišćen analitički metod koji se bazira na formiranju odgovarajućih tačnih frekventnih jednačina. U analizi oscilacija stepenastih štapova u literaturi su korišćeni razni metodi kao što su adomeni dekompozicioni metod [9,10], metod diferencijalne transformacije [11], Rejli-Ricov metod [12], metod kompozitnih elemenata [13], metod modifikovane diskretne singularne konvolucije [14], Čebiševljevi tau metod [15]. Problem poprečnih oscilacija Timošenkovog stepenastog štapa je rešen u [16] primenom formalizma Lagranževih množitelja.

U ovom radu je modifikovani Hencky-ev metod lanca štapova, razvijen u [17], iskorišćen za analizu slobodnih poprečnih oscilacija jednostepenastih Timošenkovih i Ojler-Bernulijevih štapova. Napomenimo da modifikovani Hencky-ev metod lanca štapova [17] predstavlja poboljšanu varijantu klasičnog Hencky-evog modela lanca štapova [18]. Naime, Hencky-ev model lanca štapova za razmatrani stepenasti štap se dobija na osnovu Hencky-evih modela lanca štapova segmenata koji predstavljaju homogene štapove konstantnih poprečnih preseka. Kroz numeričke primere date u Sekciji 4 demonstrirano je da se predloženi Hencky-ev metod lanca štapova može uspešno koristiti sa visokom tačnošću za određivanje kružnih frekvencija i modova oscilovanja stepenastih štapova.

2. TIMOŠENKOV JEDNOSTEPENASTI ŠTAP

Razmotrimo slobodne poprečne oscilacije elastično oslonjenog Timošenkovog štapa sa slobodnim krajevima

koji nosi dve koncentrisane mase m_l i m_r na svojim krajevima kao što je to prikazano na Sl. 1.

Slika 1: Elastično oslonjen Timošenkov štap sa jednom stepenastom promenom poprečnog preseka i sa koncentrisanim masama na krajevima

Štap je sastavljen od dva homogenog segmenta konstantnih poprečnih preseka sa Jungovim modulima elastičnosti E_1 i E_2 , gustinama ρ_1 i ρ_2 , površinama poprečnih preseka A_1 i A_2 , i aksijalnim momentima inercije preseka I_{z1} i I_{z2} . Dužine segmenata su L_1 i L_2 (videti Sl. 1). Takođe, krutosti cilindričnih i spiralnih opruga na levom i desnom kraju štapa su označene, respektivno, sa $k_{L,l}$, $k_{R,l}$, $k_{L,r}$ i $k_{R,r}$. U skladu sa modifikovanim varijantom klasičnog Hencky-evog modela lanca štapova razvijenom u [17], Timošenkov homogeni štap konstantnog poprečnog preseka dužine L , prikazan na Sl. 2(a), može se podeliti na n jednakih delova dužine L/n (videti Sl. 2(b)) a zatim se svaki od delova može zameniti jednim sistemom krutih tela formiranim od lakih krutih štapova koji nose koncentrisane mase:

$$m_1 = m_3 = \frac{\rho AL}{6n}, \quad (1)$$

$$m_2 = \frac{2\rho AL}{3n}, \quad (2)$$

sa aksijalnim momentima inercije u odnosu na ose koje prolaze kroz koncentrisane mase:

$$J_1 = J_3 = \frac{\rho I_z L}{6n}, \quad (3)$$

$$J_2 = \frac{2\rho I_z L}{3n}, \quad (4)$$

povezanih glatkim zglobovima sa dva stepena slobode relativnog kretanja u njima (videti Sl. 2(c)) sa odgovarajućim cilindričnim i spiralnim oprugama kao što je to prikazano na Sl. 2(d) pri čemu je:

$$k_R = \frac{EI_z}{L/(2n)}, \quad (5)$$

$$k_L = \frac{12EI_z}{(L/(2n))^3(1+\Phi_y)}, \quad (6)$$

$$\Phi_y = \frac{12EI_z}{GAk(L/(2n))^2}. \quad (7)$$

Na kraju, sistemi krutih tela oblika prikazanog na Sl. 2(c) su kruto povezani jedan za drugi formirajući na taj način modifikovani Hencky-ev model lanca štapova razmatranog elastičnog štapa. Primenujući opisanu proceduru modeliranja na svaki od segmenata stepenastog štapa prikazanog na Sl. 1 dobija se modifikovani Hencky-ev model lanca štapova za razmatrati Timošenkov štap kao što je to prikazano na Sl. 3. Napomenimo da je tokom procedure diskretizacije segmenata štapa, segment 1 podeljen na n_1 delova jednake dužine, a segment 2 na n_2 delova takođe jednake dužine.

Slika 2: Proces diskretizacije homogenog štapa konstantnog poprečnog preseka

Na osnovu [17], odgovarajući parametri dobijenog Hencky-evog modela lanca štapova su dati kao što sledi:

$$m_1 = \frac{\rho_1 A_1 L_1}{6n_1} + m_l, \quad (8)$$

$$m_{2i_1} = \frac{2\rho_1 A_1 L_1}{3n_1} (i_1 = 1, \dots, n_1), \quad (9)$$

$$m_{2j_1+1} = \frac{\rho_1 A_1 L_1}{3n_1} (j_1 = 1, \dots, n_1 - 1), \quad (10)$$

$$m_{2n_1+1} = \frac{\rho_1 A_1 L_1}{6n_1} + \frac{\rho_2 A_2 L_2}{6n_2}, \quad (11)$$

$$m_{2n_1+2i_2} = \frac{2\rho_2 A_2 L_2}{3n_2} (i_2 = 1, \dots, n_2), \quad (12)$$

$$m_{2n_1+2j_2+1} = \frac{\rho_2 A_2 L_2}{3n_2} (j_2 = 1, \dots, n_2 - 1), \quad (13)$$

$$m_{2n_1+2n_2+1} = \frac{\rho_2 A_2 L_2}{6n_2} + m_r, \quad (14)$$

$$J_1 = \frac{\rho_1 I_{z1} L_1}{6n_1}, \quad (15)$$

$$J_{2i_1} = \frac{2\rho_1 I_{z1} L_1}{3n_1} (i_1 = 1, \dots, n_1), \quad (16)$$

$$J_{2j_1+1} = \frac{\rho_1 I_{z1} L_1}{3n_1} (j_1 = 1, \dots, n_1 - 1), \quad (17)$$

$$J_{2n_1+1} = \frac{\rho_1 I_{z1} L_1}{6n_1} + \frac{\rho_2 I_{z2} L_2}{6n_2}, \quad (18)$$

$$J_{2n_1+2i_2} = \frac{2\rho_2 I_{z2} L_2}{3n_2} (i_2 = 1, \dots, n_2), \quad (19)$$

$$J_{2n_1+2j_2+1} = \frac{\rho_2 I_{z2} L_2}{3n_2} (j_2 = 1, \dots, n_2 - 1), \quad (20)$$

$$J_{2n_1+2n_2+1} = \frac{\rho_2 I_{z2} L_2}{6n_2}, \quad (21)$$

$$k_{L,i_1} = \frac{12E_1 I_{z1}}{(L_1/(2n_1))^3(1+\Phi_{y1})} (i_1 = 1, \dots, 2n_1), \quad (22)$$

$$k_{R,i_1} = \frac{E_1 I_{z1}}{L_1/(2n_1)} (i_1 = 1, \dots, 2n_1), \quad (23)$$

$$k_{L,i_2} = \frac{12E_2 I_{z2}}{(L_2/(2n_2))^3(1+\Phi_{y2})} (i_2 = 2n_1 + 1, \dots, 2n_1 + 2n_2), \quad (24)$$

$$k_{R,i_2} = \frac{E_2 I_{z2}}{L_2/(2n_2)} (i_2 = 2n_1 + 1, \dots, 2n_1 + 2n_2), \quad (25)$$

$$\Phi_{y1} = \frac{12E_1 I_{z1}}{G_1 A_1 k_1 (L_1/(2n_1))^2}, \quad (26)$$

$$\Phi_{y2} = \frac{12E_2 I_{z2}}{G_2 A_2 k_2 (L_2/(2n_2))^2}, \quad (27)$$

$$x_{l,1} = 0, \quad (28)$$

$$x_{l,j} = \frac{L_1}{4n_1} (j = 2, \dots, 2n_1 + 1), \quad (29)$$

$$x_{r,i_1} = \frac{L_1}{4n_1} (i_1 = 1, \dots, 2n_1), \quad (30)$$

$$x_{r,2n_1+1} = \frac{L_2}{4n_2}, \quad (31)$$

$$x_{l,2n_1+1+i_2} = \frac{L_2}{4n_2} (i_2 = 1, \dots, 2n_2), \quad (32)$$

$$x_{r,2n_1+2n_2+1} = 0, \quad (33)$$

$$x_{r,2n_1+1+i_2} = \frac{L_2}{4n_2} (i_2 = 1, \dots, 2n_2 - 1), \quad (34)$$

gde su $G_i (i=1,2)$ i $k_i (i=1,2)$ moduli smicanja i koeficijenti smicanja, respektivno, segmenata 1 i 2, respektivno. Na osnovu prethodnih razmatranja, kinetička energija razmatranog sistema krutih tela glasi:

Slika 3: Hencky-ev model lanca štapova koji odgovara Timošenkovom jednostepenastom štapu

$$T = \frac{1}{2} \sum_{i=1}^{2n_1+2n_2+1} m_i \dot{y}_i^2 + \frac{1}{2} \sum_{i=1}^{2n_1+2n_2+1} J_i \dot{\phi}_i^2, \quad (35)$$

gde je potencijalna energija sistema određena sledećim izrazom (videti [17]):

$$\begin{aligned} \Pi = & \frac{1}{2} \sum_{i=1}^{2n_1+2n_2} k_{R,i} (\phi_{i+1} - \phi_i)^2 + \frac{1}{2} k_{R,l} \phi_1^2 + \frac{1}{2} k_{R,r} \phi_{2n_1+2n_2+1}^2 \\ & + \frac{1}{2} k_{L,l} y_1^2 + \frac{1}{2} k_{L,r} y_{2n_1+2n_2+1}^2 \\ & + \frac{1}{2} \sum_{i=1}^{2n_1+2n_2} k_{L,i} (y_{i+1} - x_{l,i+1} \phi_{i+1} - y_i - x_{r,i} \phi_i)^2. \end{aligned} \quad (36)$$

Uvodeći vektor generalisanih koordinata $\mathbf{q} = [q_1, \dots, q_{4n_1+4n_2+2}]^T \equiv [\mathbf{q}_y^T, \mathbf{q}_\phi^T]^T$, gde su $\mathbf{q}_y = [y_1, \dots, y_{2n_1+2n_2+1}]^T$ i $\mathbf{q}_\phi = [\phi_1, \dots, \phi_{2n_1+2n_2+1}]^T$, izrazi za kinetičku energiju (35) i potencijalnu energiju (36) mogu se zapisati u sledećim matricnim oblicima:

$$T = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}, \quad (37)$$

$$\Pi = \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q}, \quad (38)$$

gde su $\mathbf{M} \in R^{(4n_1+4n_2+2) \times (4n_1+4n_2+2)}$ i $\mathbf{K} \in R^{(4n_1+4n_2+2) \times (4n_1+4n_2+2)}$ matrica inercije i matrica krutosti sistema, respektivno, definisane kao što sledi:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_y & \mathbf{0}_{(2n_1+2n_2+1) \times (2n_1+2n_2+1)} \\ \mathbf{0}_{(2n_1+2n_2+1) \times (2n_1+2n_2+1)} & \mathbf{M}_\phi \end{bmatrix}, \quad (39)$$

$$\mathbf{M}_y = \text{diag}(m_1, m_2, \dots, m_{2n_1+2n_2+1}), \quad (40)$$

$$\mathbf{M}_\phi = \text{diag}(J_1, J_2, \dots, J_{2n_1+2n_2+1}), \quad (41)$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_y^{(1)} & \mathbf{K}_\phi^{(1)} \\ \mathbf{K}_y^{(2)} & \mathbf{K}_\phi^{(2)} \end{bmatrix}, \quad (42)$$

gde je $\mathbf{0}_{(2n_1+2n_2+1) \times (2n_1+2n_2+1)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$ nula matrica, a komponente matrica $\mathbf{K}_y^{(1)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$, $\mathbf{K}_\phi^{(1)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$, $\mathbf{K}_y^{(2)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$ i $\mathbf{K}_\phi^{(2)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$ su određene sledećim izrazima [17]:

$$K_{y1,1}^{(1)} = k_{L,1} + k_{L,l}, \quad (43)$$

$$K_{y,i,i}^{(1)} = k_{L,i-1} + k_{L,i} \quad (i=2, \dots, 2n_1+2n_2), \quad (44)$$

$$K_{y,2n_1+2n_2+1,2n_1+2n_2+1}^{(1)} = k_{L,2n_1+2n_2} + k_{L,r}, \quad (45)$$

$$K_{y,i,i+1}^{(1)} = -k_{L,i} \quad (i=1, \dots, 2n_1+2n_2), \quad (46)$$

$$K_{y,i,i-1}^{(1)} = -k_{L,i-1} \quad (i=2, \dots, 2n_1+2n_2+1), \quad (47)$$

$$K_{y1,1}^{(2)} = k_{L,1} x_{r,1}, \quad (48)$$

$$K_{y,i,i}^{(2)} = -k_{L,i-1} x_{l,i} + k_{L,i} x_{r,i} \quad (i=2, \dots, 2n_1+2n_2), \quad (49)$$

$$K_{y,2n_1+2n_2+1,2n_1+2n_2+1}^{(2)} = -k_{L,2n_1+2n_2} x_{l,2n_1+2n_2+1}, \quad (50)$$

$$K_{y,i,i+1}^{(2)} = -k_{L,i} x_{r,i} \quad (i=1, \dots, 2n_1+2n_2), \quad (51)$$

$$K_{y,i,i-1}^{(2)} = k_{L,i-1} x_{l,i} \quad (i=2, \dots, 2n_1+2n_2+1), \quad (52)$$

$$K_{\phi 1,1}^{(1)} = k_{L,1} x_{r,1}, \quad (53)$$

$$K_{\phi i,i}^{(1)} = -k_{L,i-1} x_{l,i} + k_{L,i} x_{r,i} \quad (i=2, \dots, 2n_1+2n_2), \quad (54)$$

$$K_{\phi 2n_1+2n_2+1,2n_1+2n_2+1}^{(1)} = -k_{L,2n_1+2n_2} x_{l,2n_1+2n_2+1}, \quad (55)$$

$$K_{\phi i,i+1}^{(1)} = k_{L,i} x_{l,i+1} \quad (i=1, \dots, 2n_1+2n_2), \quad (56)$$

$$K_{\phi i,i-1}^{(1)} = -k_{L,i-1} x_{r,i-1} \quad (i=2, \dots, 2n_1+2n_2+1), \quad (57)$$

$$K_{\phi 1,1}^{(2)} = k_{R,1} + k_{R,l} + k_{L,1} x_{r,1}^2, \quad (58)$$

$$K_{\phi 2n_1+2n_2+1,2n_1+2n_2+1}^{(2)} = k_{R,2n_1+2n_2} + k_{R,r} + k_{L,2n_1+2n_2} x_{l,2n_1+2n_2+1}^2, \quad (59)$$

$$K_{\phi i,i}^{(2)} = k_{R,i-1} + k_{R,i} + k_{L,i-1} x_{l,i}^2 + k_{L,i} x_{r,i}^2 \quad (i=2, \dots, 2n_1+2n_2), \quad (60)$$

$$K_{\phi i,i+1}^{(2)} = -k_{R,i} + k_{L,i} x_{l,i+1} x_{r,i} \quad (i=1, \dots, 2n_1+2n_2), \quad (61)$$

$$K_{\phi,i-1}^{(2)} = -k_{R,i-1} + k_{L,i-1}x_{r,i-1}x_{l,i} \quad (i = 2, \dots, 2n_1 + 2n_2 + 1). \quad (62)$$

Korišćenjem Lagranževih jednačina druge vrste [19]:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial \Pi}{\partial \mathbf{q}} = \mathbf{0}_{(4n_1+4n_2+2) \times (4n_1+4n_2+2)}, \quad (63)$$

gde tačka iznad simbola označava diferenciranje po vremenu, mogu se dobiti difrencijalne jednačine kretanja sistema u sledećem matričnom obliku:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}_{(4n_1+4n_2+2) \times (4n_1+4n_2+2)} \quad (64)$$

gde je $\mathbf{0}_{(4n_1+4n_2+2) \times (4n_1+4n_2+2)}$ nula matrica.

Konačno, problem sopstvenih vrednosti [20] koji odgovara difrencijalnoj jednačini (64) glasi:

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{u} = \mathbf{0}_{(4n_1+4n_2+2) \times (4n_1+4n_2+2)} \quad (65)$$

gde je ω kružna frekvencija poprečnih oscilacija stepenastog štapa koja se određuje iz frekventne jednačine:

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0, \quad (66)$$

a $\mathbf{u}$ predstavlja sopstveni vektor pridružen kružnoj frekvenciji ω . Takođe, diskretne vrednosti funkcija glavnih oblika oscilovanja, $Y_r(x) (r = 1, \dots, 4n_1 + 4n_2 + 2)$, mogu se odrediti na sličan način kao u [17] kao što sledi:

$$Y_r(x_s) = u_{r,s} \quad (s = 1, \dots, 2n_1 + 2n_2 + 1), \quad (67)$$

ili, korišćenjem normalizovanih oblika oscilovanja:

$$\bar{Y}_r(x_s) = u_{r,s} / u_{r,1} \quad (s = 1, \dots, 2n_1 + 2n_2 + 1), \quad (68)$$

gde je $\mathbf{u}_r = [u_{r,1}, u_{r,2}, \dots, u_{r,4n_1+4n_2+2}]^T$ r -ti sopstveni vektor koji odgovara r -toj kružnoj frekvenciji ω_r .

3. OJLER-BERNULIJEV JEDNOSTEPENASTI ŠTAP

U slučaju stepenastog Ojler-Bernulijevog štapa, oblika prikazanog na Sl. 1, odgovarajući Hencky-ev model lanca štapova je prikazan na Sl. 4. Za razliku od Hencky-evog modela lanca štapova Timošenskog stepenastog štapa prikazanog na Sl.3, kod ovog model imamo da je $J_s = 0 (s = 1, \dots, 2n_1 + 2n_2 + 1)$ i $\Phi_{y1} = \Phi_{y2} = 0$ dok su ostali parametri nepromenjeni. Kinetička energija sistema krutih tela prikazanog na Sl. 4 je:

$$T = \frac{1}{2} \sum_{i=1}^{2n_1+2n_2+1} m_i \dot{y}_i^2, \quad (69)$$

gde je izraz za potencijalnu energiju određen sa (36). Sada, na osnovu (69), matrični izraz za kinetičku energiju glasi:

$$T = \frac{1}{2} \dot{\mathbf{q}}_y^T \mathbf{M}_y \dot{\mathbf{q}}_y, \quad (70)$$

gde je $\mathbf{M}_y \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$ matrica inercije čije su komponente:

$$\mathbf{M}_y = \text{diag}(m_1, m_2, \dots, m_{2n_1+2n_2+1}), \quad (71)$$

Slika 4: Hencky-ev model lanca štapova koji odgovara Ojler-Bernulijevom jednostepenastom štapu

Primenom Lagranževih jednačina druge vrste (63) slede difrencijalne jednačine kretanja sistema zapisane u matričnom obliku:

$$\mathbf{M}_y \ddot{\mathbf{q}}_y + \mathbf{K}_y^{(1)} \mathbf{q}_y + \mathbf{K}_\phi^{(1)} \mathbf{q}_\phi = \mathbf{0}_{(2n_1+2n_2+1) \times 1}, \quad (72)$$

$$\mathbf{K}_y^{(2)} \mathbf{q}_y + \mathbf{K}_\phi^{(2)} \mathbf{q}_\phi = \mathbf{0}_{(2n_1+2n_2+1) \times 1}, \quad (73)$$

gde su komponente matrica $\mathbf{K}_y^{(1)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$, $\mathbf{K}_\phi^{(1)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$, $\mathbf{K}_y^{(2)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$ i $\mathbf{K}_\phi^{(2)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$ određene izrazima (43)-(62) u kojima treba uzeti da je $\Phi_{y1} = \Phi_{y2} = 0$.

Za analizu oscilacija stepenastog štapa prikazanog na Sl. 4, matrica $\mathbf{q}_\phi$ u (73) može se izraziti preko $\mathbf{q}_y$ kao što sledi:

$$\mathbf{q}_\phi = -(\mathbf{K}_\phi^{(2)})^{-1} \mathbf{K}_y^{(2)} \mathbf{q}_y, \quad (74)$$

i zameniti u (72). Nakon toga dobija se sledeća difrencijalna jednačina po $\mathbf{q}_y$:

$$\mathbf{M}_y \ddot{\mathbf{q}}_y + \mathbf{K}_1 \mathbf{q}_y = \mathbf{0}_{(2n_1+2n_2+1) \times 1}, \quad (75)$$

gde je

$$\mathbf{K}_1 = \mathbf{K}_y^{(1)} - \mathbf{K}_\phi^{(1)} (\mathbf{K}_\phi^{(2)})^{-1} \mathbf{K}_y^{(2)}. \quad (76)$$

Problem sopstvenih vrednosti koji odgovara difrencijalnoj jednačini (75) glasi:

$$(\mathbf{K}_1 - \omega^2 \mathbf{M}_y) \mathbf{u}_1 = \mathbf{0}_{(2n_1+2n_2+1) \times 1} \quad (77)$$

gde je ω kružna frekvencija poprečnih oscilacija jednostepenastog Ojler-Bernulijevog štapa određena na osnovu sledeće frekventne jednačine:

$$\det(\mathbf{K}_1 - \omega^2 \mathbf{M}_y) = 0, \quad (78)$$

i gde $\mathbf{u}_1$ predstavlja sopstveni vektor koji je pridružen kružnoj frekvenciji ω . Dalje, diskretne vrednosti funkcija glavnih oblika oscilovanja, $Y_r(x) (r = 1, \dots, 2n_1 + 2n_2 + 1)$, određene su sa:

$$Y_r(x_s) = u_{r,s} \quad (s = 1, \dots, 2n_1 + 2n_2 + 1), \quad (79)$$

ili, korišćenjem normalizovanih oblika oscilovanja:

$$\bar{Y}_r(x_s) = u_{r,s} / u_{r,1} \quad (s = 1, \dots, 2n_1 + 2n_2 + 1), \quad (80)$$

gde je $\mathbf{u}_r = [u_{r,1}, u_{r,2}, \dots, u_{r,2n_1+2n_2+1}]^T$ r ti sopstveni vektor koji odgovara r toj kružnoj frekvenciji ω_r .

4. NUMERIČKI PRIMERI

U ovoj sekciji biće korišćeni sledeći bezdimenzionalni parametri:

$$\beta_l = \frac{m_l}{m}, \quad \beta_r = \frac{m_r}{m}, \quad (81)$$

$$\bar{k}_{R,l} = \frac{L}{E_1 I_{z1}} k_{R,l}, \quad \bar{k}_{R,r} = \frac{L}{E_1 I_{z1}} k_{R,r}, \quad (82)$$

$$\bar{k}_{L,l} = \frac{L^3}{E_1 I_{z1}} k_{L,l}, \quad \bar{k}_{L,r} = \frac{L^3}{E_1 I_{z1}} k_{L,r}, \quad (83)$$

gde je $L = L_1 + L_2$ i $m = \rho_1 A_1 L_1 + \rho_2 A_2 L_2$. Osim toga, greške sračunavanja vrednosti kružnih frekvencija mogu se odrediti u procentima primenom sledećeg izraza:

$$\frac{|\text{approximative value} - \text{exact value}|}{\text{exact value}} \cdot 100\%. \quad (84)$$

4.1. Primer 1

U ovoj podsekciji za Ojler-Bernulijev stepenasti štap, izvršeno je poređenje tačnosti između metoda prikazanog u ovom radu i analitičkog pristupa datog u [5]. Razmatrana su tri tipa jednostepenastih štapova i to: Tip 1 (štapovi konstantne debljine i stepenaste promene širine segmenata), Tip 2 (štapovi konstantne širine i stepenaste promene debljine segmenata), i Tip 3 (štapovi kružnog poprečnog preseka sa stepenastom promenom prečnika). Korišćene su sledeće fizičke karakteristike: Jungov modul elastičnosti $E_1 = E_2 = 2.068 \cdot 10^{11} \text{ N/m}^2$ i gustina $\rho_1 = \rho_2 = 7850 \text{ kg/m}^3$. Geometrijske karakteristike razmatranih tipova jednostepenastih štapova su: (Tip 1) $L = 1 \text{ m}$, $L_1 = 0.25L$, $L_2 = 0.75L$, $b_1 = 0.025 \text{ m}$, $b_2 = 0.5b_1$, $h_1 = h_2 = 0.02 \text{ m}$; (Tip 2) $L = 1 \text{ m}$, $L_1 = 0.25L$, $L_2 = 0.75L$, $b_1 = b_2 = 0.025 \text{ m}$, $h_1 = 0.02 \text{ m}$, $h_2 = 0.5h_1$; (Tip 3) $L = 1 \text{ m}$, $L_1 = 0.25L$, $L_2 = 0.75L$, $d_1 = 0.02 \text{ m}$, $d_2 = 0.5d_1$, gde su h_1 i h_2 debljine segmenata 1 i 2, respektivno, b_1 i b_2 su širine segmenata 1 i 2, respektivno. Osim toga, uzeto je da broj podela segmenata štapova bude $n_1 = n_2 = 15$ kao i sledeće skraćenice za granične uslove na krajevima štapova:

Uklješteno-Slobodni krajevi	C-F
Uklješteno-Uklješteni krajevi	C-C
Uklješteno-Oslonjeni krajevi	C-P
Oslonjeno-Oslonjeni krajevi	P-P

Takođe, u ovoj podsekciji i ta bezdimenzionalan kružna frekvencija Ω_i je u korelaciji sa i tom kružnom frekvencijom ω_i (rad/s) posredstvom izraza:

$$\Omega_i = \sqrt[4]{\frac{\rho_1 A_1 L^4 \omega_i^2}{E_1 I_{z1}}}. \quad (85)$$

Tabela 1: Bezdimenzionalne kružne frekvencije Ojler-Bernulijevog jednostepenastog štap Tipa 1 sračunate za razne granične uslove

Granični Uslovi	Ovaj rad	[5]
C-F	2.08492	2.0849
	4.91534	4.9153
	7.90447	7.9045
C-C	4.9579	4.9579
	7.90424	7.9043
	10.9708	10.9708
C-P	4.16925	4.1692
	7.16494	7.1650
	10.1856	10.1857
P-P	3.11602	3.1160
	6.22181	6.2218
	9.42479	9.4248

Tabela 2: Bezdimenzionalne kružne frekvencije Ojler-Bernulijevog jednostepenastog štap Tipa 2 sračunate za razne granične uslove

Granični Uslovi	Ovaj rad	[5]
C-F	1.66872	1.6687
	3.95285	3.9528
	6.16349	6.1635
C-C	3.99062	3.9906
	6.16176	6.1618
	8.30529	8.3054
C-P	3.3603	3.3603
	5.64571	5.6457
	7.74037	7.7404
P-P	2.22703	2.2270
	4.57059	4.5706
	7.09833	7.0984

Tabela 3: Bezdimenzionalne kružne frekvencije Ojler-Bernulijevog jednostepenastog štap Tipa 3 sračunate za razne granične uslove

Granični Uslovi	Ovaj rad	[5]
C-F	1.71509	1.7151
	4.13954	4.1395
	6.38411	6.3841
C-C	4.17665	4.1766
	6.38235	6.3824
	8.21542	8.2155
C-P	3.50359	3.5036
	5.90229	5.9023
	7.7142	7.7143
P-P	2.1602	2.1602
	4.39954	4.3995
	6.96604	6.9661

4.2. Primer 2

Cilj ove podsekcije je da se ispita efekat koji koncentrisana masa m_r i krutosti opruga $k_{L,r}$ i $k_{R,r}$ imaju na vrednosti bezdimenzionalnih kružnih frekvencija (85) jednostepenaste Ojler-Bernulijeve konzole sledećih materijalnih i geometrijskih karakteristika: $E_1 = E_2 = 2.068 \cdot 10^{11} \text{ N/m}^2$, $\rho_1 = \rho_2 = 7850 \text{ kg/m}^3$, $h_1 = 19.05 \text{ mm}$, $h_2 = 5.49 \text{ mm}$, $b_1 = b_2 = 25.4 \text{ mm}$, $L_1 = 254 \text{ mm}$, $L_2 = 140 \text{ mm}$. Napomenimo da su ove geometrijske karakteristike identične onima datim u [10, 13]. Numerički proračuni su izvršeni za $n_1 = n_2 = 15$.

Tabela 4: Bezdimenzionalne kružne frekvencije $\Omega_i (i=1, \dots, 5)$ Ojler-Bernulijeve jednostepenaste konzole dobijene za $\bar{k}_{L,r} = \bar{k}_{R,r} = 50$ i razne vrednosti parametra β_r

β_r	0.0	0.2	0.4	0.6	0.8
Ω_1	3.17988	3.15373	3.10506	3.01962	2.90701
Ω_2	5.67839	4.20623	3.69895	3.47185	3.3733
Ω_3	7.1018	6.54992	6.50199	6.48529	6.47683
Ω_4	9.04833	8.06672	8.01557	7.99807	7.98924
Ω_5	11.7556	11.3215	11.2924	11.282	11.2767

Tabela 5: Bezdimenzionalne kružne frekvencije $\Omega_i (i=1, \dots, 5)$ Ojler-Bernulijeve jednostepenaste konzole dobijene za $\beta_r = 0.6$ i razne vrednosti bezdimenzionalnih krutosti $\bar{k}_{L,r}$ i $\bar{k}_{R,r}$

$k_{T,r}$	20	0	50	100	400
$k_{R,r}$	0	20	100	200	400
Ω_1	2.54214	1.54377	3.01979	3.18537	3.23749
Ω_2	2.96215	3.30788	3.47245	3.88533	5.35207
Ω_3	5.78771	6.47771	6.48716	6.49034	6.51388
Ω_4	7.46398	7.98827	8.00108	8.00358	8.01159
Ω_5	10.5672	11.2713	11.2856	11.2875	11.2894

4.3. Primer 3

U ovoj podsekciji, izvršeno je vrednovanje tačnosti rezultata koje daje opisani metod za Timošenkov jednostepenasti štap kroz poređenje sa vrednostima dobijenim analitičkim pristupom datim u [4]. Za ovaj slučaj numerički proračuni su odrađeni za $n_1 = n_2 = 25$. Korišćene su sledeće materijalne i geometrijske karakteristike štapa: $E_1 = E_2 = 2.068 \cdot 10^{11} \text{ N/m}^2$, $\rho_1 = \rho_2 = 7850 \text{ kg/m}^3$, Poasonov koeficijent $\nu = 0.3$, koeficijent smicanja $k = 5/6$ (pravougaoni poprečni presek), moduli smicanja $G_1 = E_1 / [2(1+\nu)]$ i $G_2 = E_2 / [2(1+\nu)]$, $L = 1 \text{ m}$, $L_1 = (2/3)L$, $L_2 = (1/3)L$, $h_1 = \sqrt{12\eta}L_1$, $h_2 = 0.8h_1$, $b_1 = b_2 = 0.2h_1$. Bezdimenzionalne kružne frekvencije u ovoj podsekciji se određuju na osnovu izraza:

$$\Omega_i = \sqrt{\frac{\rho_1 A_1 L^4}{E_1 I_{z1}}} \omega_i. \quad (86)$$

Tabela 6: Bezdimenzionalne kružne frekvencije $\Omega_i (i=1, \dots, 5)$ Timošenkovke jednostepenaste konzole dobijene za $\eta = 0.0036$ i $\bar{k}_{L,r} = \bar{k}_{R,r} = 0$ kao i za razne vrednosti parametra β_r

β_r	Ovaj rad	[4]
0.0	3.77304	3.77
	19.8044	19.80
	47.3495	47.35
	84.1198	84.14
	124.994	125.06
0.2	2.73086	2.73
	15.9197	15.92
	41.3668	41.36
	77.0646	77.08
	116.249	116.30
0.4	2.24179	2.24
	14.9888	14.98
	40.4193	40.42
	76.1572	76.17
	115.311	115.36
0.6	1.94574	1.94
	14.5765	14.57
	40.0404	40.04
	75.8071	75.82
	114.96	115.01
0.8	1.74235	1.74
	14.3443	14.34
	39.8368	39.83
	75.6218	75.63
	114.775	114.82
1.0	1.59167	1.59
	14.1955	14.19
	39.7097	39.71
	75.507	75.52
	114.662	114.71

Tabela 7: Bezdimenzionalne kružne frekvencije $\Omega_i (i = 1, \dots, 5)$ Timošenkovke jednostepenaste konzole dobijene za $\eta = 0.01$ i $\bar{k}_{L,r} = \bar{k}_{R,r} = 0$ kao i za razne vrednosti parametra β_r

β_r	Ovaj rad	[4]
0.0	3.67728	3.67
	17.5286	17.52
	38.8053	38.81
	64.1336	64.15
	90.4665	90.53
0.2	2.67888	2.67
	14.4002	14.40
	34.3773	34.38
	59.505	59.52
	85.5861	85.64
0.4	2.20359	2.20
	13.6049	13.60
	33.6057	33.60
	58.837	58.85
	84.9603	85.02
0.6	1.91444	1.91
	13.2479	13.24
	33.2923	33.29
	58.5749	58.59
	84.7194	84.77
0.8	1.71529	1.71
	13.0457	13.04
	33.1227	33.12
	58.4352	58.45
	84.592	84.65
1.0	1.56753	1.56
	12.9156	12.91
	33.0166	33.01
	58.3484	58.36
	84.5132	84.57

5. ZAKLJUČCI

U ovom radu je modifikovani Hencky-ev model lanca štapova, razvijen u [17], primenjen u analizi slobodnih poprečnih oscilacija Ojler-Bernulijevih i Timošenkovih jednostepenastih štapova. Na taj način je postignuto proširenje opsega primene modifikovanog Hencky-evog modela lanca štapova na polje oscilacija nehomogenih štapova promenljivog poprečnog preseka. Kroz numeričke primere u Sekciji 4 pokazana je tačnost modifikovanog Hencky-evog modela lanca štapova poređenjem sa dostupnim u literaturi rezultatima dobijenim analitičkim postupcima. Na osnovu numeričkih rezultata može se reći da modifikovani Hencky-ev model lanca štapova za relativno mali broj podela segmenata na delove daje rezultate koji se odlično poklapaju sa do sada publikovanim rezultatima. Razmatranja u radu pokazuju da se prikazani pristup može lako prilagoditi za proučavanje slobodnih poprečnih oscilacija višestepenastih

Ojler-Bernulijevih i Timošenkovih štapova. Takođe, u radu su dati neki novi rezultati koji oslikavaju uticaj koncentrisanih masa i opruga na krajevima stepenastog štapa na vrednosti njegovih kružnih frekvencija.

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LITERATURA

- [1] X. Tong, B. Tabarrok, and K.Y. Yeh, "Vibration Analysis of Timoshenko Beams with Non-Homogeneity and Varying Cross-Section," J. Sound Vib., Vol. 186(5), pp. 821–835, (1995)
- [2] S.K. Jang and C.W. Bert, "Free vibrations of stepped beams: exact and numerical solutions," J. Sound Vib., Vol. 130, pp. 342–346, (1989)
- [3] S.K. Jang and C.W. Bert, "Free vibrations of stepped beams: higher mode frequencies and effects of steps on frequency," J. Sound Vib., Vol. 132, pp. 164–168, (1989)
- [4] R.E. Rossi, P.A.A. Laura and R.H. Gutierrez, "A Note on Transverse Vibrations of a Timoshenko Beam of Non-Uniform Thickness Clamped at One End and Carrying a Concentrated Mass at the Other," J. Sound Vib., Vol. 143(3), pp. 491–502, (1990)
- [5] S. Naguleswaran, "Natural Frequencies, Sensitivity and Mode Shape Details of an Euler–Bernoulli Beam with One-Step Change in Cross-Section and with Ends on Classical Supports," J. Sound Vib., Vol. 252(4), pp. 751–767, (2002)
- [6] S.C. Stanton and B.P. Mann, "On the Dynamic Response of Beams with Multiple Geometric or Material Discontinuities," Mech. Syst. Signal Pr., Vol. 24, pp. 1409–1419, (2010)
- [7] Janaína da Cunha Vaz and José Juliano de Lima Junior, "Vibration Analysis of Euler-Bernoulli Beams in Multiple Steps and Different Shapes of Cross Section," J. Vib. Control, Vol. 22(1), pp. 193–204, (2016)
- [8] M.S. Cao, W. Xu, Z. Su, W. Ostachowicz, and N. Xia, "Local Coordinate Systems-Based Method to Analyze High-Order Modes of n-step Timoshenko Beam," J. Vib. Control, Vol. 23(1), pp. 89–102, (2017)
- [9] Q. Mao and S. Pietrzko, "Free Vibration Analysis of Stepped Beams by Using Adomian Decomposition Method," Appl. Math. Comput., Vol. 217, pp. 3429–3441, (2010)
- [10] Q. Mao, "Free vibration analysis of multiple-stepped beams by using Adomian decomposition method," Math. Comput. Model., Vol. 54(1-2), pp. 756–764, (2011)
- [11] K. Suddoung, J. Charoensuk and N. Wattanasakulpong, "Application of the Differential Transformation Method to Vibration Analysis of Stepped Beams with Elastically Constrained Ends," J. Vib. Control, Vol. 19(16), pp. 2387–2400, (2013)

- [12] K. Mazanoglu, "Natural Frequency Analyses of Segmented Timoshenko-Euler Beams Using the Rayleigh-Ritz Method," *J. Vib. Control*, Vol. 23(13), pp. 2135-2154, (2017)
- [13] Z.R. Lu, M. Huang, J.K. Liu, W.H. Chen and W.Y. Liao, "Vibration Analysis of Multiple-Stepped Beams With the Composite Element Model," *J. Sound Vib.*, Vol. 322(4-5), pp. 1070-1080, (2009)
- [14] G. Duan and X. Wang, "Free Vibration Analysis of Multiple-Stepped Beams by the Discrete Singular Convolution," *Appl. Math. Comput.*, Vol. 219, pp. 11096–11109, (2013)
- [15] J. Lee, "Application of Chebyshev-tau Method to the Free Vibration Analysis of Stepped Beams," *Int. J. Mech. Sci.*, Vol. 101-102, pp. 411-420, (2015)
- [16] D. Cekus, "Use of Lagrange Multiplier Formalism to Solve Transverse Vibrations Problem of Stepped Beams According to Timoshenko Theory," *Scientific Research of the Institute of Mathematics and Computer Science*, Vol. 2(10), pp. 49-56, (2011)
- [17] S. Šalinić, "An Improved Variant of Hencky Bar-Chain Model for Buckling and Bending Vibration of Beams With End Masses and Springs," *Mech. Syst. Signal Pr.*, Vol. 90, pp. 30-43, (2017)
- [18] C.M. Wang, H. Zhang, R.P. Gao, W.H. Duan and N. Challamel, "Hencky Bar-Chain Model for Buckling and Vibration of Beams with Elastic End Restraints," *Int. J. Struct. Stab. Dy.*, Vol. 15(7), pp. 1540007, (2015)
- [19] A.I. Lurie, "Analytical Mechanics", Springer-Verlag, New York, (2002)
- [20] L. Meirovitch, "Fundamentals of Vibrations", McGraw-Hill, New York, (2001)

Free Transverse Vibration Analysis of Single-stepped Timoshenko and Euler-Bernoulli Beams using the Modified Hencky Bar-chain Model

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In this paper, free transverse vibration of single-stepped Timoshenko and Euler-Bernoulli beams consisting of two uniform homogeneous segment with attached springs and lumped masses at the beam ends is analysed by using the modified Hencky bar-chain model. The effects of the springs and masses at the beam ends on the natural frequencies and mode shapes of the stepped beam is examined. Theoretical considerations are accompanied by numerical examples to illustrate the accuracy of the modified Hencky bar-chain model. The paper extends the application range of the modified Hencky bar-chain model in the field of vibration analysis of mechanical systems.

Keywords: Transverse vibration, Timoshenko beam, Euler-Bernoulli beam, Hencky bar-chain, stepped beam, natural frequencies

1. INTRODUCTION

Stepped beams represents an important type of structural elements in various engineering applications. Often, in the vibration analysis of tapered beams, tapered beams are approximately represented by stepped beams (see [1]).

For the determination of natural frequencies of stepped beams, in [2-8] was used analytical approach based on forming the corresponding exact determinant equation. In the vibration analysis of stepped beams the adomian decomposition method [9,10], the differential transformation method [11], the Rayleigh-Ritz method [12], the composite element model method [13], the modified discrete singular convolution method [14], the Chebyshev-tau method [15] were used. The transverse vibrations problem of the Timoshenko stepped beam was solved in [16] by means of the Lagrange multiplier formalism.

In this paper, the modified Hencky bar-chain method developed in [17] is used to analyse the free transverse vibration of single-stepped Timoshenko and Euler-Bernoulli beams. Note that the modified Hencky bar-chain model [17] represents the improved variant of the classical Hencky bar-chain model [18]. Namely, the Hencky bar-chain model of the stepped beam is obtained based on Hencky bar-chain models of each segments of the stepped beam where the segments represent homogeneous uniform beams. Through the numerical examples given in Section 4 it is demonstrated that the proposed Hencky bar-chain method can be used for the approximate determination of natural frequencies and mode shapes of stepped beams with high accuracy.

2. SINGLE-STEPPED TIMOSHENKO BEAM

Consider transverse free vibration of an elastically end supported free-free Timoshenko beam carrying two lumped masses m_l and m_r at the ends as shown in Fig. 1.

Figure 1: Timoshenko beam with one step change in cross section elastically end supported carrying two tip masses

The beam is composed of two uniform homogeneous segments with Young's modulus E_1 and E_2 , densities ρ_1 and ρ_2 , cross-sectional areas A_1 and A_2 , and second moments of area I_{z1} and I_{z2} . Lengths of the segments are L_1 and L_2 (see Fig. 1). Also, the stiffness of the lateral and rotational springs at the left and the right beam end are denoted, respectively, by $k_{L,l}$, $k_{R,l}$, $k_{L,r}$, and $k_{R,r}$. According to modified variant of the classical Hencky bar-chain model developed in [17], a Timoshenko uniform homogeneous beam of length L shown in Fig. 2(a) may be divided into n equal segments of length L/n (see Fig. 2(b)) and then each of the segments are replaced by a rigid multibody system composed of massless rigid beams carrying lumped masses:

$$m_1 = m_3 = \frac{\rho AL}{6n}, \quad (1)$$

$$m_2 = \frac{2\rho AL}{3n}, \quad (2)$$

with rotary inertias:

$$J_1 = J_3 = \frac{\rho I_z L}{6n}, \quad (3)$$

$$J_2 = \frac{2\rho I_z L}{3n}, \quad (4)$$

connected through frictionless two-degrees of freedom joints (see Fig. 2(c)) with appropriate lateral and rotational springs in them as it is shown in Fig. 2(d) where:

$$k_R = \frac{EI_z}{L/(2n)}, \quad (5)$$

$$k_L = \frac{12EI_z}{(L/(2n))^3(1+\Phi_y)}, \quad (6)$$

$$\Phi_y = \frac{12EI_z}{GAk(L/(2n))^2}. \quad (7)$$

At the end, the rigid multibody systems of the form shown in Fig. 2(c) are fixed to each other to form a modified Hencky bar-chain model of the flexible beam. Applying described modelling procedure to the segments of the stepped beam shown in Fig. 1, a modified Hencky bar-chain model of the Timoshenko stepped beam is obtained as it is shown in Fig. 3. Note that in procedure of discretization of the beam segments, the segment 1 is divided into n_1 equal length segments and the segment 2 into n_2 ones.

Figure 2: The proces of discretization of a homogeneous uniform beam

Based on [17], the corresponding parameters of the obtained Hencky bar-chain model are given below:

$$m_1 = \frac{\rho_1 A_1 L_1}{6n_1} + m_l, \quad (8)$$

$$m_{2i_1} = \frac{2\rho_1 A_1 L_1}{3n_1} (i_1 = 1, \dots, n_1), \quad (9)$$

$$m_{2j_1+1} = \frac{\rho_1 A_1 L_1}{3n_1} (j_1 = 1, \dots, n_1 - 1), \quad (10)$$

$$m_{2n_1+1} = \frac{\rho_1 A_1 L_1}{6n_1} + \frac{\rho_2 A_2 L_2}{6n_2}, \quad (11)$$

$$m_{2n_1+2i_2} = \frac{2\rho_2 A_2 L_2}{3n_2} (i_2 = 1, \dots, n_2), \quad (12)$$

$$m_{2n_1+2j_2+1} = \frac{\rho_2 A_2 L_2}{3n_2} (j_2 = 1, \dots, n_2 - 1), \quad (13)$$

$$m_{2n_1+2n_2+1} = \frac{\rho_2 A_2 L_2}{6n_2} + m_r, \quad (14)$$

$$J_1 = \frac{\rho_1 I_{z1} L_1}{6n_1}, \quad (15)$$

$$J_{2i_1} = \frac{2\rho_1 I_{z1} L_1}{3n_1} (i_1 = 1, \dots, n_1), \quad (16)$$

$$J_{2j_1+1} = \frac{\rho_1 I_{z1} L_1}{3n_1} (j_1 = 1, \dots, n_1 - 1), \quad (17)$$

$$J_{2n_1+1} = \frac{\rho_1 I_{z1} L_1}{6n_1} + \frac{\rho_2 I_{z2} L_2}{6n_2}, \quad (18)$$

$$J_{2n_1+2i_2} = \frac{2\rho_2 I_{z2} L_2}{3n_2} (i_2 = 1, \dots, n_2), \quad (19)$$

$$J_{2n_1+2j_2+1} = \frac{\rho_2 I_{z2} L_2}{3n_2} (j_2 = 1, \dots, n_2 - 1), \quad (20)$$

$$J_{2n_1+2n_2+1} = \frac{\rho_2 I_{z2} L_2}{6n_2}, \quad (21)$$

$$k_{L,i_1} = \frac{12E_1 I_{z1}}{(L_1/(2n_1))^3(1+\Phi_{y1})} (i_1 = 1, \dots, 2n_1), \quad (22)$$

$$k_{R,i_1} = \frac{E_1 I_{z1}}{L_1/(2n_1)} (i_1 = 1, \dots, 2n_1), \quad (23)$$

$$k_{L,i_2} = \frac{12E_2 I_{z2}}{(L_2/(2n_2))^3(1+\Phi_{y2})} (i_2 = 2n_1 + 1, \dots, 2n_1 + 2n_2), \quad (24)$$

$$k_{R,i_2} = \frac{E_2 I_{z2}}{L_2/(2n_2)} (i_2 = 2n_1 + 1, \dots, 2n_1 + 2n_2), \quad (25)$$

$$\Phi_{y1} = \frac{12E_1 I_{z1}}{G_1 A_1 k_1 (L_1/(2n_1))^2}, \quad (26)$$

$$\Phi_{y2} = \frac{12E_2 I_{z2}}{G_2 A_2 k_2 (L_2/(2n_2))^2}, \quad (27)$$

$$x_{l,1} = 0, \quad (28)$$

$$x_{l,j} = \frac{L_1}{4n_1} (j = 2, \dots, 2n_1 + 1), \quad (29)$$

$$x_{r,i_1} = \frac{L_1}{4n_1} (i_1 = 1, \dots, 2n_1), \quad (30)$$

$$x_{r,2n_1+1} = \frac{L_2}{4n_2}, \quad (31)$$

$$x_{l,2n_1+1+i_2} = \frac{L_2}{4n_2}(i_2 = 1, \dots, 2n_2), \quad (32)$$

$$x_{r,2n_1+2n_2+1} = 0, \quad (33)$$

$$x_{r,2n_1+1+i_2} = \frac{L_2}{4n_2}(i_2 = 1, \dots, 2n_2 - 1), \quad (34)$$

where $G_i (i=1,2)$ and $k_i (i=1,2)$ are the shear modulus and the shear coefficients, respectively, of the segments 1 and 2, respectively. Based on the above consideration, the kinetic energy of the considered rigid multibody system reads:

Figure 3: The Hencky bar-chain model of the Timoshenko single-stepped beam

$$T = \frac{1}{2} \sum_{i=1}^{2n_1+2n_2+1} m_i \dot{y}_i^2 + \frac{1}{2} \sum_{i=1}^{2n_1+2n_2+1} J_i \dot{\phi}_i^2, \quad (35)$$

whereas the potential energy of the system is determined by the following expression (see [17]):

$$\begin{aligned} \Pi = & \frac{1}{2} \sum_{i=1}^{2n_1+2n_2} k_{R,i} (\phi_{i+1} - \phi_i)^2 + \frac{1}{2} k_{R,l} \phi_1^2 + \frac{1}{2} k_{R,r} \phi_{2n_1+2n_2+1}^2 \\ & + \frac{1}{2} k_{L,l} y_1^2 + \frac{1}{2} k_{L,r} y_{2n_1+2n_2+1}^2 \\ & + \frac{1}{2} \sum_{i=1}^{2n_1+2n_2} k_{L,i} (y_{i+1} - x_{l,i+1} \phi_{i+1} - y_i - x_{r,i} \phi_i)^2. \end{aligned} \quad (36)$$

Introducing the vector of generalized coordinates $\mathbf{q} = [q_1, \dots, q_{4n_1+4n_2+2}]^T \equiv [\mathbf{q}_y^T, \mathbf{q}_\phi^T]^T$, where $\mathbf{q}_y = [y_1, \dots, y_{2n_1+2n_2+1}]^T$ and $\mathbf{q}_\phi = [\phi_1, \dots, \phi_{2n_1+2n_2+1}]^T$, the expressions for the kinetic energy (35) and the potential energy (36) can be written in the following matrix form:

$$T = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}, \quad (37)$$

$$\Pi = \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q}, \quad (38)$$

where $\mathbf{M} \in R^{(4n_1+4n_2+2) \times (4n_1+4n_2+2)}$ and $\mathbf{K} \in R^{(4n_1+4n_2+2) \times (4n_1+4n_2+2)}$ are the mass and stiffness matrices of the system, respectively, defined as follows:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_y & \mathbf{0}_{(2n_1+2n_2+1) \times (2n_1+2n_2+1)} \\ \mathbf{0}_{(2n_1+2n_2+1) \times (2n_1+2n_2+1)} & \mathbf{M}_\phi \end{bmatrix}, \quad (39)$$

$$\mathbf{M}_y = \text{diag}(m_1, m_2, \dots, m_{2n_1+2n_2+1}), \quad (40)$$

$$\mathbf{M}_\phi = \text{diag}(J_1, J_2, \dots, J_{2n_1+2n_2+1}), \quad (41)$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_y^{(1)} & \mathbf{K}_\phi^{(1)} \\ \mathbf{K}_y^{(2)} & \mathbf{K}_\phi^{(2)} \end{bmatrix}, \quad (42)$$

where $\mathbf{0}_{(2n_1+2n_2+1) \times (2n_1+2n_2+1)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$ is a zero matrix and the components of the matrices $\mathbf{K}_y^{(1)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$, $\mathbf{K}_\phi^{(1)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$, $\mathbf{K}_y^{(2)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$, and $\mathbf{K}_\phi^{(2)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$ are determined by the following expressions [17]:

$$K_{y1,1}^{(1)} = k_{L,l} + k_{L,l}, \quad (43)$$

$$K_{yi,i}^{(1)} = k_{L,i-1} + k_{L,i} \quad (i = 2, \dots, 2n_1 + 2n_2), \quad (44)$$

$$K_{y2n_1+2n_2+1, 2n_1+2n_2+1}^{(1)} = k_{L,2n_1+2n_2} + k_{L,r}, \quad (45)$$

$$K_{yi,i+1}^{(1)} = -k_{L,i} \quad (i = 1, \dots, 2n_1 + 2n_2), \quad (46)$$

$$K_{yi,i-1}^{(1)} = -k_{L,i-1} \quad (i = 2, \dots, 2n_1 + 2n_2 + 1), \quad (47)$$

$$K_{y1,1}^{(2)} = k_{L,l} x_{r,1}, \quad (48)$$

$$K_{yi,i}^{(2)} = -k_{L,i-1} x_{l,i} + k_{L,i} x_{r,i} \quad (i = 2, \dots, 2n_1 + 2n_2), \quad (49)$$

$$K_{y2n_1+2n_2+1, 2n_1+2n_2+1}^{(2)} = -k_{L,2n_1+2n_2} x_{l,2n_1+2n_2+1}, \quad (50)$$

$$K_{yi,i+1}^{(2)} = -k_{L,i} x_{r,i} \quad (i = 1, \dots, 2n_1 + 2n_2), \quad (51)$$

$$K_{yi,i-1}^{(2)} = k_{L,i-1} x_{l,i} \quad (i = 2, \dots, 2n_1 + 2n_2 + 1), \quad (52)$$

$$K_{\phi 1,1}^{(1)} = k_{L,l} x_{r,1}, \quad (53)$$

$$K_{\phi i,i}^{(1)} = -k_{L,i-1} x_{l,i} + k_{L,i} x_{r,i} \quad (i = 2, \dots, 2n_1 + 2n_2), \quad (54)$$

$$K_{\phi 2n_1+2n_2+1, 2n_1+2n_2+1}^{(1)} = -k_{L,2n_1+2n_2} x_{l,2n_1+2n_2+1}, \quad (55)$$

$$K_{\phi i,i+1}^{(1)} = k_{L,i} x_{l,i+1} \quad (i = 1, \dots, 2n_1 + 2n_2), \quad (56)$$

$$K_{\phi i,i-1}^{(1)} = -k_{L,i-1} x_{r,i-1} \quad (i = 2, \dots, 2n_1 + 2n_2 + 1), \quad (57)$$

$$K_{\phi 1,1}^{(2)} = k_{R,l} + k_{R,l} + k_{L,l} x_{r,1}^2, \quad (58)$$

$$K_{\phi 2n_1+2n_2+1, 2n_1+2n_2+1}^{(2)} = k_{R,2n_1+2n_2} + k_{R,r} + k_{L,2n_1+2n_2} x_{l,2n_1+2n_2+1}^2, \quad (59)$$

$$K_{\phi i,i}^{(2)} = k_{R,i-1} + k_{R,i} + k_{L,i-1} x_{l,i}^2 + k_{L,i} x_{r,i}^2 \quad (i = 2, \dots, 2n_1 + 2n_2), \quad (60)$$

$$K_{\phi i,i+1}^{(2)} = -k_{R,i} + k_{L,i} x_{l,i+1} x_{r,i} \quad (i = 1, \dots, 2n_1 + 2n_2), \quad (61)$$

$$K_{\phi i,i-1}^{(2)} = -k_{R,i-1} + k_{L,i-1} x_{r,i-1} x_{l,i} \quad (i = 2, \dots, 2n_1 + 2n_2 + 1). \quad (62)$$

Using the Lagrange equations of the second kind [19]:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial \Pi}{\partial \mathbf{q}} = \mathbf{0}_{(4n_1+4n_2+2) \times (4n_1+4n_2+2)}, \quad (63)$$

where the differentiations with respect to time t are denoted by the overdot symbol, the equations of motion of the system may be obtained in the following matrix form:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}_{(4n_1+4n_2+2) \times (4n_1+4n_2+2)} \quad (64)$$

where $\mathbf{0}_{(4n_1+4n_2+2) \times (4n_1+4n_2+2)}$ is a zero matrix.

Finally, the eigenvalue problem [20] corresponding to the differential equation (64) reads:

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{u} = \mathbf{0}_{(4n_1+4n_2+2) \times (4n_1+4n_2+2)} \quad (65)$$

where ω is the circular natural frequency of the transverse vibration of the stepped beam determined from the frequency equation:

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0, \quad (66)$$

and $\mathbf{u}$ represents the eigenvector associated with the natural frequency ω . Also, the discrete values of the mode shape functions, $Y_r(x) (r=1, \dots, 4n_1+4n_2+2)$, can be obtained in a similar manner as in [17] as follows:

$$Y_r(x_s) = u_{r,s} \quad (s=1, \dots, 2n_1+2n_2+1), \quad (67)$$

or, using normalized mode shapes:

$$\bar{Y}_r(x_s) = u_{r,s} / u_{r,1} \quad (s=1, \dots, 2n_1+2n_2+1), \quad (68)$$

where $\mathbf{u}_r = [u_{r,1}, u_{r,2}, \dots, u_{r,4n_1+4n_2+2}]^T$ is the r -th eigenvector corresponding to the r -th natural frequency ω_r .

3. SINGLE-STEPPED EULER-BERNOULLI BEAM

In the case of a stepped Euler-Bernoulli beam of the shape shown in Fig. 1 the corresponding Hencky bar-chain model is depicted in Fig. 4. In contrast to the Hencky bar-chain model of the Timoshenko stepped beam shown in Fig. 3, for this model one has that $J_s = 0 (s=1, \dots, 2n_1+2n_2+1)$ and $\Phi_{y1} = \Phi_{y2} = 0$ whereas the other parameters are unchanged. The kinetic energy of the rigid multibody system shown in Fig. 4 is:

$$T = \frac{1}{2} \sum_{i=1}^{2n_1+2n_2+1} m_i \dot{y}_i^2, \quad (69)$$

whereas the expression for the potential energy is determined by (36). Now, based on (69), the matrix expression for the kinetic energy reads:

$$T = \frac{1}{2} \dot{\mathbf{q}}_y^T \mathbf{M}_y \dot{\mathbf{q}}_y, \quad (70)$$

where $\mathbf{M}_y \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$ is the mass matrix whose components are defined as:

$$\mathbf{M}_y = \text{diag}(m_1, m_2, \dots, m_{2n_1+2n_2+1}), \quad (71)$$

Figure 4: The Hencky bar-chain model of the Euler-Bernoulli single-stepped beam

Using the Lagrange equations of the second kind (63) yields the following differential equations of the system written in the matrix form:

$$\mathbf{M}_y \ddot{\mathbf{q}}_y + \mathbf{K}_y^{(1)} \mathbf{q}_y + \mathbf{K}_\phi^{(1)} \mathbf{q}_\phi = \mathbf{0}_{(2n_1+2n_2+1) \times 1}, \quad (72)$$

$$\mathbf{K}_y^{(2)} \mathbf{q}_y + \mathbf{K}_\phi^{(2)} \mathbf{q}_\phi = \mathbf{0}_{(2n_1+2n_2+1) \times 1}, \quad (73)$$

where the components of the matrices $\mathbf{K}_y^{(1)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$, $\mathbf{K}_\phi^{(1)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$, $\mathbf{K}_y^{(2)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$, and $\mathbf{K}_\phi^{(2)} \in R^{(2n_1+2n_2+1) \times (2n_1+2n_2+1)}$ are determined by the expressions (43)-(62) in which $\Phi_{y1} = \Phi_{y2} = 0$ should be taken out.

For the vibration analysis of the stepped beam shown in Fig. 4, the matrix $\mathbf{q}_\phi$ from (73) can be expressed in term of $\mathbf{q}_y$ as follows:

$$\mathbf{q}_\phi = -(\mathbf{K}_\phi^{(2)})^{-1} \mathbf{K}_y^{(2)} \mathbf{q}_y, \quad (74)$$

and substituted into (72). After that the following differential equation in $\mathbf{q}_y$ is obtained:

$$\mathbf{M}_y \ddot{\mathbf{q}}_y + \mathbf{K}_1 \mathbf{q}_y = \mathbf{0}_{(2n_1+2n_2+1) \times 1}, \quad (75)$$

where

$$\mathbf{K}_1 = \mathbf{K}_y^{(1)} - \mathbf{K}_\phi^{(1)} (\mathbf{K}_\phi^{(2)})^{-1} \mathbf{K}_y^{(2)}. \quad (76)$$

The eigenvalue problem corresponding to the differential equation (75) reads:

$$(\mathbf{K}_1 - \omega^2 \mathbf{M}_y) \mathbf{u}_1 = \mathbf{0}_{(2n_1+2n_2+1) \times 1} \quad (77)$$

where ω is the natural frequencies of the single-stepped Euler-Bernoulli beam in transverse vibration determined from the following frequency equation:

$$\det(\mathbf{K}_1 - \omega^2 \mathbf{M}_y) = 0, \quad (78)$$

and $\mathbf{u}_1$ represents the eigenvector associated with the natural frequency ω . Further, the discrete values of the mode shape functions, $Y_r(x) (r=1, \dots, 2n_1+2n_2+1)$, are defined by:

$$Y_r(x_s) = u_{r,s} \quad (s=1, \dots, 2n_1+2n_2+1), \quad (79)$$

or, using normalized mode shapes:

$$\bar{Y}_r(x_s) = u_{r,s} / u_{r,1} \quad (s=1, \dots, 2n_1+2n_2+1), \quad (80)$$

where $\mathbf{u}_r = [u_{r,1}, u_{r,2}, \dots, u_{r,2n_1+2n_2+1}]^T$ is the r th eigenvector corresponding to the r th natural frequency ω_r .

4. NUMERICAL EXAMPLES

In this section the following dimensionless parameters will be used:

$$\beta_l = \frac{m_l}{m}, \quad \beta_r = \frac{m_r}{m}, \quad (81)$$

$$\bar{k}_{R,l} = \frac{L}{E_1 I_{z1}} k_{R,l}, \quad \bar{k}_{R,r} = \frac{L}{E_1 I_{z1}} k_{R,r}, \quad (82)$$

$$\bar{k}_{L,l} = \frac{L^3}{E_1 I_{z1}} k_{L,l}, \quad \bar{k}_{L,r} = \frac{L^3}{E_1 I_{z1}} k_{L,r}, \quad (83)$$

where $L = L_1 + L_2$ and $m = \rho_1 A_1 L_1 + \rho_2 A_2 L_2$. In addition, the percentage frequency errors are calculated as:

$$\frac{|\text{approximative value} - \text{exact value}|}{\text{exact value}} \times 100\%. \quad (84)$$

4.1. Example 1

In this subsection, for the Euler-Bernoulli stepped beams, the comparison of accuracies between the approach proposed in this study and the exact approach given in [5] are performed. The three types of single-stepped beams are considered, that is: Type 1 (beams of constant thickness but with a step change in width), Type 2 (beams of constant width but with a step change in thickness), and Type 3 (beams of circular cross-section with a step change in diameter). The following physical characteristics of the beams are used: Young's modulus $E_1 = E_2 = 2.068 \cdot 10^{11} \text{ N/m}^2$, and mass density $\rho_1 = \rho_2 = 7850 \text{ kg/m}^3$. The geometric characteristics of the single-stepped beams are: (Type 1) $L = 1 \text{ m}$, $L_1 = 0.25L$, $L_2 = 0.75L$, $b_1 = 0.025 \text{ m}$, $b_2 = 0.5b_1$, $h_1 = h_2 = 0.02 \text{ m}$; (Type 2) $L = 1 \text{ m}$, $L_1 = 0.25L$, $L_2 = 0.75L$, $b_1 = b_2 = 0.025 \text{ m}$, $h_1 = 0.02 \text{ m}$, $h_2 = 0.5h_1$; (Type 3) $L = 1 \text{ m}$, $L_1 = 0.25L$, $L_2 = 0.75L$, $d_1 = 0.02 \text{ m}$, $d_2 = 0.5d_1$, where h_1 and h_2 are thickness of the segments 1 and 2, respectively, b_1 and b_2 are width of the segments 1 and 2, respectively. In addition, the numbers of divisions of beam sections are chosen to be $n_1 = n_2 = 15$ and the following abbreviations for the boundary conditions at the beam ends are used:

Clamped-Free ends	C-F
Clamped-Clamped ends	C-C
Clamped-Pinned ends	C-P
Pinned-Pinned ends	P-P

Also, in this subsection the i th dimensionless natural frequency Ω_i is related to the i th natural frequency ω_i (rad/s) as follows:

$$\Omega_i = \sqrt[4]{\frac{\rho_1 A_1 L^4 \omega_i^2}{E_1 I_{z1}}}. \quad (85)$$

Table 1: The dimensionless natural frequencies of the Euler-Bernoulli single-stepped beam of Type 1 for various kinds of boundary conditions

Boundary conditions	Present study	[5]
C-F	2.08492	2.0849
	4.91534	4.9153
	7.90447	7.9045
C-C	4.9579	4.9579
	7.90424	7.9043
	10.9708	10.9708
C-P	4.16925	4.1692
	7.16494	7.1650
	10.1856	10.1857
P-P	3.11602	3.1160
	6.22181	6.2218
	9.42479	9.4248

Table 2: The dimensionless natural frequencies of the Euler-Bernoulli single-stepped beam of Type 2 for various kinds of boundary conditions

Boundary conditions	Present study	[5]
C-F	1.66872	1.6687
	3.95285	3.9528
	6.16349	6.1635
C-C	3.99062	3.9906
	6.16176	6.1618
	8.30529	8.3054
C-P	3.3603	3.3603
	5.64571	5.6457
	7.74037	7.7404
P-P	2.22703	2.2270
	4.57059	4.5706
	7.09833	7.0984

Table 3: The dimensionless natural frequencies of the Euler-Bernoulli single-stepped beam of Type 3 for various kinds of boundary conditions

Boundary conditions	Present study	[5]
C-F	1.71509	1.7151
	4.13954	4.1395
	6.38411	6.3841
C-C	4.17665	4.1766
	6.38235	6.3824
	8.21542	8.2155
C-P	3.50359	3.5036
	5.90229	5.9023
	7.7142	7.7143
P-P	2.1602	2.1602
	4.39954	4.3995
	6.96604	6.9661

4.2. Example 2

The aim of this subsection is to examine the effects of tip mass m_r and springs $k_{L,r}$ and $k_{R,r}$ on the values of dimensionless natural frequencies (85) of the cantilever single-stepped Euler-Bernoulli beam with following material and geometric characteristics: $E_1 = E_2 = 2.068 \cdot 10^{11} \text{ N/m}^2$, $\rho_1 = \rho_2 = 7850 \text{ kg/m}^3$, $h_1 = 19.05 \text{ mm}$, $h_2 = 5.49 \text{ mm}$, $b_1 = b_2 = 25.4 \text{ mm}$, $L_1 = 254 \text{ mm}$, $L_2 = 140 \text{ mm}$. Note that these geometric characteristics are the same as in [10, 13]. Numerical calculations are carried out for $n_1 = n_2 = 15$.

Table 4: The dimensionless natural frequencies $\Omega_i (i=1, \dots, 5)$ of the cantilever Euler-Bernoulli single-stepped beam for $\bar{k}_{L,r} = \bar{k}_{R,r} = 50$ and various values of the parameter β_r

β_r	0.0	0.2	0.4	0.6	0.8
Ω_1	3.17988	3.15373	3.10506	3.01962	2.90701
Ω_2	5.67839	4.20623	3.69895	3.47185	3.3733
Ω_3	7.1018	6.54992	6.50199	6.48529	6.47683
Ω_4	9.04833	8.06672	8.01557	7.99807	7.98924
Ω_5	11.7556	11.3215	11.2924	11.282	11.2767

Table 5: The dimensionless natural frequencies $\Omega_i (i=1, \dots, 5)$ of the cantilever Euler-Bernoulli single-stepped beam for $\beta_r = 0.6$ and various values of the dimensionless stiffnesses $\bar{k}_{L,r}$ and $\bar{k}_{R,r}$

$k_{T,r}$	20	0	50	100	400
$k_{R,r}$	0	20	100	200	400
Ω_1	2.54214	1.54377	3.01979	3.18537	3.23749
Ω_2	2.96215	3.30788	3.47245	3.88533	5.35207
Ω_3	5.78771	6.47771	6.48716	6.49034	6.51388
Ω_4	7.46398	7.98827	8.00108	8.00358	8.01159
Ω_5	10.5672	11.2713	11.2856	11.2875	11.2894

4.3. Example 3

In this subsection, the accuracy of the approach presented in the case of a cantilever Timoshenko single-stepped beam is examined through the comparison with the exact results given in [4]. For this case numerical calculations are carried out for $n_1 = n_2 = 25$. The following material and geometric characteristics are used: $E_1 = E_2 = 2.068 \cdot 10^{11} \text{ N/m}^2$, $\rho_1 = \rho_2 = 7850 \text{ kg/m}^3$, Poisson's ratio $\nu = 0.3$, shear coefficient $k = 5/6$ (rectangular cross-section), shear modulus $G_1 = E_1 / [2(1+\nu)]$ and $G_2 = E_2 / [2(1+\nu)]$, $L = 1 \text{ m}$, $L_1 = (2/3)L$, $L_2 = (1/3)L$, $h_1 = \sqrt{12\eta}L_1$, $h_2 = 0.8h_1$, $b_1 = b_2 = 0.2h_1$. Dimensionless natural frequencies are calculated as:

$$\Omega_i = \sqrt{\frac{\rho_1 A_1 L^4}{E_1 I_{z1}}} \omega_i. \quad (86)$$

Table 6: The dimensionless natural frequencies $\Omega_i (i=1, \dots, 5)$ of the cantilever Timoshenko single-stepped beam for $\eta = 0.0036$ and $\bar{k}_{L,r} = \bar{k}_{R,r} = 0$ as well as various values of the parameter β_r

β_r	Present study	[4]
0.0	3.77304	3.77
	19.8044	19.80
	47.3495	47.35
	84.1198	84.14
	124.994	125.06
0.2	2.73086	2.73
	15.9197	15.92
	41.3668	41.36
	77.0646	77.08
	116.249	116.30
0.4	2.24179	2.24
	14.9888	14.98
	40.4193	40.42
	76.1572	76.17
	115.311	115.36
0.6	1.94574	1.94
	14.5765	14.57
	40.0404	40.04
	75.8071	75.82
	114.96	115.01
0.8	1.74235	1.74
	14.3443	14.34
	39.8368	39.83
	75.6218	75.63
	114.775	114.82
1.0	1.59167	1.59
	14.1955	14.19
	39.7097	39.71
	75.507	75.52
	114.662	114.71

Table 7: The dimensionless natural frequencies $\Omega_i (i = 1, \dots, 5)$ of the cantilever Timoshenko single-stepped beam for $\eta = 0.01$ and $\bar{k}_{L,r} = \bar{k}_{R,r} = 0$ as well as various values of the parameter β_r

β_r	Present study	[4]
0.0	3.67728	3.67
	17.5286	17.52
	38.8053	38.81
	64.1336	64.15
	90.4665	90.53
0.2	2.67888	2.67
	14.4002	14.40
	34.3773	34.38
	59.505	59.52
	85.5861	85.64
0.4	2.20359	2.20
	13.6049	13.60
	33.6057	33.60
	58.837	58.85
	84.9603	85.02
0.6	1.91444	1.91
	13.2479	13.24
	33.2923	33.29
	58.5749	58.59
	84.7194	84.77
0.8	1.71529	1.71
	13.0457	13.04
	33.1227	33.12
	58.4352	58.45
	84.592	84.65
1.0	1.56753	1.56
	12.9156	12.91
	33.0166	33.01
	58.3484	58.36
	84.5132	84.57

5. CONCLUSIONS

In this paper the modified Hencky bar-chain model developed in [17] has been applied to the free transverse vibration analysis of Euler-Bernoulli and Timoshenko single-stepped beams. In this manner, the application range of the modified Hencky bar-chain model in the field of vibration of nonhomogeneous nonuniform beams has been achieved. Through the numerical examples given in Section 4 the accuracy of the the modified Hencky bar-chain model approach in comparison to exact results available in the literature has been proved. Based on numerical results it can be said that the modified Hencky bar-chain model approach, for relatively small number of divisions of the segments, gives results which are in excellent agreement with published results. The considerations in this paper show that the this approach can be easy adapted to study the free transverse vibration of multiple-stepped both Euler-Bernoulli and Timoshenko

beams. Also, some new results regarding to the influence of end masses and stiffness of end springs on the values of natural frequencies of single-stepped beams have been given.

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REFERENCES

- [1] X. Tong, B. Tabarrok, and K.Y. Yeh, "Vibration Analysis of Timoshenko Beams with Non-Homogeneity and Varying Cross-Section," J. Sound Vib., Vol. 186(5), pp. 821–835, (1995)
- [2] S.K. Jang and C.W. Bert, "Free vibrations of stepped beams: exact and numerical solutions," J. Sound Vib., Vol. 130, pp. 342–346, (1989)
- [3] S.K. Jang and C.W. Bert, "Free vibrations of stepped beams: higher mode frequencies and effects of steps on frequency," J. Sound Vib., Vol. 132, pp. 164–168, (1989)
- [4] R.E. Rossi, P.A.A. Laura and R.H. Gutierrez, "A Note on Transverse Vibrations of a Timoshenko Beam of Non-Uniform Thickness Clamped at One End and Carrying a Concentrated Mass at the Other," J. Sound Vib., Vol. 143(3), pp. 491–502, (1990)
- [5] S. Naguleswaran, "Natural Frequencies, Sensitivity and Mode Shape Details of an Euler–Bernoulli Beam with One-Step Change in Cross-Section and with Ends on Classical Supports," J. Sound Vib., Vol. 252(4), pp. 751–767, (2002)
- [6] S.C. Stanton and B.P. Mann, "On the Dynamic Response of Beams with Multiple Geometric or Material Discontinuities," Mech. Syst. Signal Pr., Vol. 24, pp. 1409–1419, (2010)
- [7] Janaína da Cunha Vaz and José Juliano de Lima Junior, "Vibration Analysis of Euler-Bernoulli Beams in Multiple Steps and Different Shapes of Cross Section," J. Vib. Control, Vol. 22(1), pp. 193–204, (2016)
- [8] M.S. Cao, W. Xu, Z. Su, W. Ostachowicz, and N. Xia, "Local Coordinate Systems-Based Method to Analyze High-Order Modes of n-step Timoshenko Beam," J. Vib. Control, Vol. 23(1), pp. 89–102, (2017)
- [9] Q. Mao and S. Pietrzko, "Free Vibration Analysis of Stepped Beams by Using Adomian Decomposition Method," Appl. Math. Comput., Vol. 217, pp. 3429–3441, (2010)
- [10] Q. Mao, "Free vibration analysis of multiple-stepped beams by using Adomian decomposition method," Math. Comput. Model., Vol. 54(1-2), pp. 756–764, (2011)
- [11] K. Suddoung, J. Charoensuk and N. Wattanasakulpong, "Application of the Differential Transformation Method to Vibration Analysis of Stepped Beams with Elastically Constrained Ends," J. Vib. Control, Vol. 19(16), pp. 2387–2400, (2013)

- [12] K. Mazanoglu, "Natural Frequency Analyses of Segmented Timoshenko-Euler Beams Using the Rayleigh-Ritz Method," *J. Vib. Control*, Vol. 23(13), pp. 2135-2154, (2017)
- [13] Z.R. Lu, M. Huang, J.K. Liu, W.H. Chen and W.Y. Liao, "Vibration Analysis of Multiple-Stepped Beams With the Composite Element Model," *J. Sound Vib.*, Vol. 322(4-5), pp. 1070-1080, (2009)
- [14] G. Duan and X. Wang, "Free Vibration Analysis of Multiple-Stepped Beams by the Discrete Singular Convolution," *Appl. Math. Comput.*, Vol. 219, pp. 11096–11109, (2013)
- [15] J. Lee, "Application of Chebyshev-tau Method to the Free Vibration Analysis of Stepped Beams," *Int. J. Mech. Sci.*, Vol. 101-102, pp. 411-420, (2015)
- [16] D. Cekus, "Use of Lagrange Multiplier Formalism to Solve Transverse Vibrations Problem of Stepped Beams According to Timoshenko Theory," *Scientific Research of the Institute of Mathematics and Computer Science*, Vol. 2(10), pp. 49-56, (2011)
- [17] S. Šalinić, "An Improved Variant of Hencky Bar-Chain Model for Buckling and Bending Vibration of Beams With End Masses and Springs," *Mech. Syst. Signal Pr.*, Vol. 90, pp. 30-43, (2017)
- [18] C.M. Wang, H. Zhang, R.P. Gao, W.H. Duan and N. Challamel, "Hencky Bar-Chain Model for Buckling and Vibration of Beams with Elastic End Restraints," *Int. J. Struct. Stab. Dy.*, Vol. 15(7), pp. 1540007, (2015)
- [19] A.I. Lurie, "Analytical Mechanics", Springer-Verlag, New York, (2002)
- [20] L. Meirovitch, "Fundamentals of Vibrations", McGraw-Hill, New York, (2001)