$$\begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 & -l_1 \\ 0 & 0 & 0 & 0 \\ \sin(\theta_1) & -\cos(\theta_1) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_x & o_x & a_x & p_{xD} \\ n_y & o_y & a_y & p_{yD} \\ n_z & o_z & a_z & p_{zD} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & l_2 \cdot \cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & l_2 \cdot \sin(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & l_3 \cdot \cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & l_3 \cdot \sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5)$$

ili

$$\begin{pmatrix} n_x \cdot \cos(\theta_1) + n_y \cdot \sin(\theta_1) & o_x \cdot \cos(\theta_1) + o_y \cdot \sin(\theta_1) & a_x \cdot \cos(\theta_1) + a_y \cdot \sin(\theta_1) & p_{xD} \cdot \cos(\theta_1) + p_{yD} \cdot \sin(\theta_1) - l_1 \\ n_z & o_z & a_z & p_{zD} \\ n_x \cdot \sin(\theta_1) - n_y \cdot \cos(\theta_1) & o_x \cdot \sin(\theta_1) - o_y \cdot \cos(\theta_1) & a_x \cdot \sin(\theta_1) - a_y \cdot \cos(\theta_1) & p_{xD} \cdot \sin(\theta_1) - p_{yD} \cdot \cos(\theta_1) \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_2 + \theta_3) & -\sin(\theta_2 + \theta_3) & 0 & l_2 \cdot \cos(\theta_2) + l_3 \cdot \cos(\theta_2 + \theta_3) \\ \sin(\theta_2 + \theta_3) & \cos(\theta_2 + \theta_3) & 0 & l_2 \cdot \sin(\theta_2) + l_3 \cdot \sin(\theta_2 + \theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6)$$

Poređenjem elemenata 3 vrste i 4 kolone obe matrice dobija se:

$$p_{xD} \cdot \cos(\theta_1) - p_{yD} \cdot \sin(\theta_1) = 0 \quad (7)$$

Rešavanjem jednačine dobija se:

$$\theta_1 = \arctg\left(\frac{p_{yD}}{p_{xD}}\right) \quad (8)$$

Poređenjem elemenata iz 4 kolone, 1 i 2 vrste dobija se sledeći sistem jednačina:

$$\begin{cases} p_{xD} \cdot \cos(\theta_1) + p_{yD} \cdot \sin(\theta_1) - l_1 = l_2 \cdot \cos(\theta_2) + l_3 \cdot \cos(\theta_2 + \theta_3) \\ p_{zD} = l_2 \cdot \sin(\theta_2) + l_3 \cdot \sin(\theta_2 + \theta_3) \end{cases} \quad (9)$$

Premeštanjem na levu stranu izraza za θ_2 , kvadriranjem svake jednačine, a zatim sabiranjem jednačina i zamenom:

$$A = \frac{l_1^2 + l_2^2 - l_3^2 - 2l_1(p_{xD} \cos(\theta_1) - p_{yD} \sin(\theta_1)) + 2p_{xD}p_{yD} \sin(\theta_1) \cos(\theta_1) + (p_{xD}^2 - p_{yD}^2) \cos^2(\theta_1) + p_{yD}^2 + p_{zD}^2}{2l_2 p_{zD}} \quad (10)$$

$$B = \frac{-2l_1 l_2 + 2l_2(p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1))}{2l_2 p_{zD}}$$

dobija se jednakost:

$$A - B \cos(\theta_2) = \sin \theta_2 \quad (11)$$

čije rešenje je dato u obliku :

$$\theta_2 = \arccos\left(\frac{AB \pm \sqrt{1 + B^2 - A^2}}{1 + B^2}\right) \quad (12)$$

Takođe iz izraza (9), premeštanjem na levu stranu izraza za θ_2 , deljenjem druge jednačine sa prvom dobija se izraz za θ_3 :

$$\begin{pmatrix} \cos(\theta_2) \cdot (n_x \cos(\theta_1) + n_y \sin(\theta_1)) + n_z \sin(\theta_2) & \cos(\theta_2) \cdot (o_x \cos(\theta_1) + o_y \sin(\theta_1)) + o_z \sin(\theta_2) & \cos(\theta_2) \cdot (a_x \cos(\theta_1) + a_y \sin(\theta_1)) & \cos(\theta_2) \cdot (p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1) - l_1) + p_{zD} \sin(\theta_2) - l_2 \\ -\sin(\theta_2) \cdot (n_x \cos(\theta_1) + n_y \sin(\theta_1)) + n_z \cos(\theta_2) & -\sin(\theta_2) \cdot (o_x \cos(\theta_1) + o_y \sin(\theta_1)) + o_z \cos(\theta_2) & -\sin(\theta_2) \cdot (a_x \cos(\theta_1) + a_y \sin(\theta_1)) & -\sin(\theta_2) \cdot (p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1) - l_1) + p_{zD} \cos(\theta_2) \\ n_x \sin(\theta_1) - n_y \cos(\theta_1) & o_x \sin(\theta_1) - o_y \cos(\theta_1) & a_x \sin(\theta_1) - a_y \cos(\theta_1) & p_{xD} \sin(\theta_1) - p_{yD} \cos(\theta_1) \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & l_3 \cdot \cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & l_3 \cdot \sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (15)$$

Balansirajući izraze iz 4 kolone, 1 i 2 vrste i njihovim deljenjem, dobija se:

$$\theta_3 = \arctg\left(\frac{-\sin(\theta_2) \cdot (p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1) - l_1) + p_{zD} \cos(\theta_2)}{\cos(\theta_2) \cdot (p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1) - l_1) + p_{zD} \sin(\theta_2) - l_2}\right) \quad (16)$$

Nakon toga, prelazeći na mehanizam orijentisanja, poznavajući ugao kašike u odnosu na referentni koordinatni sistem α , slika 1, može se napisati:

$$\theta_2 + \theta_3 + \theta_4 = \alpha \quad (17)$$

$$p_x - p_{xD} = l_4 \cos(\alpha) \cos(\theta_1)$$

$$p_y - p_{yD} = l_4 \cos(\alpha) \sin(\theta_1) \quad (18)$$

$$p_z - p_{zD} = l_4 \sin(\alpha)$$

Iz jednačina (17) i (18) sledi:

$$\theta_4 = \alpha - \theta_2 - \theta_3 \quad (20)$$

$$p_{xD} = p_x - l_4 \cos(\alpha) \cos(\theta_1)$$

$$p_{yD} = p_y - l_4 \cos(\alpha) \sin(\theta_1) \quad (21)$$

$$p_{zD} = p_z - l_4 \sin(\alpha)$$

Prema tome, inverzni kinematički model radnog uređaja bagera sa kašikom je:

$$\theta_1 = \arctg\left(\frac{p_y}{p_x}\right)$$

$$\theta_2 = \arccos\left(\frac{AB \pm \sqrt{1 + B^2 - A^2}}{1 + B^2}\right) \quad (22)$$

$$\theta_3 = \arctg\left(\frac{-\sin(\theta_2) \cdot (p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1) - l_1) + p_{zD} \cos(\theta_2)}{\cos(\theta_2) \cdot (p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1) - l_1) + p_{zD} \sin(\theta_2) - l_2}\right)$$

$$\theta_4 = \alpha - \theta_2 - \theta_3$$

na koje dodajemo izraze (10) i (21).

3. INVERZNI KINEMATSKI MODEL DOBIJEN GEOMETRIJSKIM METODOM

Kinematska šema otvorenog kinematskog lanca radnog uređaja bagera, za proizvoljan položaj dat je na slici 2.

2. Za dati kinematski lanac poznate su dužine elemenata l_1 , l_2 , l_3 , l_4 , položaj vrha zuba kašike, p_x , p_y , p_z i njegova orijentacija α . Za ove uslove potrebno je definisanje uglova θ_1 , θ_2 , θ_3 , θ_4 .

$$\theta_1 = \arctg\left(\frac{p_y}{p_x}\right) \quad (23)$$

Ugao θ_3 je određen iz trougla BCD kosinusnom teoremom:

$$BD^2 = BC^2 + CD^2 - 2BC \cdot CD \cdot \cos(\pi - \theta_3) \quad (24)$$

Uvođenjem zavisnosti sa slike 2:

$$BD = r = \sqrt{r_o^2 + r_v^2} \quad (25)$$

$$r_o = \sqrt{p_x^2 + p_y^2} - l_1 - l_4 \cdot \cos(\alpha)$$

$$r_v = p_z + l_4 \cdot \sin(\alpha)$$

jednačina (23) postaje:

$$r_o^2 + r_v^2 = l_2^2 + l_3^2 - 2l_2l_3 \cos(\pi - \theta_3) \quad (26)$$

i dobija se:

$$\theta_3 = \pi + \arccos\left(\frac{l_2^2 + l_3^2 - r_o^2 - r_v^2}{2l_2l_3}\right) \quad (27)$$

ako je

$$l_2 - l_3 \leq r \leq l_2 + l_3 \quad (28)$$

ugao θ_2 definisan je 2 koraka iz trouglova BD_0D i BCD ,

$$\theta_2 = \theta_{21} + \theta_{22} \quad (29)$$

Iz trougla BD_0D dobija se:

$$\theta_{21} = \arctg\left(\frac{r_v}{r_o}\right) \quad (30)$$

Iz trougla BCD , sinusnom teoremom dobija se

$$\theta_{22} = \arcsin\left(\frac{l_3 \sin(\theta_3)}{r}\right) \quad (31)$$

i na kraju

$$\theta_2 = \arctg\left(\frac{r_v}{r_o}\right) + \arcsin\left(\frac{l_3 \sin(\theta_3)}{r}\right) \quad (32)$$

Za ugao $\theta_4 = \theta_4(\alpha)$ korišćena je jednačina (20).

Prema tome, inverzni kinematski model radnog uređaja bagera sa kašikom dobijen geometrijskom metodom je:

$$\theta_1 = \arctg\left(\frac{p_y}{p_x}\right) \quad (33)$$

$$\theta_2 = \arctg\left(\frac{r_v}{r_o}\right) + \arcsin\left(\frac{l_3 \sin(\theta_3)}{r}\right)$$

$$\theta_3 = \pi + \arccos\left(\frac{l_2^2 + l_3^2 - r_o^2 - r_v^2}{2l_2l_3}\right)$$

$$\theta_4 = \alpha - \theta_2 - \theta_3$$

u koji uvodimo izraz (25).

Slika 2: Kinematska šema otvorenog kinematskog lanca radnog uređaja bagera sa kašikom

4. ODNOSI IZMEĐU PROMENLJIVIH KINEMATSKIH VEZA I POGONSKIH HIDRAULIČKIH CILINDARA

Posmatrajući sliku 1, može se za cilindar strele FG dobiti jednačina:

$$\varphi_2 = \pi - \beta_2 - \gamma_2 - \theta_2$$

$$s_2 = \sqrt{BG^2 + BF^2 + 2BG \cdot BF \cdot \cos(\varphi_2)} \quad (34)$$

Slično tome, za cilindar držača kašike HK, dobija se jednačina:

$$\varphi_3 = \theta_3 + \beta_3 + \gamma_3 - 2\pi$$

$$s_3 = \sqrt{HC^2 + KC^2 + 2HC \cdot KC \cdot \cos(\varphi_3)} \quad (35)$$

Za cilindar kašike, prvo se rešava polužni mehanizam DQPN. Ako se krene od ugla:

$$\psi_4 = \theta_4 + \gamma_4 - 2\pi$$

i izraza:

$$a = \frac{DQ^2 - PQ^2 + NP^2 + ND^2 - 2 \cdot NP \cdot ND \cdot \cos(\psi_4)}{2 \cdot DQ \cdot NP \cdot \sin(\psi_4)} \quad (36)$$

$$b = \frac{NP \cdot \cos(\psi_4) + ND}{NP \cdot \sin(\psi_4)} \quad)$$

dobija se

$$\varphi_p = \arccos \left(\frac{a \cdot b \pm \sqrt{1 + b^2 - a^2}}{1 + b^2} \right) \quad (37)$$

i konačno:

$$\varphi_4 = \beta_4 + \varphi_p + \delta_4$$

$$s_4 = \sqrt{NM^2 + LN^2 + 2NM \cdot LM \cdot \cos(\varphi_4)} \quad (38)$$

5. ALGORITAM ZA KORIŠĆENJE MATRIČNOG METODA

a) Željena putanja vrha zuba kašike je data u parametarskoj formi u odnosu na referentni system $O_0x_0y_0z_0$,

$$x = x(u)$$

$$y = y(u)$$

$$z = z(u)$$

b) Orijentacija kašike je izražena tokom pomeranja na putanji u odnosu na tačke na istoj,

$$\alpha = \alpha(x(u), y(u), z(u))$$

c) Postaviti koordinata putanje za koordinate vrha zuba kašike $p_x = x(u)$

$$p_y = y(u)$$

$$p_z = z(u)$$

d) Ugao θ_1 je određen na osnovu izraza (22).

e) Koordinate tačke D definišu se iz izraza (21).

f) Uglovi $\theta_2, \theta_3, \theta_4$ definisani su izrazima (22).

g) Dužine hidrauličkih cilindara definisani su sa (34), (35) i (38).

6. ZAKLJUČAK

- Inverzni kinematski model radnog uređaja bagera sa kašikom može biti definisan primenom kinematskog modela ili primenom geometrijskog metoda;
- Poređenjem rezultata ova dva postupka mogu se konstatovati identične zavisnosti;
- Može se konstatovati da jednačine sadrže inverzne trigonometrijske funkcije, koje su uzrok poteškoća u programiranju i zahtevaju posebnu pažnju.

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Determining the Inverse Kinematics Model of a Bucket Excavator's Digging Equipment

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This paper presents a method to build the inverse kinematics model of a bucket excavator's digging equipment. The model is determined by using two different methods, the matrix method applied on the decoupled forward kinematics model and respectively the geometric model.

Keywords: Inverse kinematics; Bucket excavator; Matrix method

1. INTRODUCTION

The inverse kinematics model is especially important because it's the base for programming command and control of mechatronic systems. In specialty literature are presented some methods for determining the inverse kinematics model, out of which are used more often: matrix method, geometric method and Jacobian matrix method. In this paper the determination is carried out using the first 2 methods.

The attached problem can be expressed as follows: determine the variables of the driving joints while knowing the position and orientation of the end-effector.

2. INVERSE KINEMATICS MODEL DETERMINED BY USING THE MATRIX METHOD

In this method, the global transformation matrix of the end-effector is known:

$$U = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} x_0 \cdot x_4 & x_0 \cdot y_4 & x_0 \cdot z_4 & p_x \\ y_0 \cdot x_4 & y_0 \cdot y_4 & y_0 \cdot z_4 & p_y \\ z_0 \cdot x_4 & z_0 \cdot y_4 & z_0 \cdot z_4 & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1)$$

which is the position of the bucket's tooth tip and its orientation related to the fixed reference frame, figure 1.

Applying this method is easier if the entire mechanism is split up in two parts, positioning mechanism having the first 3 degrees of mobility and orientation mechanism having the remaining degrees of mobility.

Figure 1: The kinematic scheme of the bucket excavator's digging equipment

According to this division, the forward kinematics model for the excavator's mechanism is expressed as:

$${}_0T^4 = {}_0T^3 \cdot {}_3T^4 \quad (2)$$

First, the problem regarding the positioning mechanism is solved for which we have the expression:

$$U_D = {}_0T^1 \cdot {}_1T^2 \cdot {}_2T^3 \quad (3)$$

This relation is multiplied to the right with the inverse matrix of the ${}_0T^1$, which is ${}_1T^0$ and is obtained:

$${}_1T^0 \cdot U_D = {}_1T^2 \cdot {}_2T^3 \quad (4)$$

In the right side of the equation is only one unknown, θ_1 , and in the left side are the other unknowns.

Doing the calculations:

$$\begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 & -l_1 \\ 0 & 0 & 0 & 0 \\ \sin(\theta_1) & -\cos(\theta_1) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_x & o_x & a_x & p_{xD} \\ n_y & o_y & a_y & p_{yD} \\ n_z & o_z & a_z & p_{zD} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & l_2 \cdot \cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & l_2 \cdot \sin(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & l_3 \cdot \cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & l_3 \cdot \sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5)$$

or

$$\begin{pmatrix} n_x \cdot \cos(\theta_1) + n_y \cdot \sin(\theta_1) & o_x \cdot \cos(\theta_1) + o_y \cdot \sin(\theta_1) & a_x \cdot \cos(\theta_1) + a_y \cdot \sin(\theta_1) & p_{xD} \cdot \cos(\theta_1) + p_{yD} \cdot \sin(\theta_1) - l_1 \\ n_z & o_z & a_z & p_{zD} \\ n_x \cdot \sin(\theta_1) - n_y \cdot \cos(\theta_1) & o_x \cdot \sin(\theta_1) - o_y \cdot \cos(\theta_1) & a_x \cdot \sin(\theta_1) - a_y \cdot \cos(\theta_1) & p_{xD} \cdot \sin(\theta_1) - p_{yD} \cdot \cos(\theta_1) \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_2 + \theta_3) & -\sin(\theta_2 + \theta_3) & 0 & l_2 \cdot \cos(\theta_2) + l_3 \cdot \cos(\theta_2 + \theta_3) \\ \sin(\theta_2 + \theta_3) & \cos(\theta_2 + \theta_3) & 0 & l_2 \cdot \sin(\theta_2) + l_3 \cdot \sin(\theta_2 + \theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6)$$

By comparing the elements from row 3, column 4 of both matrices we obtain:

$$p_{xD} \cdot \cos(\theta_1) - p_{yD} \cdot \sin(\theta_1) = 0 \quad (7)$$

And by solving the equation is obtained:

$$\theta_1 = \arctg\left(\frac{p_{yD}}{p_{xD}}\right) \quad (8)$$

By comparing the elements from columns 4, rows 1 and 2 the following system of equations is obtained:

$$\begin{cases} p_{xD} \cdot \cos(\theta_1) + p_{yD} \cdot \sin(\theta_1) - l_1 = l_2 \cdot \cos(\theta_2) + l_3 \cdot \cos(\theta_2 + \theta_3) \\ p_{zD} = l_2 \cdot \sin(\theta_2) + l_3 \cdot \sin(\theta_2 + \theta_3) \end{cases} \quad (9)$$

by moving to the left side the θ_2 terms, pick up the squared of each equation, adding together the equations and substituting:

$$A = \frac{l_1^2 + l_2^2 - l_3^2 - 2l_1(p_{xD} \cos(\theta_1) - p_{yD} \sin(\theta_1)) + 2p_{xD}p_{yD} \sin(\theta_1) \cos(\theta_1) + (p_{xD}^2 - p_{yD}^2) \cos^2(\theta_1) + p_{yD}^2 + p_{zD}^2}{2l_2 p_{zD}} \quad (10)$$

$$B = \frac{-2l_1 l_2 + 2l_2(p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1))}{2l_2 p_{zD}}$$

the equation is obtained:

$$A - B \cos(\theta_2) = \sin(\theta_2) \quad (11)$$

which can be solved :

$$\theta_2 = \arccos\left(\frac{AB \pm \sqrt{1 + B^2 - A^2}}{1 + B^2}\right) \quad (12)$$

Also from (9), by moving to the left side the θ_2 terms, and dividing the second equation to the first is obtained the θ_3 expression:

$$\begin{pmatrix} \cos(\theta_2) \cdot (n_x \cos(\theta_1) + n_y \sin(\theta_1)) + n_z \sin(\theta_2) & \cos(\theta_2) \cdot (o_x \cos(\theta_1) + o_y \sin(\theta_1)) + o_z \sin(\theta_2) & \cos(\theta_2) \cdot (a_x \cos(\theta_1) + a_y \sin(\theta_1)) & \cos(\theta_2) \cdot (p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1) - l_1) + p_{zD} \sin(\theta_2) - l_2 \\ -\sin(\theta_2) \cdot (n_x \cos(\theta_1) + n_y \sin(\theta_1)) + n_z \cos(\theta_2) & -\sin(\theta_2) \cdot (o_x \cos(\theta_1) + o_y \sin(\theta_1)) + o_z \cos(\theta_2) & -\sin(\theta_2) \cdot (a_x \cos(\theta_1) + a_y \sin(\theta_1)) & -\sin(\theta_2) \cdot (p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1) - l_1) + p_{zD} \cos(\theta_2) \\ n_x \sin(\theta_1) - n_y \cos(\theta_1) & o_x \sin(\theta_1) - o_y \cos(\theta_1) & a_x \sin(\theta_1) - a_y \cos(\theta_1) & p_{xD} \sin(\theta_1) - p_{yD} \cos(\theta_1) \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & l_3 \cdot \cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & l_3 \cdot \sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (15)$$

Balancing the terms from column 4, rows 1 and 2 and dividing them, we obtain:

$$\theta_3 = \arctg\left(\frac{-\sin(\theta_2) \cdot (p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1) - l_1) + p_{zD} \cos(\theta_2)}{\cos(\theta_2) \cdot (p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1) - l_1) + p_{zD} \sin(\theta_2) - l_2}\right) \quad (16)$$

Now, going to the orientation mechanism, knowing the angle of the bucket in relation to the fixed reference frame naming it α , figure 1, we can write:

$$\theta_2 + \theta_3 + \theta_4 = \alpha \quad (17)$$

$$p_x - p_{xD} = l_4 \cos(\alpha) \cos(\theta_1)$$

$$p_y - p_{yD} = l_4 \cos(\alpha) \sin(\theta_1) \quad (18)$$

$$p_z - p_{zD} = l_4 \sin(\alpha)$$

From equation (17) and (18) follows:

$$\theta_4 = \alpha - \theta_2 - \theta_3 \quad (20)$$

$$p_{xD} = p_x - l_4 \cos(\alpha) \cos(\theta_1)$$

$$p_{yD} = p_y - l_4 \cos(\alpha) \sin(\theta_1) \quad (21)$$

$$p_{zD} = p_z - l_4 \sin(\alpha)$$

Accordingly inverse kinematics model of the backhoe excavator's digging equipment is:

$$\theta_1 = \arctg\left(\frac{p_y}{p_x}\right)$$

$$\theta_2 = \arccos\left(\frac{AB \pm \sqrt{1 + B^2 - A^2}}{1 + B^2}\right) \quad (22)$$

$$\theta_3 = \arctg\left(\frac{-\sin(\theta_2) \cdot (p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1) - l_1) + p_{zD} \cos(\theta_2)}{\cos(\theta_2) \cdot (p_{xD} \cos(\theta_1) + p_{yD} \sin(\theta_1) - l_1) + p_{zD} \sin(\theta_2) - l_2}\right)$$

$$\theta_4 = \alpha - \theta_2 - \theta_3$$

to which we add the substitutions (10) and (21).

3. INVERSE KINEMATICS MODEL DETERMINED BY USING THE GEOMETRIC METHOD

The kinematic scheme of the open loop kinematic chain of the backhoe excavator's digging equipment, for a random position is presented figure 2. For this kinematic chain the lengths of the bars are known l_1, l_2, l_3, l_4 , position of the bucket's tooth tip, p_x, p_y, p_z and its orientation α . For these conditions the determination of the angles $\theta_1, \theta_2, \theta_3, \theta_4$ of the joints is required.

$$\theta_1 = \arctg\left(\frac{p_y}{p_x}\right) \quad (23)$$

The angle θ_3 is determined from the triangle BCD the cosines law:

$$BD^2 = BC^2 + CD^2 - 2BC \cdot CD \cdot \cos(\pi - \theta_3) \quad (24)$$

Having the substitutions in figure 2,

$$BD = r = \sqrt{r_o^2 + r_v^2} \quad (25)$$

$$r_o = \sqrt{p_x^2 + p_y^2} - l_1 - l_4 \cdot \cos(\alpha) \quad (25)$$

$$r_v = p_z + l_4 \cdot \sin(\alpha)$$

equation (23) becomes:

$$r_o^2 + r_v^2 = l_2^2 + l_3^2 - 2l_2l_3 \cos(\pi - \theta_3) \quad (26)$$

and is obtained:

$$\theta_3 = \pi + \arccos\left(\frac{l_2^2 + l_3^2 - r_o^2 - r_v^2}{2l_2l_3}\right) \quad (27)$$

if

$$l_2 - l_3 \leq r \leq l_2 + l_3 \quad (28)$$

Angle θ_2 is determined in 2 steps from the triangles

BD_0D and BCD ,

$$\theta_2 = \theta_{21} + \theta_{22} \quad (29)$$

From triangle BD_0D is obtained

$$\theta_{21} = \arctg\left(\frac{r_v}{r_o}\right) \quad (30)$$

And from triangle BCD by using the sine law is obtained

$$\theta_{22} = \arcsin\left(\frac{l_3 \sin(\theta_3)}{r}\right) \quad (31)$$

and in the end

$$\theta_2 = \arctg\left(\frac{r_v}{r_o}\right) + \arcsin\left(\frac{l_3 \sin(\theta_3)}{r}\right) \quad (32)$$

For angle $\theta_4 = \theta_4(\alpha)$ the equation (20) is used.

Accordingly inverse kinematics model of the backhoe excavator's digging equipment determined by using the geometric method is:

$$\theta_1 = \arctg\left(\frac{p_y}{p_x}\right) \quad (33)$$

$$\theta_2 = \arctg\left(\frac{r_v}{r_o}\right) + \arcsin\left(\frac{l_3 \sin(\theta_3)}{r}\right)$$

$$\theta_3 = \pi + \arccos\left(\frac{l_2^2 + l_3^2 - r_o^2 - r_v^2}{2l_2l_3}\right)$$

$$\theta_4 = \alpha - \theta_2 - \theta_3$$

to which we add the substitutions (25).

Figure 2: The kinematic scheme of the open loop kinematic chain of the backhoe excavator's digging equipment

4. THE RELATION BETWEEN THE VARIABLES OF THE DRIVING KINEMATIC JOINTS AND HYDRAULIC CYLINDERS STROKES

Having the substitutions from figure 1, for the hydraulic cylinder FG that is driving the boom we have the equations:

$$\begin{aligned}\varphi_2 &= \pi - \beta_2 - \gamma_2 - \theta_2 \\ s_2 &= \sqrt{BG^2 + BF^2 + 2BG \cdot BF \cdot \cos(\varphi_2)}\end{aligned}\quad (34)$$

Similarly, for the hydraulic cylinder that drives the stick, HK, we have the equations:

$$\begin{aligned}\varphi_3 &= \theta_3 + \beta_3 + \gamma_3 - 2\pi \\ s_3 &= \sqrt{HC^2 + KC^2 + 2HC \cdot KC \cdot \cos(\varphi_3)}\end{aligned}\quad (35)$$

For the hydraulic cylinder of the bucket, the four-bar mechanism DQPN is solved firstly. Beginning with the angle:

$$\psi_4 = \theta_4 + \gamma_4 - 2\pi$$

and substitutions:

$$\begin{aligned}a &= \frac{DQ^2 - PQ^2 + NP^2 + ND^2 - 2 \cdot NP \cdot ND \cdot \cos(\psi_4)}{2 \cdot DQ \cdot NP \cdot \sin(\psi_4)} \\ b &= \frac{NP \cdot \cos(\psi_4) + ND}{NP \cdot \sin(\psi_4)}\end{aligned}\quad (36)$$

is obtained

$$\varphi_p = \arccos\left(\frac{a \cdot b \pm \sqrt{1 + b^2 - a^2}}{1 + b^2}\right)\quad (37)$$

and finally:

$$\begin{aligned}\varphi_4 &= \beta_4 + \varphi_p + \delta_4 \\ s_4 &= \sqrt{NM^2 + LN^2 + 2NM \cdot LM \cdot \cos(\varphi_4)}\end{aligned}\quad (38)$$

5. ALGORITHM FOR USING THE MATRIX METHOD

a) The desired path of the bucket's tooth tip is bringing in parametric form and related to the fixed reference frame

$$O_0x_0y_0z_0,$$

$$x = x(u)$$

$$y = y(u)$$

$$z = z(u)$$

b) The orientation of the bucket is expressed during the movement on the path in relation to the points on the path,

$$\alpha = \alpha(x(u), y(u), z(u))$$

c) Let the values of the path's coordinates be the coordinates of the bucket's tooth tip

$$p_x = x(u)$$

$$p_y = y(u)$$

$$p_z = z(u)$$

d) The angle θ_1 is determined from the first equation (22).

e) The coordinates of the point D are determined from equations (21).

f) The angles θ_2 , θ_3 , θ_4 are determined with equations (22).

g) The lengths of the hydraulic cylinders are determined with (34), (35) and (38).

6. CONCLUSIONS

- Inverse kinematics model of a bucket excavator's digging equipment can be determined either by using the forward kinematic model or by using the geometric method;
- By comparing the results of the two methods we can observe that 2 of the equations are identical;
- We can also observe that the equations contain inverse trigonometric functions, which means that there are difficulties for the programmer and attention on the numeric results is necessary.

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