ITERATIVE LEARNING CONTROL WITH UNKNOWN TIME-DELAY

The paper presents a new algorithm for iterative learning control (ILC) called "natural" ILC. ILC is developed on the basis of a biological analog — the principle of self-adaptability. Sufficient conditions for the convergence of a new type of learning control algorithm for a class of time-varying delayed uncertain, non-linear systems — a process plant with unknown pure time-delay are presented. A new algorithm has benefits which include control of an object with unknown time-delay, improving the properties of tracking, as well as the speed of convergence of ILC.

PRELIMINARIES

A dynamic model of a process plant with unknown pure-time-delay with uncertainties can be presented in the form of state space and output equations as a class of time-varying, non-linear systems (3, 4):

\[ \dot{x}_i = f(x_i, x_i(t - \tau), t) + B(x_i)u_i + v_i \]

\[ y_i = g(x_i) \]  

(1)

In these equations \( t \) denotes time, \( t \in [0, T], t \in \mathbb{R} \), \( \tau \) presents the terminal time which in known; \( x_i \) the state vector, \( x_i \in \mathbb{R}^n \), \( u_i \) the control vector; \( v_i \) the vector bounded uncertainties or disturbances to the system, \( u_i \in \mathbb{R}^m \), \( y_i \) the output vector of the system; \( y_i(t) \) \( \tau \) denotes the \( i \)-th repetitive operation of the system. Also, \( \tau \) denotes the unknown pure time-delay and \( \tau \leq T \). For later use in proving the convergence of the proposed learning control, the following norms are introduced (5):

- the vector norm as

\[ ||x|| = \max_{1 \leq i \leq n} |x_i| \quad x = [x_1, x_2, \ldots, x_n]^T \]  

(2a)

- the matrix norm as

\[ ||A|| = \max_{1 \leq i \leq m} \left( \sum_{j=1}^n |a_{ij}| \right) \quad A = [a_{ij}]_{m \times n} \]  

(2b)

- the \( \lambda \)-norm for a real function:

\[ h(t), t \in [0, T], h:[0,T] \rightarrow \mathbb{R}^n \] as

\[ ||h(t)||_\lambda = \sup_{t \leq T} e^{-\lambda t} ||h(t)||, \quad \lambda > 0 \]  

(2c)

Before presenting a new iterative learning control algorithm, the following assumptions on the system are imposed.

**PT.** Functions \( f(\cdot), B(\cdot) : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n \) and \( g(\cdot) : \mathbb{R}^m \times [0, T] \rightarrow \mathbb{R}^m \), belonging to a set of functions defined on \( [0, T] \), are piecewise continuous and satisfy the Lipschitzian continuity conditions, i.e.
\[ \nabla t \in [0, T] \]
\[ \| [f(x_{i+1}(t), x_i(t), t) - f(x_i(t), x_i(t), t) \| \leq k_f \| x_{i+1}(t) - x_i(t) \| \] (3)
\[ \| B(x_{i+1}(t)) - B(x_i(t)) \| \leq k_B \| x_{i+1}(t) - x_i(t) \| \]
\[ \| g(x_{i+1}(t)) - g(x_i(t)) \| \leq k_g \| x_{i+1}(t) - x_i(t) \| \]
where \( k_f, k_B, k_g, k_x > 0 \) are the Lipschitz constants and the partial derivative \( g_u(\cdot) = \partial g(x_i(t))/\partial x_i(t) \) is differentiable in \( x \).

**P2.** The system is causal. Furthermore, for a given output there exists a unique bounded input and a unique bounded state \( u_d(t), x_d(t), t \in [0, T] \) such that:
\[ x_d(t) = f(x_d(t), x_d(t), t) + B(x_d(t))u_d(t) \] (4)
\[ y_d(t) = g(x_d(t)) \]

**P3.** Functions \( g_u, B(\cdot), \dot{x}_d, u_d \) are uniformly bounded:
\[ a_g = \sup_{t \in [0, T]} \| g(x) \|, \text{ and } a_B = \sup_{t \in [0, T]} \| B(x) \|, \text{ in } \mathbb{R}^n \]

\[ a_{u_d} = \sup_{t \in [0, T]} \| u_d(t) \|, a_{u_d} = \sup_{t \in [0, T]} \| x_d(t) \|, a_v = \sup_{t \in [0, T]} \| v(t) \| \]

**P4.** It is assumed that when \( t < 0, x_i(t) = 0 \).

The learning controller for generating the present control input is based on the previous control history and a learning mechanism. Motivated by human learning, the basic idea of iterative learning control is to use information from previous execution of the task in order to improve performance from trial to trial in the sense that the learning error is sequentially reduced.

The task is synthesis control \( u(t) \) applying the learning concept. For a given output trajectory \( y_d(t) \), the control objective is to find a control input \( u(t) \) such that when \( i \rightarrow \infty \), the system output \( y_d(t) \) will track the desired output trajectory as close as possible. A learning control based on iterative learning can be found in the literature in the following manner [5, 6]:
\[ u_{i+1}(t) = L(t) u_i(t) + K(t) \dot{e}_i(t) \] (6)

Also the tracking errors and their first derivative are defined as:
\[ e_i(t) = y_d(t) - y_i(t) \]
\[ \dot{e}_i(t) = d e_i(t)/d t = \dot{y}_d(t) - \dot{y}_i(t) \] (7)

and \( L(t) \), \( K(t) \), are the gain matrices of appropriate dimensions.

**MAIN RESULTS**

In this paper a new algorithm for iterative learning control is suggested which differs from existing learning algorithms. For improving the properties of tracking, as well as the speed of convergence and, especially, control of a process plant with unknown time-delay, it is proposed to apply a biological analog – the principle of self-adaptability [7] which introduces local negative feedback on the control with great amplification. In the simplest case the learning control law can be shown as (Fig. 1):

\[ u_{i+1}(t) = -\Delta u_i(t) + u_i(t) + K(t) \dot{e}_i(t) \] (8)

where \( \Delta, K \) denote matrices of appropriate dimensions; \( u(t) \) the value of the function at time \( t \) and \( u(t) \); \( \dot{e}_i(t) \) – \( \dot{e}_i \rightarrow 0^+ \) denotes a control vector of the just realized control at time \( t \). If the feedback delay can be neglected then:
\[ u(t) = u(t) \] (9)
yields:
\[ u_{i+1}(t) = -\Delta u_i(t) + u_i(t) + K(t) \dot{e}_i(t) \] (10)
or:
\[ u_{i+1}(t) [1 + \Delta] = u_i(t) + K(t) \dot{e}_i(t) \] (11)

In ILC a fundamental problem is to guarantee the ILC convergence property, i.e. to guarantee the system is output trajectory converging to the desired one within a prescribed desired accuracy as the number of ILC iterations increases. A sufficient condition for the convergence of a new iterative learning control is given in the following theorem.

**Theorem 1:**

If an existing non-linear repetitive system satisfied the previously introduced assumptions (P1–P4) with the proposed new iterative learning control and matrices \( \Delta \) and \( K \) exist such that \( \| l + \Delta \|^{-1} \| I - Kg \| \leq p < 1 \) then, when \( i \rightarrow \infty \) the bounds of the tracking errors \( \| u_i(t) - u_d(t) \|, \| x_i(t) - x_d(t) \|, \| y_i(t) - y_d(t) \| \) converge asymptotically to a residual ball centered at the origin.

**Proof:**

Let
\[ g_i = g_d \dot{x}_d(t) - g(x_i(t)) \]
\[ \dot{u}_i = u_d(t) - u_i(t) \]
\[ \Delta u = B_d(t) - B(x_i(t)) \]
\[ x_i = \dot{x}_i(t) - \dot{x}_i(t) \]
\[ f_i = f_i \left( x_i, \dot{x}_i, \ddot{x}_i \right) \]
\[ \mathcal{G}_n = \mathcal{G}_n \left( \mathbf{x}(t) \right) - \mathbf{G}_n \left( \mathbf{K}(t) \right) \]

Also, the tracking error can be presented as:
\[ \hat{e}_i = g_{x_i} \dot{x}_i + g_{\dot{x}_i} \ddot{x}_i \]
\[ \text{or,} \]
\[ \dot{x}_i = \hat{e}_i + \Delta \mathbf{B}_i \mathbf{u} + \mathbf{B}_i \hat{e}_i - v_i \]

Taking the control law is given as:
\[ u_i(t) = \left[ I + \Delta \right] \hat{e}_i(t) + \left( K \hat{e}_i(t) \right) \]

it yields:
\[ \left[ I + \Delta \right] \Delta \mathbf{u}_i = \Delta \mathbf{u}_i + \Delta \mathbf{u}_i - K_e \mathbf{i} = 0 \]
\[ \Rightarrow u_i = \left[ I + \Delta \right]^{-1} \Delta \mathbf{u}_i + \left[ I + \Delta \right]^{-1} \Delta \mathbf{u}_i - K_e \mathbf{i} \]
where \( \Delta = \text{diag} \left( \hat{e}_i, i = 1, 2, 3, ... n \right) \) and:
\[ \left[ I + \Delta \right]^{-1} \Delta = \text{diag} \left( \frac{1}{1 + \hat{e}_i} \right) \]
\[ a_i^* = \sup_{t \in [0, T]} \left\| \left[ I + \Delta \right]^{-1} \mathbf{u}_i \right\| \quad a_i = \sup_{t \in [0, T]} \left\| \left[ I + \Delta \right]^{-1} \Delta \mathbf{u}_i \right\| \]
also:
\[ u_i = \left[ I + \Delta \right]^{-1} \Delta \mathbf{u}_i + \left[ I + \Delta \right]^{-1} \Delta \mathbf{u}_i - K_e \mathbf{i} \]

Estimating the norms of \( \left[ \mathcal{G} \right] \) and using assumptions P1–P4 and the condition of theorem one may obtain:
\[ \left\| \mathcal{G}_n \right\| \leq \left\| \left[ I + \Delta \right]^{-1} \left( -K e \mathbf{B}_i \right) \right\| \left\| \Delta \mathbf{u}_i \right\| + a_{i_a} a_{u_d} + a_{i_a} a_{u_d} k_e + \mathbf{e}_x^T \mathbf{a}_{x_e} k_e \mathbf{e}_x + a_{i_a} a_{u_d} k_e + a_{i_a} a_{u_d} k_e + \mathbf{e}_x^T \mathbf{a}_{x_e} k_e \mathbf{e}_x + a_{i_a} a_{u_d} k_e \mathbf{e}_x \]
\[ + a_{i_a} a_{u_d} k_e \mathbf{e}_x \]
\[ \left\| \mathcal{G}_n \right\| \leq \left\| \left[ I + \Delta \right]^{-1} \left( -K e \mathbf{B}_i \right) \right\| \left\| \Delta \mathbf{u}_i \right\| + \mathbf{e}_x^T \mathbf{a}_{x_e} k_e \mathbf{e}_x + \gamma \]
\[ \left( \mathcal{G}_n \right) = a_{i_a} a_{u_d} k_e + \mathbf{e}_x^T \mathbf{a}_{x_e} k_e \mathbf{e}_x \]
\[ \gamma = a_{i_a} a_{u_d} k_e + \mathbf{e}_x^T \mathbf{a}_{x_e} k_e \mathbf{e}_x \]
\[ \mathbf{e}_x = \mathbf{e}_x \]
\[ \mathbf{x}(t) = \mathbf{x}(O) + \int_0^t \mathbf{x} \mathbf{x}(t) \mathbf{d} \mathbf{t} \]
\[ \mathbf{e}_x = \mathbf{e}_x + \mathbf{e}_x \]
\[ \mathbf{x}(t) = \mathbf{x}(O) + \mathbf{e}_x \]
\[ \left\| \mathcal{G}_n \right\| \leq \left\| \left[ I + \Delta \right]^{-1} \left( -K e \mathbf{B}_i \right) \right\| \left\| \Delta \mathbf{u}_i \right\| + \mathbf{e}_x^T \mathbf{a}_{x_e} k_e \mathbf{e}_x + \gamma \]
\[ \gamma = a_{i_a} a_{u_d} k_e + \mathbf{e}_x^T \mathbf{a}_{x_e} k_e \mathbf{e}_x \]
\[ \mathbf{e}_x = \mathbf{e}_x + \mathbf{e}_x \]
\[ \mathbf{x}(t) = \mathbf{x}(O) + \mathbf{e}_x \]
\[ \left\| \mathcal{G}_n \right\| \leq \left\| \left[ I + \Delta \right]^{-1} \left( -K e \mathbf{B}_i \right) \right\| \left\| \Delta \mathbf{u}_i \right\| + \mathbf{e}_x^T \mathbf{a}_{x_e} k_e \mathbf{e}_x + \gamma \]
\[ \gamma = a_{i_a} a_{u_d} k_e + \mathbf{e}_x^T \mathbf{a}_{x_e} k_e \mathbf{e}_x \]
\[ \mathbf{e}_x = \mathbf{e}_x + \mathbf{e}_x \]
\[ \mathbf{x}(t) = \mathbf{x}(O) + \mathbf{e}_x \]

Taking the \( \lambda \)-norm of \( \mathbf{x}(t) \) is:
\[ \sup_{t \in [0, T]} e^ {-\lambda t} \left\| \mathbf{x}(t) \right\| = \sup_{t \in [0, T]} e^{-\lambda t} \left\| \mathbf{x}(t) \right\| e^ {-\lambda t} \]
\[ \left\| \mathcal{G}_n \right\| \leq \left\| \left[ I + \Delta \right]^{-1} \left( -K e \mathbf{B}_i \right) \right\| \left\| \Delta \mathbf{u}_i \right\| + \mathbf{e}_x^T \mathbf{a}_{x_e} k_e \mathbf{e}_x \]
where using a suitable choice of matrices $\Delta$ and $K$ i.e., one can find a sufficiently large $\lambda$ such that $\rho^* < 1$ so that
\[
\|\bar{u}_{i+1}\| \leq \rho^* \|\bar{u}_i\| + \varepsilon^*
\]  
(33)

Then, according to Lemma 1 it can be concluded that
\[
\lim_{i \to \infty} \|\bar{u}_{i+1}\| \leq \varepsilon^*/(1 - \rho^*)
\]  
(34)

Also,
\[
\|\bar{x}_i\| \leq \|\bar{u}_i\|, \quad O_U(\lambda^{-1}) + \varepsilon^*
\]  
(35)

now, it can be easily shown that
\[
\lim_{i \to \infty} \|\bar{x}_i\| \leq O_U(\lambda^{-1})\varepsilon^*/(1 - \rho^*) + \varepsilon^*
\]  
(36)

or
\[
\lim_{i \to \infty} \|\bar{e}_i\| \leq k\|O_U(\lambda^{-1})\varepsilon^*/(1 - \rho^*) + \varepsilon^*\|
\]  
(37)

This completes the proof of Theorem 1

Example

A simple example of a Siso system
\[
x(t) = 0.9x(t) + 0.1u(t) + 0.05 + u(t)
\]
\[
y(t) = x(t)
\]
is used to illustrate the previous approach. The given matrices $\Delta$ and $K$ are $\Delta x: 0.02$ and $0.4$ and the output trajectory is $y_d(t) = t$.

CONCLUSION

A new iterative learning algorithm utilizing a biological analog – the principle of self-adaptability for a process plant is considered. It is shown that the new algorithm of ILC improves the speed of convergence and also has better properties of tracking which makes the suggested algorithm more attractive from the viewpoint of practical applications. Also, sufficient conditions for the convergence of ILC are obtained.

REFERENCES