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A CONVERGENCE CRITERION OF SERIES

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Abstract. In this article is given a convergence criterion of series, which is similar to classical criterions of Abel and Dirichlet, and also some of its consequences and applications.

1. Results

Theorem 1. *Series* $p_1a_1 + p_2a_2 + \cdots + p_na_n + \cdots$ *is convergent if the following conditions are satisfied:*

1. *Sequence* $a_1, a_2, \dots, a_n, \dots$ *is nonincreasing and series* $a_1 + a_2 + \cdots + a_n + \cdots$ *is convergent.*
2. *Sequence* $p_1, \frac{p_1+p_2}{2}, \dots, \frac{p_1+p_2+\cdots+p_n}{n}, \dots$ *is bounded.*

Proof. We have

$$\sum_{k=1}^{n} p_k a_k = \sum_{k=1}^{n} (a_k - a_{k+1}) \sum_{\nu=1}^{k} p_\nu + a_{n+1} \sum_{k=1}^{n} p_k. \tag{1}$$

For $p_n = 1 \quad (n = 1, 2, ...)$ this equality becomes

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k(a_k - a_{k+1}) + n a_{n+1}. \tag{2}$$

The first condition of the theorem implies that the sequence $a_1, a_2, ..., a_n, \dots$ is nonnegative and converges to zero, and further, by Pringsheim's theorem [1, Page 27], that the series with nonnegative members $\sum_{k=1}^{n} k(a_k - a_{k+1})$ converges. This and the second condition of the theorem imply that the series $\sum_{k=1}^{\infty} k(a_k - a_{k+1}) \frac{1}{k} \sum_{\nu=1}^{k} p_\nu$, i.e. the series $\sum_{k=1}^{\infty} (a_k - a_{k+1}) \sum_{\nu=1}^{k} p_\nu$ (absolutely) converges, and that

$$\lim_{n\to\infty} a_{n+1} \sum_{k=1}^{n} p_k = \lim_{n\to\infty} n a_{n+1} \frac{1}{n} \sum_{k=1}^{n} p_k = 0.$$

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From (1) we then conclude that the series $\sum_{k=1}^{\infty} p_k a_k$ converges. □

From the proof of this theorem it can be easily seen that the theorem holds true if the condition (1) is changed by the weaker condition (1′).

1′) Series $\sum\limits_{n=1}^{\infty} n\,(a_n - a_{n+1})$ is absolutely convergent and $\lim\limits_{n\to\infty} na_n = 0$.

Corollary 8. *If $a_1, a_2, \dots, a_n, \dots$ is nonincreasing sequence and series $a_1 + \dots + a_n + \cdots$ is convergent, then the series*

$$\sum_{n=1}^{\infty}(-1)^n na_n, \quad \sum_{n=1}^{\infty} na_{n^2}, \quad \sum_{n=1}^{\infty} 2^n a_{2^n}$$

are convergent.

Proof. According to the proved criterion, it is sufficient to prove that the second condition of the previous theorem is satisfied for corresponding coefficients at a_n in these series.

If we take $p_n = (-1)^n a_n$, then $0 \leqslant \dfrac{p_1 + \dots + p_n}{n} \leqslant 1$, thus the series $\sum_{n=1}^{\infty}(-1)^n na_n$ is convergent. For the second series we can take $q_n = \begin{cases} \sqrt{n}, & \sqrt{n} \in N \\ 0, & \sqrt{n} \notin N \end{cases}$, and then we have

$$0 \leqslant \frac{q_1 + \dots + q_n}{n} = \frac{1 + 2 + \dots + [\sqrt{n}]}{n} \leqslant \frac{n + \sqrt{n}}{2n} \leqslant 1.$$

Thus, the series $\sum_{n=1}^{\infty} na_{n^2}$ is convergent.

For the third series, we take $r_n = \begin{cases} n, & \log_2 n \in N \\ 0, & \log_2 n \notin N. \end{cases}$ Hence, we have

$$0 \leqslant \frac{r_1 + \dots + r_n}{n} = \frac{1 + 2 + \dots + 2^{[\log_2 n]}}{n} \leqslant \frac{2 \cdot 2^{\log_2 n}}{n} = 2,$$

therefore, the series $\sum_{n=1}^{\infty} 2^n a_{2^n}$ is convergent. □

Let us remark that the convergence of the third series is a part of the claim of Cauchy's condensational criterion, here proved in a different way.

Theorem 2. *Let the series $\sum_{n=1}^{\infty} \dfrac{a_n}{n}$ and $\sum_{n=1}^{\infty} \dfrac{b_n}{n}$ be convergent with the corresponding nonincreasing sequences $\left(\dfrac{a_n}{n}\right)_1^{\infty}$, $\left(\dfrac{b_n}{n}\right)_1^{\infty}$. Then trigonometric series*

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges for every real x.

Proof. The statement is trivial for $x = 2k\pi \quad (k \in Z)$. The given trigonometric series can be written in the form

$$\sum_{n=1}^{\infty}(n\cos nx)\frac{a_n}{n} + \sum_{n=1}^{\infty}(n\sin nx)\frac{b_n}{n}.$$

According to the theorem 1, if we take into account conditions of this theorem, it is sufficient to prove that sequences

$$\frac{\cos x + 2\cos 2x + \cdots + n\cos nx}{n}, \quad \frac{\sin x + 2\sin 2x + \cdots + n\sin nx}{n}$$

are bounded. This last fact is direct implication of boundedness of the sequence

$$\frac{1}{n}\left|\sum_{k=1}^{n} ke^{ikx}\right|,$$

which can be proved as follows:

We have, for $\zeta \neq 1$

$$\sum_{k=1}^{n} k\zeta^k = \zeta\sum_{k=1}^{n} k\zeta^{k-1} = \zeta\left(\sum_{k=1}^{n}\zeta^k\right)' =$$

$$= \zeta\left(\frac{1-\zeta^{n-1}}{1-\zeta}\right)' = \zeta\frac{-(n+1)\zeta^n + n\zeta^{n+1} + 1}{(1-\zeta)^2}.$$

Hence

$$\frac{1}{n}\left|\sum_{k=1}^{n} ke^{ikx}\right| = \frac{1}{n}\left|e^{ix}\frac{-(n+1)e^{inx} + ne^{i(n+1)x} + 1}{(1-e^{ix})^2}\right| \leq$$

$$\leq \frac{2(n+1)}{n\left|1-e^{ix}\right|^2}, \quad x \neq 2k\pi \quad (k \in Z). \quad \square$$

2. References

[1] D.S. Mitrinović, D.D. Adamović, *Nizovi i redovi,* Naučna knjiga, Belgrade, 1980.

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