STABILITY AND VIBRATION IN CIVIL ENGINEERING

STABILNOST I VIBRACIJE U GRAĐEVINARSTVU

Ján RAVINGER

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1 INTRODUCTION

Taking into account the stiffness and inertia forces, dynamic behaviour of structures can be investigated. Dynamic investigation usually starts with an example of free vibration. It means to evaluate the natural frequency. The simplest stability problem of structures is buckling of a column. This problem can be arranged preparing the equilibrium conditions on a deformed structure. In general, however, for the evaluation of the stability problems strains should be evaluated for a deformed differential element what means to apply geometric non-linear theory.

Combination of dynamics and stability yields in a lot of problems: dynamic buckling, dynamic post buckling behaviour, parametric resonance, etc. Introduction example – vibration of a column loaded in compression is simple but its investigation still represents a lot of problems.

The natural frequency can be measured by using rather simple equipment. The comparison of frequencies measured experimentally and evaluated numerically is the basis of non-destructive methods for investigation of structure properties. Generally, it can be said that in structural design stability effects have to be taken into consideration. These two ideas are the reason for our investigation of the combination of vibration and stability.

Leonard Euler was probably the first scientist who had analyzed stability problems. The former solutions are supposed to be the linear stability. It means that we suppose an ideal structure. The differences between theory and reality inspired researchers to search for more accurate models. Especially the slender web as the main part of thin-walled structure has significant post-buckling reserves and it is necessary to accept a geometric non-linear theory for their description. The problem of the vibration of the non-linear system was

formulated by Bolotin². Burgreen³ analysed the problem of the vibration of an imperfect column in early 50's. Some valuable results have been achieved by Volmir⁷. Combination of dynamics and stability is still a subject of research all over the world.

2 DYNAMIC POST-BUCKLING BEHAVIOUR OF SLENDER WEB

2.1 Post-buckling behaviour of slender web – displacement model

As it was already mentioned, slender web is the main constructional element of thin-walled structure. If we assume an "ideal" slender web and a distribution of the in-plane stresses are not the function of the out-of plane (the plate) displacements, the problem leads to eigenvalues and eigenvectors. From the obtained eigenvalues elastic critical load can be evaluated and eigenvector characterizes the mode of buckling.

Post-buckling behaviour can be assumed as follows (Fig.1)

Displacements of the point of the middle surface are

$$\mathbf{q} = [u, v, w]^T \quad (1)$$

In the post-buckling behaviour of the slender web the plate displacements are much larger than in-plane (web) displacements ($w \gg u, v$) and so the strains are

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} w_{,x}^2 \\ w_{,y}^2 \\ 2w_{,x}w_{,y} \end{Bmatrix} - z \begin{Bmatrix} w_{,xx} \\ w_{,yy} \\ w_{,xy} \end{Bmatrix} \quad (2)$$

where "z" is the coordinate of the thickness. The indexes "x, y" denote partial derivations.

Dr.h.c. prof. Ing. Ján Ravinger, DrSc. Slovak University of Technology, Faculty of Civil Engineering
Radlinského 11, 813 68 Bratislava, Slovakia. E-mail
jan.ravinger@stuba.sk

Figure 1. Notation of the quantities of slender web

For the next investigation, slender web with initial deformations is assumed. Initial deformations are the plate types only.

$$\mathbf{q}_0 = [0, 0, w_0]^T \quad (3)$$

Due to that the initial strains are

$$\boldsymbol{\varepsilon}_0 = \frac{1}{2} \begin{Bmatrix} w_{0,x}^2 \\ w_{0,y}^2 \\ 2w_{0,x}w_{0,y} \end{Bmatrix} - z \begin{Bmatrix} w_{0,xx} \\ w_{0,yy} \\ 2w_{0,xy} \end{Bmatrix} \quad (4)$$

The “ w ” represents the global displacements and “ w_0 ” is part related to the initial displacement. The linear elastic material has been assumed

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{Bmatrix} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \boldsymbol{\sigma}_w,$$

where

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & \\ \nu & 1 & \\ & & \frac{1-\nu}{2} \end{bmatrix} \quad (5)$$

E , ν are the Young's modulus and Poisson's ratio, $\boldsymbol{\sigma}_w = [\sigma_{xw}, \sigma_{yw}, \tau_w]^T$ are the residual stresses.

The global potential energy of the slender web is

$$U = U_i + U_e \quad (6)$$

where

$U_i = \frac{1}{2} \int_V (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0)^T \boldsymbol{\sigma} dV$ - is the potential energy of the

internal forces,

$U_e = - \int_{\Gamma} (\mathbf{q} - \mathbf{q}_0)^T \mathbf{p} d\Gamma$ - the potential energy of the

external forces,

where V is the volume of the slender web, Γ is the in-plane surface.

The displacements are assumed as the product of the variational functions and the displacements parameters

$$\mathbf{q} = \mathbf{B} \cdot \boldsymbol{\alpha} \quad (7)$$

The minimum of the global potential energy gives the system of conditional equation

$$\mathbf{K}_G(\boldsymbol{\alpha}) \boldsymbol{\alpha} = \mathbf{f} \quad (8)$$

where $\mathbf{K}_G$ is the stiffness matrix as the function of the displacement parameters – non-linear stiffness matrix, $\mathbf{f}$ is the vector of the external load.

2.2 Post-buckling behaviour of slender web loaded in compression – illustrative example

For the simplification we suppose the square rectangular slender web loaded in compression simply supported all around.

We do not need to suppose the external load as the constant along the edge. But the external force must be

defined as $F = \int_0^b t \sigma dy$. Consequently, the average

stress can be defined as $\sigma = \frac{F}{b \cdot t}$. For the approximate

solution, we take displacement functions as

$$w = \alpha S_{x1} S_{y1},$$

$$w_0 = \alpha_0 S_{x1} S_{y1},$$

$$u = \beta_1 \left(1 - \frac{2x}{b} \right) + \beta_2 S_{x2} C_{y2} + \beta_3 S_{x2},$$

$$v = \gamma_1 \left(1 - \frac{2y}{b} \right) + \gamma_2 C_{x2} S_{y2} + \gamma_3 S_{y2},$$

$$\text{where } S_{xi} = \sin \frac{i\pi x}{b}, \dots, C_{yi} = \cos \frac{i\pi y}{b}.$$

We have divided the variational parameters into:

- plate $\boldsymbol{\alpha}_D = \boldsymbol{\alpha}$,

- in-plane $\boldsymbol{\alpha}_S = [\beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3]^T$.

The in-plane displacements parameters are

$$\alpha_S = \frac{\pi}{16b} (\alpha^2 - \alpha_0^2) [\pi, l, -l + \nu, \pi, l, -l + \nu]^T$$

Introducing $\sigma_E = \frac{\pi^2 Et^2}{12(1-\nu^2)b^2}$ (Euler's elastic critical stress), the dimensionless load as $\bar{\sigma} = \frac{\sigma'}{4\sigma_E} = \frac{\sigma'}{\sigma_{cr}}$, ($\sigma_{cr} = 4\sigma_E$) and the dimensionless parameters of the displacements function $\bar{\alpha} = \frac{\alpha}{t}$, $\bar{\alpha}_0 = \frac{\alpha_0}{t}$, the result can be arranged into the final equation

$$0.34125(\bar{\alpha}^2 - \bar{\alpha}_0^2) + 1 - \frac{\bar{\alpha}_0}{\bar{\alpha}} = \bar{\sigma} \quad (9)$$

The parameters α and α_0 represent the amplitudes of the out of plate displacements of the slender web. Eq. (9) is arranged in Fig. 2.

It is evident that slender web could be loaded above the level of the elastic critical load. Due to that "the post-buckling behaviour" can be introduced.

It has to be noted that the presented example represents an approximate solution.

2.3 System of non-linear algebraic equations

First, we present a note related to the solution of geometric non-linear problems.

We use (for example) the Ritz variational method. The functions of the displacements are sums of the products of the basic functions and the variational coefficients.

$$q = B \cdot \alpha \quad (10)$$

These equations could be written in the mode

$$u, w \Rightarrow \alpha \uparrow 1 \quad (11)$$

The sign „ $\uparrow$ “ is used as the exponent.

The elongations taking into account non-linear parts have the variational coefficients in quadrates and can be recorded as

$$\varepsilon = u_{,x} + \frac{1}{2} w_{,x}^2 - z \cdot w_{,xx} \Rightarrow \alpha \uparrow 2 \quad (12)$$

Assuming the linear elastic material, the stresses are in quadrates as well.

$$\sigma = E(\varepsilon - \varepsilon_0) \Rightarrow \alpha \uparrow 2 \quad (13)$$

The potential energy of the internal forces is a product of the elongations and the stresses, then, finally, the variational coefficients are of the fourth power

$$U_i = \frac{1}{2} \varepsilon^T \cdot \sigma = (\alpha \uparrow 2) \cdot (\alpha \uparrow 2) = \alpha \uparrow 4 \quad (14)$$

The system of conditional equations may be arranged as a partial derivation according to the variational coefficients

$$\frac{\partial U}{\partial \alpha_i} = \dots = \alpha \uparrow 3 \quad (15)$$

Finally, we obtain the **system of cubic algebraic equations**.

A partial approval of our explanation can be seen in the example of the post buckling behaviour of the slender web (Part 2.2, Eq.(9) where we have got the cubic algebraic equation).

Note. In the example of the buckling of the column, the cubic terms have been eliminated. This "special case" is the consequence of the constant normal force along the column.

Let us continue with our former considerations.

The system of linear algebraic equations can be arranged as a matrix (two dimensional area). The system of quadratic algebraic equations could be arranged as a three dimensional matrix. The cubic algebraic equations are a four dimensional matrix. We are not able to imagine the four dimensional matrix, but modern computers are able to compile it.

One typical property of the finite element method is a large number of parameters (many thousands). To arrange 1000 cubic algebraic equations represents in computer memory $1000^4 = 1 \cdot 10^{12}$ real numbers and this is beyond possibilities.

Figure 2. Post-buckling behaviour of slender web loaded in compression

The way how to solve these non-linear systems has been found. The idea is to use the Newton-Raphson iteration without compilation of the system of non-linear (cubic) algebraic equations. It will be explained in the following parts.

2.4 Incremental formulation

As it has been already explained in the previous part, we are forced to arrange the iterative method. It can be prepared from the incremental formulation, and so we must prepare all the regulars in increments.

Note. All the rules for one dimensional problem (beams, columns) are prepared. For the solution of the two dimensional problems (webs, plates) all the steps are similar.

As the first step, the increments and variations for the elongations must be prepared. If we have the linear function as

$$f\left(\frac{du}{dx}\right) = \frac{du}{dx} = u_{,x} \quad (16)$$

For the increments $u + \Delta u$, we get the increments of the function

$$\begin{aligned} \Delta f &= f\left(\frac{d(u + \Delta u)}{dx}\right) - f\left(\frac{du}{dx}\right) = \frac{du}{dx} + \frac{d\Delta u}{dx} - \frac{du}{dx} = \\ &= \frac{d\Delta u}{dx} = \Delta u_{,x} \end{aligned} \quad (17)$$

We do the same steps for the non-linear function

$$f\left(\frac{dw}{dx}\right) = \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = \frac{1}{2}w_{,x}^2 \quad (18)$$

We have for the increment of this function

$$\begin{aligned} \Delta f &= f\left(\frac{d(w + \Delta w)}{dx}\right) - f\left(\frac{dw}{dx}\right) = \frac{1}{2}\left(\frac{d(w + \Delta w)}{dx}\right)^2 - \\ &- \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = \frac{1}{2}\left(\left(\frac{dw}{dx}\right)^2 + 2\frac{dw}{dx} \cdot \frac{d\Delta w}{dx} + \left(\frac{d\Delta w}{dx}\right)^2\right) - \\ &- \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = \frac{dw}{dx} \cdot \frac{d\Delta w}{dx} + \frac{1}{2}\left(\frac{d\Delta w}{dx}\right)^2 = \\ &= w_{,x} \cdot \Delta w_{,x} + \frac{1}{2}\Delta w_{,x}^2 \end{aligned} \quad (19)$$

According to these rules the increment of the strain can be arranged as follows

$$\Delta \varepsilon_{,x} = \Delta u_{,x} + w_{,x} \cdot \Delta w_{,x} + \frac{1}{2}\Delta w_{,x}^2 - z \cdot \Delta w_{,xx} \quad (20)$$

Then the variation of the increment of the elongation is prepared

$$\delta \Delta \varepsilon_{,x} = \delta \Delta u_{,x} + w_{,x} \cdot \delta \Delta w_{,x} + \delta \Delta w_{,x} \cdot \Delta w_{,x} - z \cdot \delta \Delta w_{,xx} \quad (21)$$

2.5 The Hamilton's principle

In this step, we prepare the rules for the dynamic process. In order to neglect the inertial forces, we get the static problems.

The Hamilton's principle means: in each time interval, the variation of the kinetic and potential energy and the variation of the work of the external forces is equal zero. This rule is valid for the increments as well:

$$\int_{t_0}^{t_1} \delta(\Delta T - \Delta U) dt + \int_{t_0}^{t_1} \delta \Delta W dt = 0 \quad (22)$$

where $\Delta T = \int_V \frac{1}{2} \rho \Delta \dot{q}^T \Delta \dot{q} dV$ is the increment of the

kinetic energy, $\Delta U = \int_V \left(\frac{1}{2} \Delta \varepsilon \cdot \Delta \sigma + \Delta \varepsilon \cdot \sigma \right) dV$ – the

increment of the potential energy of the internal forces, $\Delta W = \int_V \Delta q^T \cdot (\mathbf{p} + \Delta \mathbf{p}) dV$ – the increment of the work of

the external forces, t_0, t_1 – the time intervals, ρ – the mass density, V – the volume (in our case it is the volume of the beam – column), $\mathbf{p}, \Delta \mathbf{p}$ – the external load, the increment of the external load.

The dots mean the time derivation.

We assume the linear elastic material (Eq. (5)). For the increments, we have

$$\Delta \sigma = \mathbf{D} \Delta \varepsilon$$

In the case of the beam type of structures, the volume integration can be changed into the integration over the cross section and the integration over the length:

A, I – the cross section area, the moment of inertia.

The longitudinal axis is situated into centre of the gravity of the cross section.

We use the Ritz variational method

$$u = \mathbf{B}_S \cdot \alpha_S, \quad w = \mathbf{B}_D \cdot \alpha_D, \quad (23)$$

We have the incremental model and the variational coefficients α_S a α_D are timeless functions.

For the increments of the displacements functions, the independent basic variational functions can be used. The increments of the variational coefficients are the function of the time

$$\Delta u = \mathbf{B}_{SI} \cdot \Delta \alpha_S(t), \quad \Delta w = \mathbf{B}_{DI} \cdot \Delta \alpha_D(t) \quad (24)$$

Note. In some dynamic processes where there can be different boundary condition for the static behaviour and for the vibration, it is useful to have different basic variational functions for the displacements and for the increment of the displacements.

Finally, Eq. (22) leads to the system of conditional equation. This system could be arranged into the mode

$$\begin{aligned} K_{M-D}\Delta\ddot{\alpha}_D + K_{INC-D}\Delta\alpha_D + K_{INC-DS}\Delta\alpha_S + f_{INT-D} - f_{EXT-D} - \Delta f_{EXT-D} &= 0 \\ K_{M-S}\Delta\ddot{\alpha}_S + K_{INC-S}\Delta\alpha_S + K_{INC-SD}\Delta\alpha_D + f_{INT-S} - f_{EXT-S} - \Delta f_{EXT-S} &= 0 \end{aligned} \quad (25)$$

where $K_{M-D} = \int_0^a B_{DI}^T \rho A B_{DI} dx$ is the mass matrix of

the “bending” displacements,

$K_{INC-D} = K_{INC-DL} + K_{INC-DG}$ □ the incremental stiffness matrix of the bending,

$K_{INC-DL} = \int_0^a B_{DIX}^T EI B_{DIX} dx$ □ the linear part,

$K_{INC-DG} = \int_0^a B_{DIX}^T EA \left(\frac{3}{2} w_{,x}^2 - \frac{1}{2} w_{0,x}^2 \right) B_{DIX} dx$ □ the

non-linear part of the incremental stiffness matrix of the bending stiffness,

$K_{INC-DS} = \int_0^a B_{DIX}^T EA (w_{,x} + u_{,x} w_{,x}) B_{SIX} dx$ □ the

incremental “bending – axial” stiffness matrix,

$f_{INT-D} = \int_0^a B_{DIX}^T EI (w_{,xx} - w_{0,xx}) dx + \int_0^a B_{DIX}^T EA (u_{,x} + \frac{1}{2} w_{,x}^2 u_{,x} - \frac{1}{2} w_{0,x}^2 u_{,x} + \frac{1}{2} w_{,x}^2 u_{,x} - w_{,x} w_{0,x}^2) dx$ □ the vector of the bending internal

forces,

$\Delta f_{EXT-D} = \int_0^a B_{DI}^T \Delta p_D dx$ □ the increment of the vector of

the bending external forces,

$K_{M-S} = \int_0^a B_{SI}^T \rho A B_{SI} dx$ □ the mass matrix of the

“axial” displacements,

$K_{INC-S} = \int_0^a B_{SXI}^T EA B_{SXI} dx$ □ the incremental stiffness

matrix of the axial stiffness.

It can be proved that $K_{INC-SD} = K_{INC-DS}^T$ – the incremental “axial – bending” stiffness matrix,

$f_{INT-S} = \int_0^a B_{SIX}^T EA \left(u_{,x} + \frac{1}{2} w_{,x}^2 - \frac{1}{2} w_{0,x}^2 \right) dx$ □ the

vector of the axial internal forces,

$f_{EXT-S} = \int_0^a B_{SI}^T p_S dx$ – the vector of the axial external

forces, $\Delta f_{EXT-S} = \int_0^a B_{SI}^T \Delta p_S dx$ – the increment of the

vector of the axial external forces.

It is evident that Eq.(25) represents the system of the differential equations of the second degree.

The axial and the bending displacement can be joined as

$$\Delta\alpha = \begin{Bmatrix} \Delta\alpha_D \\ \Delta\alpha_S \end{Bmatrix}, \quad \alpha = \begin{Bmatrix} \alpha_D \\ \alpha_S \end{Bmatrix}$$

The system of conditional equations (Eq. (25)) could be written as

$$K_M \Delta\ddot{\alpha} + K_{INC} \Delta\alpha + f_{INT} - f_{EXT} - \Delta f_{EXT} = 0 \quad (26)$$

where

$$K_M = \begin{bmatrix} K_{M-D} & \\ & K_{M-S} \end{bmatrix},$$

$$K_{INC} = \begin{bmatrix} K_{INC-D} & K_{INC-DS} \\ K_{INC-SD} & K_{INC-S} \end{bmatrix}$$

$$\Delta f_{EXT} = \begin{Bmatrix} \Delta f_{EXT-D} \\ \Delta f_{EXT-S} \end{Bmatrix}, \quad f_{EXT} = \begin{Bmatrix} f_{EXT-D} \\ f_{EXT-S} \end{Bmatrix}, \quad f_{INT} = \begin{Bmatrix} f_{INT-D} \\ f_{INT-S} \end{Bmatrix}$$

Static behaviour

The inertial forces can be neglected for the solution of the static behaviour of the structure

$$K_M \Delta\ddot{\alpha} \equiv 0 \quad (27)$$

Note. In the case of the static behaviour, except the Hamilton's principle, (Eq.(28)) the principle of the minimum of the increment of the global potential energy can be applied.

The system of the differential equations (Eq. (25)) will be changed into the system of the linear algebraic equation related to the increments of the displacements

$$K_{INC} \Delta\alpha + f_{INT} - f_{EXT} - \Delta f_{EXT} = 0 \quad (28)$$

If the problem is not established in the increments, but in the displacement parameters, we get the system of the cubic algebraic equations in the mode

$$f_{INT} - f_{EXT} = 0 \quad (29)$$

As previously explained in the introduction Part 2.3, this system of cubic algebraic equations cannot be compiled. (Note. This system can be arranged in some simple examples only.)

Eq. (28) is the basis for the incremental solution and for the Newton-Raphson iteration as well.

2.6 Incremental solution

We assume the system in equilibrium represented by the parameters of the displacements “ α ”. Then it is valid that

$$f_{INT} - f_{EXT} = 0 \quad (30)$$

The increment of the external load is obtained. The increments of the parameters of the displacements can be obtained from Eq. (28)

$$\Delta \alpha = K_{INC}^{-1} \Delta f_{EXT} \quad (31)$$

The displacement parameters of the new level are

$$\alpha_D^i = \alpha_D + \Delta \alpha_D \quad (32)$$

2.7 Newton-Raphson iteration

We do not assume any system in equilibrium represented by the parameters of the displacements " α^i ". Then we have the vector of residuum

$$r^i = f_{INT} - f_{EXT} \quad (33)$$

For the correction of the roots (displacement parameters), we assume the constant level of the external load ($\Delta f_{EXT} = 0$). Then it can be evaluated from Eq. (28)

$$\Delta \alpha^i = -K_{INC}^{-1} \cdot r^i \quad (34)$$

The new approximation of the displacement parameters is

$$\alpha^{i+1} = \alpha^i + \Delta \alpha^i \quad (35)$$

Eqs. (33 – 35) represent the Newton-Raphson iteration.

We have a large amount of parameters. For the completing the iterative process, it is necessary to use suitable norms. One of them could be

$$\|n\| = \frac{(\alpha^{i+1})^T \cdot \alpha^{i+1} - (\alpha^i)^T \cdot \alpha^i}{(\alpha^{i+1})^T \cdot \alpha^i} \leq 0.001, (0.0001) \quad (36)$$

Using the terminology of the Newton-Raphson iteration, we have

$$K_{INC} = J \quad (37)$$

The incremental stiffness matrix is the same as the Jacoby matrix of the Newton-Raphson iteration. The Jacoby matrix characterizes the tangent plane to the non-linear surface and is defined as

$$J_{ij} = \frac{\partial}{\partial \alpha_i} K_{Gnel}^* \quad (38)$$

where K_{Gnel}^* is the system of non-linear (in our case cubic) algebraic equations.

2.8 Bifurcation point

In the case of the non-linear problems, many results can be obtained represented by many paths (curves) illustrating relation of load versus the displacement parameters. Especially in the case of the stability problems, stable and unstable paths should be distinguished.

The global potential energy represents the surface. The local minimum of this surface is the point of stable path of the non-linear solution. From the theory of the quadratic surfaces for the local minimum, the Jacoby matrix (in our case, the incremental stiffness matrix) must be positively defined and all the principle minors must be positive as well

$$D = \left| K_{INC} \right|_{det} > 0, D_k > 0. \quad (39)$$

If any condition of Eq. (39) is not satisfied, the path is unstable. The point between the stable and unstable paths is called **the bifurcation point**. In the bifurcation point, we have

$$D = \left| K_{INC} \right|_{det} = 0 \quad (40)$$

2.9 Vibration of the structure

The conditional equations have been arranged as a dynamic process. The static behaviour is taken as a partial problem. From the viewpoint of the dynamic, we consider only the problem of the vibration. We are able to evaluate the vibration of the structure in different load levels including the effects of initial imperfections. We assume the structure in equilibrium and zero increment of the load

$$\Delta f_{EXT} = 0 \quad (41)$$

The system of conditional equations (Eq.(25) will be reduced

$$K_M \Delta \ddot{\alpha} + K_{INC} \Delta \alpha = 0 \quad (42)$$

Related to the increments of the displacements parameters, this system represents a homogeneous differential equation with constant coefficient. The solution has the mode

$$\Delta \alpha = \Delta \bar{\alpha} \sin(\omega t) \quad (43)$$

where ω is the circular frequency.

Putting this into Eq. (42), we get

$$-\omega^2 K_M \Delta \bar{\alpha} \sin(\omega t) + K_{INC} \Delta \bar{\alpha} \sin(\omega t) = 0 \quad (44)$$

The non-trivial solution leads to the problem of eigenvalues and eigenvectors

$$\left| K_{INC} - \omega^2 K_M \right|_{det} = 0. \quad (45)$$

The eigenvalues represent the squares of circular frequencies, and eigenvectors are the parameters of the modes of the vibration.

Note. Incremental stiffness matrix includes level of the load, deformation of structure and initial imperfections as well.

3 STABILITY AND VIBRATION

3.1 Vibration of simply supported column loaded in compression

In Part 2.5, the derivation has been started by using the Hamilton's principle and generally prepared the conditional equation for the dynamic process. In Part 2.10., we have arranged the equations for the evaluation of the vibration.

Another simple and interesting example is the vibration of the imperfect column. For the application of the action of the force, we must suppose one support as the hinge and the other support as the roller (the sliding support (Fig 3.)). (Note: The column is displayed in horizontal position.)

The axial inertial forces are neglected and the displacement functions are

$$w = \alpha_1 \sin \frac{\pi x}{l}, \quad w_0 = \alpha_0 \sin \frac{\pi x}{l}, \quad u = \left[x, \sin \frac{2\pi x}{l} \right] \begin{Bmatrix} \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

The parameters of axial displacements are

$$\alpha_2 = -\frac{F}{EA} - \frac{\pi^2}{l^2}(\alpha_1^2 - \alpha_0^2), \quad \alpha_3 = -\frac{\pi}{8l}(\alpha_1^2 - \alpha_0^2)$$

The equation of the static behaviour can be arranged in the form

$$\bar{F} = \left(1 - \frac{\alpha_0}{\alpha_1} \right), \quad \text{where } \bar{F} = \frac{F}{F_{EU}}, \quad F_{EU} = \frac{\pi^2 EI}{l^2} \text{ is}$$

Euler's elastic critical force.

The incremental stiffness matrix is

$$K_{INC} = \frac{\pi^4 EI}{l^4} \frac{l}{2} - \frac{\pi^2}{l^2} \frac{l}{2} F$$

Putting this into Eq.(9.59), obtained result is

$$\omega^2 = \omega_0^2 (1 - \bar{F})$$

where $\omega_0^2 = \frac{\pi^4 EI}{\rho A l^4}$ is the square of the circular frequency of the simply supported column

(46)

We have obtained a trivial result of the linear relation of the square of the circular frequency and the internal force. It can be seen that during the free vibration the initial displacements do not affect the free vibration.

3.2 Vibration of simply supported column – fixed supports

The result represented by Eq.(46) in the case of the level of the load as the elastic critical load gives the zero frequency. This is out of reality. For example, the minor foreman knocks on the columns. The low tone (the low frequency) means the small force inside the column and the column must be wedged. The high tone (the high frequency) means the high level of the load and the additional columns must be used.

To improve the obtained result the following arrangement must be done (Fig.4.):

For the displacements and the initial displacements, we take

$$w = \alpha_1 \sin(\pi x / l), \quad w_0 = \alpha_0 \sin(\pi x / l),$$

$$u = [x, \sin(2\pi x / l)] [\alpha_2, \alpha_3]^T$$

But for the increment of the displacement, we assume

$$\Delta w = \Delta \alpha_1 \sin(\pi x / l), \quad \Delta u = \Delta \alpha_3 \sin(2\pi x / l)$$

Now, different basic variational functions are used for the displacements and for the initial displacements: Finally, the incremental stiffness matrix is

$$K_{INC} = \frac{\pi^4 EI}{l^4} \frac{l}{2} - \frac{\pi^2}{l^2} \frac{l}{2} F + EA \frac{\pi^4}{l^4} \frac{l}{2} \frac{\alpha_1^2}{2}$$

Then we get the expression for the square of the circular frequency

$$\omega^2 = \omega_0^2 \left(1 - \bar{F} + \frac{l}{2} \frac{\alpha_1^2}{r^2} \right) \quad \text{where } r = \sqrt{\frac{I}{A}}$$

is the radius of inertia.

(47)

Figure 3. Simply supported column with initial displacement

Figure 4. Simply supported column with initial displacement – fixed support

Figure 5. Stability and vibration of imperfect column

Thus, the result close to reality has been obtained. (Fig. 5) The displacement parameter „ α_1 “ is the function of the initial displacement and the level of the load. It means that the initial displacement enters the problem. If the load limits the level of the elastic critical load, the displacement and the frequency limits the infinity.

This example represents an advantage of the separation of the basic variational functions for the displacements and for the increments of the displacements.

3.3 Initial displacement as the second mode of buckling

A partial interesting problem is the influence of the mode of the initial displacement. In the previous part, we have supposed the initial displacement in the same mode as the first buckling mode (the mode of buckling related to the lowest elastic critical load). Due to that to obtain the solution by the analytical way was rather easy. The FEM has been used for the solution of more complicated examples.

Fig 6. presents the solution of the buckling and the vibration of the column when the initial displacement has the mode related to the second mode of buckling.

Figure 6. Stability and vibration of imperfect column with the initial displacement as the second mode of buckling

Note. A lot of examples have been solved using the FEM. The obtained results can be presented in the dimensionless mode.

These results enable us to note some peculiarities. Even the initial displacement has the same mode as the second mode of the buckling (*“the mode 2”*), the collapse mode of the column is *“the mode 1”*. The lowest elastic critical load is the maximum load. The mode of the vibration is *“the mode 1”* in all cases.

3.4 Experimental verification

The presented theoretical solutions are pointing to a substantial difference in the vibration of the beam at the moment when the critical load is reached. Considering sliding supports, the frequency should be zero. When supports are fixed, the frequency limits in infinity. This curiosity has been verified by an experiment.

The equipment for experimental verification of stability and vibration of beams loaded by pressure is shown in Fig. 7 and 8.

Figure 7. Scheme of the test set-up

Figure 8. General view of the test (the static behaviour)

The force (the load) is produced through the screw with a slight gradient (gradient 1.5 mm, average 30 mm), it means the load with the controlled deformation. The hinges are created by ball bearings in the jaw. The force is measured by manometer. The deflections are measured by mechanical displacement transducers fixed to the supporting steel structure. During measuring the frequency, the mechanical transducers are taken out and the accelerometer is attached.

Before the presentation of results, it is appropriate to make a note for specification of the mass matrixes due to end bearing (Fig. 9).

The mass matrix taking into account the effect of the end bearing will be

$$K_M = \rho A \frac{l}{2} + 2 * 0.06 * \sin \frac{\pi * 0.015}{l}$$

where the length of the beam is given in meters.

This effect of the end-bearing is dependent on the mass of the beam and is small (less than 1.5 %). To verify the dependence between the pressure force and frequency, the beams made of various types of materials have been analysed.

Steel hollow section profile Jäckl 30/15/1.5 mm

In the case of steel, the value of modulus of elasticity and the mass density are constant. When the exact dimensions of closed sections were measured, small problem occurred in measuring wall thickness.

The dimensions have been specified by measuring the weight of the profile. The rounded corners were considered in specification of cross-sectional characteristics. For further evaluation the following values were used

$$\text{Jäckl } 29.9 / 14.8 / 1.53, A = 121.4 \text{ mm}^2, I = 4286.0 \text{ mm}^4, r = 5.94 \text{ mm}, l = 1450 \text{ mm}$$

$$E = 210000 \text{ MPa}, \mu = 7850 \text{ kg / m}^3, F_{cr} = 4225.1 \text{ N}, \omega_0 = 144.2 \text{ s}^{-1}$$

Figure 10. Results from measurements of the steel hollow thin-walled section – Jäckl

Timber beams

The modulus of elasticity of wood is an open question in the analyses of timber beams. In the presented measurements the critical load is identified at the moment of the increasing of the deformation without the increase of the force. Since the cross-sectional characteristics (the cross section, the moment of inertia) as well as the length of the beam have been known, using the Euler's elastic critical force, the modulus of elasticity can be evaluated. By measuring the weight of the profile, the mass density of wood has been easily and accurately evaluated. Subsequently, the natural circular frequency has been evaluated and two timber beams investigated.

The presented results confirmed undoubtedly a phenomenon that the frequency of the beam increases when the pressure force is near the critical level.

3.5 Vibration of frame

In the examples of the vibration of the columns, the problem could be arranged in the dimensionless equation. In the case of the frame, we have to use FEM. The obtained results are arranged into the dimensionless mode.

The geometry of the investigated frame is shown in Fig. 12. When we have an example where the mode of the vibration is similar to the mode of the buckling, the relation between the load and the square of circular frequency is linear and we are not able to take into consideration the effects of initial displacements. Generally, the behaviour of the column and the behaviour of the frame are similar. (Fig. 5 – alt A – movable support).

Timber beam 47 / 47 mm , $A = 2209 \text{ mm}^2$, $I = 406640 \text{ mm}^4$, $r = 13.57 \text{ mm}$, $l = 2040 \text{ mm}$
 $E = 10200 \text{ MPa}$, $\rho = 472 \text{ kg / m}^3$, $F_{cr} = 9836.7 \text{ N}$, $\omega_0 = 147.3 \text{ s}^{-1}$
 Timber beam 42 / 32 mm , $A = 1344 \text{ mm}^2$, $I = 114888 \text{ mm}^4$, $r = 9.24 \text{ mm}$, $l = 1650 \text{ mm}$
 $E = 9750 \text{ MPa}$, $\rho = 454 \text{ kg / m}^3$, $F_{cr} = 4060.8 \text{ N}$, $\omega_0 = 1154.9 \text{ s}^{-1}$

Figure 11. Results from the measurements of the timber beams

The load conditions have been arranged to get the mode of the buckling different to the mode of the vibration. In this case, the relationship between the load and the square of the circular frequency is non-linear (Fig. 13). Analogously as in the case of the column, we suppose a different boundary condition for the static load and for the vibration. If the point of the load application is fixed during the vibration process, we can take into consideration the effects of initial displacements. (We have supposed the same mode of initial displacements as the mode of buckling.)

The modes of vibration for a different level of the load are presented as well (Fig. 13). In this case, we have supposed the frame without the initial imperfections. We can see that in the case of the higher level of the load ($F/F_{cr} > 0.6$), the mode of vibration is similar to the mode of the buckling.

3.6 Vibration and residual stresses

The residual stresses are typical in the welded steel structures, but we can observe the residual stresses even in concrete structures, timber structures and many other structures as well. The question is if the residual stresses have any influence on the circular frequency. The situation is much different in the beam type structures in comparison to the plate structures.

The residual stresses have been mentioned in Eq.(5). In the case of the beam structures, this equation will be reduced

$$\sigma_{,x} = E \cdot (\varepsilon_{,x} - \varepsilon_{0,x}) + \sigma_w$$

The residual stresses produce the addition in the increment of the potential energy

Figure 12. Load versus displacement for different values of initial displacements

Figure 13. Influence of initial displacements on the frame vibration

In the case of the beam type of structures, the volume integration can be changed into the integration over the cross section and the integration over the length.

$$\int_V \cup dV = \int_0^a \left(\int_A (\delta \Delta u_{,x} + u_{,x} \cdot \delta \Delta u_{,x} + w_{,x} \cdot \delta \Delta w_{,x} + \delta \Delta w_{,x} \cdot \Delta w_{,x} - z \cdot \delta \Delta w_{xx}) \sigma_w dA \right) dx =$$

All the derivations and the variations of the displacements functions are not any function of the cross section.

where $\omega_0^2 = \frac{\pi^4 E t^2}{3b^4(1-\nu^2)\rho}$ is the square of the circular frequency of a simply supported slender web and $K = 3(1-\nu^2)/8 = 0.34125$ (for $\nu=0.3$).

$$\int_0^a \left(\left(\delta \Delta u_{,x} + u_{,x} \cdot \delta \Delta u_{,x} + w_{,x} \cdot \delta \Delta w_{,x} + \delta \Delta w_{,x} \cdot \Delta w_{,x} \right) \int_A \sigma_w dA - \delta \Delta w_{xx} \int_A z \cdot \sigma_w dA \right) dx =$$

The residual stresses must be in equilibrium in the given cross section

$$\begin{aligned} \int_A \sigma_w dA &= 0, \quad \int_A z \cdot \sigma_w dA = 0 \Rightarrow \\ \dots \int_V \delta \Delta \varepsilon \cdot \sigma_w dV &= 0 \end{aligned} \quad (48)$$

It is evident that **the residual stresses in the case of the beam structures have no influence on the circular frequency.**

Note. In the case of the statically indeterminate structure, Eq. (48) is not valid and the residual stresses could have the influence on the vibration.

There is much different situation in the case of the plate structures. In this case, the volume integration is divided into the integration over the thickness and the integration over the neutral surface. The integration of the residual stresses over the thickness is not zero and thus,

$$\int_V \delta \Delta \varepsilon \cdot \sigma_w dV = \int_{-t/2}^{t/2} \left(\int \delta \Delta \varepsilon \cdot \sigma_w dz \right) d\Gamma = \int_{-t/2}^{t/2} \left(\delta \Delta \varepsilon \int \sigma_w dz \right) d\Gamma \neq 0$$

Finally, **in the case of the plate structures, the residual stresses have an influence on the circular frequency.**

Expanding the example from Part 2.2 we can get the result for the square of the circular frequency of the square slender web loaded in compression

$$\omega^2 = \omega_0^2 \cdot (1 - \bar{\sigma} + K(3\bar{\alpha}^2 - \bar{\alpha}_0^2)) \quad (49)$$

It is evident that, in comparison to the column, the circular frequency of the slender web is influenced by the initial displacement even in the case of the moving supports.

The influence of the fixed (unmovable) supports can be solved by the use of the following functions for the increments of the in-plane displacements

$$\Delta u = S_{x2} \cdot C_{y2} \cdot \Delta \beta_2 + S_{x2} \cdot \Delta \beta_3,$$

$$\Delta v = C_{x2} \cdot S_{y2} \cdot \Delta \gamma_2 + S_{y2} \cdot \Delta \gamma_3$$

The result can be arranged in the form

$$\omega^2 = \omega_0^2 \cdot \left(1 - \bar{\sigma} + \frac{21-12\nu-9\nu^2}{8} \bar{\alpha}^2 - \frac{3(1-\nu^2)}{8} \bar{\alpha}_0^2 \right) \quad (50)$$

Fig. 14 shows assumptions for distribution of residual stresses in the square slender web loaded in compression. We suppose the constant residual stresses through the thickness and then the approximate circular frequency can be expressed by the equation

$$\omega^2 = \omega_0^2 \cdot (1 - \bar{\sigma} - \bar{\sigma}_{xw} - \bar{\sigma}_{yw} + K(3\bar{\alpha}^2 - \bar{\alpha}_0^2)) \quad (51)$$

where $\bar{\sigma}_{xw} = \frac{\sigma_{xw}}{\sigma_{cr}}$, $\bar{\sigma}_{yw} = \frac{\sigma_{yw}}{\sigma_{cr}}$.

Figure 14. The distribution of the residual stresses in the slender web

One can see that residual stresses produce “shifting” of the level of the load.

Effect of residual stresses on circular frequency has been proved by experiment (Ravinger⁴). (Fig.15). Some results are presented in Figs. 16 and 17.

Figure 15. General view of experimental arrangement for the test of thin-walled panel

Figure 16. Comparison of theoretical and experimental results for the panel with $t=3.49 \text{ mm}$ thick web

Figure 17. Comparison of theoretical and experimental results for the panel with $t=2.505 \text{ mm}$ thick web

4 CONCLUSION

The presented theory and results prove the influence of the natural frequency on the level of the load, on the geometrical imperfections and the residual stresses, too. This knowledge can be used as an inverse idea. Measuring of the natural frequencies provides a picture of the stresses and imperfections in a thin-walled structure. One idea how we can investigate the structure is presented in Fig. 18. Many times we are not able to measure the whole structure (global vibration) but even measuring local parts of structure (local vibration) can give us valuable results.

It is true that the relation of frequencies versus stresses and imperfections represents a sophisticated theory, but it is unlikely an obstacle for further investigation.

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Figure 18. Scheme for non-destructive investigation of structure properties

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SUMMARY

STABILITY AND VIBRATION IN CIVIL ENGINEERING

Jan REVINGER

Von Kármán theory has been used for the description of the post-buckling behaviour of a thin-walled panel with imperfections and residual stresses. Using Hamilton's principle in incremental form the problem of free vibration has been established. Examples of buckling of a column, frame and a slender web loaded in compression emphasizing different types of support are presented. An influence of the mode of the geometrical imperfection is shown and an approximate solution taking into account the residual stresses is found.

Theoretical and numerical results are compared with the results from a laboratory experiment.

Key words: stability, post-buckling, vibration, finite element method, residual stresses

REZIME

STABILNOST I VIBRACIJE U GRAĐEVINARSTVU

Jan REVINGER

Teorija Von Kármán je korišćena u cilju opisivanja ponašanja tankih zidnih panela sa imperfekcijama i zaostalim naponima nakon izvijanja. Koristeći Hamiltonov princip u inkrementalnoj formi, uveden je problem slobodnih vibracija. U radu su dati primeri izvijanja stubova, ramova i tankih ploča opterećenih na pritisak, naglašavajući različite vrste oslonaca. Dat je prikaz uticaja oblika geometrijske imperfekcije kao i približno rešenje, uzimajući u obzir zaostale napone.

Teorijski i proračunski podaci upoređeni su sa rezultatima dobijenim laboratorijskim ispitivanjem.

Ključne reči: stabilnost, stability, post-izvijanje, vibracije, metoda konačnih elemenata, zaostali naponi