A NOTE ON THE POST’S COSET THEOREM

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Abstract. In this paper a proof of Post’s Coset Theorem is presented. The proof uses from Theory of \( n \)-groups, besides the definition of \( n \)-groups ([1];1.1), the description of \( n \)-group as an algebra with the laws of the type \( < n, n - 1, n - 2 > \). ( [8];1.2,1.3).

1. Preliminaries

Definition 1.1. Let \( n \geq 2 \) and let \((Q, A)\) be an \( n \)-groupoid. We say that \((Q, A)\) is a Dörnte \( n \)-group [briefly: \( n \)-group] iff is an \( n \)-semigroup and an \( n \)-quasigroup as well".

Proposition 1.2. [8] Let \( n \geq 2 \) and let \((Q, A)\) be an \( n \)-groupoid. Then the following statements are equivalent: (i) \((Q, A)\) is an \( n \)-group; (ii) there are mappings \( ^{-1} \) and \( e \) respectively of the sets \( Q^{n-1} \) and \( Q^{n-2} \) into the set \( Q \) such that the following laws hold in the algebra \((Q, \{A, ^{-1}, e\})\) [of the type \( < n, n - 1, n - 2 > \)]

\begin{align*}
(a) & \quad A(x_1^{n-2}, A(x_2^{n-2}), x_2^{n-1}) = A(x_1^{n-1}, A(x_2^{n-1})), \\
(b) & \quad A(e(a_1^{n-2}), a_1^{n-2}, x) = x \text{ and} \\
(c) & \quad A((a_1^{n-2}, a)\^{-1}, a_1^{n-2}, a) = e(a_1^{n-2}); \text{ and}
\end{align*}

(iii) there are mappings \( ^{-1} \)and \( e \) respectively of the sets \( Q^{n-1} \) and \( Q^{n-2} \) into the set \( Q \) such that the following laws hold in the algebra \((Q, \{A, ^{-1}, e\})\) [of the type \( < n, n - 1, n - 2 > \)]

\begin{align*}
(\bar{a}) & \quad A(A(x_1^n), x_{n+1}^{n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{n-1}), \\
(\bar{b}) & \quad A(x, a_1^{n-2}, e(a_1^{n-2})) = x \text{ and} \\
(\bar{c}) & \quad A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = e(a_1^{n-2}).
\end{align*}


Key words and phrases: \( n \)-groupoids, \( n \)-semigroups, \( n \)-quasigroups, \( n \)-groups, \( \{1, n\} \)-neutral operations on \( n \)-groupoids, inversing operation on \( n \)-group.

* A notion of an \( n \)-group was introduced by W. Dörnte in [1] as a generalization of the notion of a group. See, also [3–5].
Remark 1.3. \( e \) is an \( \{1, n\} \)-neutral operation of \( n \)-groupoid \((Q, A)\) iff algebra \((Q, \{A, e\})\) of type \(<n, n-2>\) satisfies the laws \((b)\) and \((b)\) from 1.2 \([6]\). Operation \(^{-1}\) from 1.2 \([c], (c)\) is a generalization of the inverting operation in a group \([7]\).

Definition 1.4. Let \((Q, B)\) be an \(n\)-groupoid and \(n \geq 2\). Then: 1) \( B \equiv B \); and 2) for every \( k \in \mathbb{N} \) and for every \( x_{(i+j)(n-1)+1}^{(k+1)(n-1)+1} \in Q \)

\[
B_{(i+j)(n-1)+1}^{(k+1)(n-1)+1} \equiv B(B(x_{(i+j)(n-1)+1}^{(k+1)(n-1)+1}), x_{(i+j)(n-1)+1}^{(k+1)(n-1)+1})
\]

Proposition 1.5. Let \((Q, B)\) be an \(n\)-semigroup, \(n \geq 2\) and \((i, j) \in \mathbb{N}^2\). Then, for every \( x_{(i+j)(n-1)+1}^{(i+j)(n-1)+1} \in Q \) and for every \( t \in \{1, \ldots, i(n-1)+1\} \) the following equality holds

\[
B_{(i+j)(n-1)+1}^{(i+j)(n-1)+1} = B(B_{(i+j)(n-1)+1}^{(i+j)(n-1)+1}, x_{(i+j)(n-1)+1}^{(i+j)(n-1)+1})
\]

2. Auxiliary proposition

Proposition 2.1. Let \(n \geq 2\) and let \((Q, A)\) be an \(n\)-group. Also, let \(a_1^{k(n-1)}, b_1^{k(n-1)}, c\) be arbitrary elements of the set \(Q\) such that the following equality holds

\[
A(a_1^{k(n-1)}, c) = A(b_1^{k(n-1)}, c) \quad (A(a_1^{k(n-1)}, b_1^{k(n-1)})) = A(c, b_1^{k(n-1)})
\]

Then for all \(x \in Q\) the following equality holds

\[
A(a_1^{k(n-1)}, x) = A(b_1^{k(n-1)}, x) \quad (A(x, a_1^{k(n-1)})) = A(x, b_1^{k(n-1)})
\]

Sketch of the proof. \(A(a_1^{k(n-1)}, c) = A(b_1^{k(n-1)}, c)\)

\[
A(a_1^{k(n-1)}, c_1^{n-2}, (c_1^{n-2}, c)^{-1}) = A(b_1^{k(n-1)}, c_1^{n-2}, (c_1^{n-2}, c)^{-1})
\]

\[
A(a_1^{k(n-1)}, c_1^{n-2}, (c_1^{n-2}, c)^{-1}) = A(b_1^{k(n-1)}, c_1^{n-2}, (c_1^{n-2}, c)^{-1})
\]

\[
A(a_1^{k(n-1)}, c_1^{n-2}, (c_1^{n-2}, c)^{-1}) = A(b_1^{k(n-1)}, c_1^{n-2}, (c_1^{n-2}, c)^{-1})
\]

\[
A(a_1^{k(n-1)}, e_1^{n-2}) = A(b_1^{k(n-1)}, e_1^{n-2})
\]

\[
A(A(a_1^{k(n-1)}, e_1^{n-2}), c_1^{n-2}, x) = A(b_1^{k(n-1)}, e_1^{n-2}, c_1^{n-2}, x)
\]

\[
A(a_1^{k(n-1)}, e_1^{n-2}, c_1^{n-2}, x) = A(b_1^{k(n-1)}, e_1^{n-2}, c_1^{n-2}, x)
\]

\[
A(a_1^{k(n-1)}, e_1^{n-2}, c_1^{n-2}, x) = A(b_1^{k(n-1)}, e_1^{n-2}, c_1^{n-2}, x)
\]

\[
A(a_1^{k(n-1)}, x) = A(b_1^{k(n-1)}, x)
\]

[1.2-1.5] \(\square\)
Proposition 2.2. Let \( n \geq 2 \) and let \((Q, A)\) be an \( n\)-group. Also, let \(a_1^{k(n-1)}, b_1^{l(n-1)}, c\) be arbitrary elements of the set \(Q\) such that the following equality holds
\[
A(a_1^{k(n-1)}, c) = A(b_1^{l(n-1)}, c)
\]

Then the following equality holds
\[
A(c, a_1^{k(n-1)}) = A(c, b_1^{l(n-1)})
\]

Sketch of the proof. \( A(a_1^{k(n-1)}, c) = A(b_1^{l(n-1)}, c) \Leftrightarrow \)
\[
A(c, a_1^{k(n-1)}, c, c_1^{n-2}) = A(c, b_1^{l(n-1)}, c, c_1^{n-2}) \Leftrightarrow
A(A(c, a_1^{k(n-1)}), c, c_1^{n-2}) = A(A(c, b_1^{l(n-1)}), c, c_1^{n-2}) \Leftrightarrow
A(c, a_1^{k(n-1)}) = A(c, b_1^{l(n-1)})
\]

[:1.1 - cancellation laws, 1.5 \( \square \)

3. A proof of the Post’s Coset Theorem

Theorem 3.1 (Post’s Coset Theorem [2]*): Every \( n\)-group has a covering group.

Proof. Let \((Q, A)\) be an \( n\)-group.

1) If \( n = 2 \), \((Q, A)\) is an ordinary group and hence is its own covering group.

2) The case \( n \geq 3 \):

Let \( \Gamma \) be the set of all sequence over \( Q \). Also, let the multiplication in \( \Gamma \) be defined as the juxtaposition:
\[
a_1^i \ast b_1^j \overset{\text{def}}{=} a_1^i, b_1^j
\]
for all \( a_1^i, b_1^j \in \Gamma; i, j \in \mathbb{N} \cup \{0\} \).

Then:
1° \((\Gamma, \ast)\) is a semigroup. Moreover, \( \emptyset \) \( \emptyset \): empty sequence is a neutral element of the semigroup \((\Gamma, \ast)\).

Now we define the relation \( \theta \) as follows:
2° For all \( \alpha, \beta \in \Gamma \):
\[
\alpha \theta \beta \overset{\text{def}}{=} (\exists \gamma \in \Gamma)(\exists \delta \in \Gamma) A(\gamma, \alpha, \delta) = A(\gamma, \beta, \delta);
|\gamma, \alpha, \delta| = k(n - 1) + 1, |\gamma, \beta, \delta| = l(n - 1) + 1.
\]

By 2°, 2.1 and 2.2, we conclude that the following statements holds:

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*See, also [3–5].
3° Let $\alpha$ and $\beta$ an arbitrary elements of the set $\Gamma$ such that the statement holds: $\alpha \theta \beta$. Then, for each $\gamma, \delta \in \Gamma$ such that $|\gamma, \alpha, \delta| = k(n - 1) + 1$ and $|\gamma, \beta, \delta| = l(n - 1) + 1$, where $k, l \in N$, the following equality holds

\[ A(\gamma, \alpha, \delta) = A(\gamma, \beta, \delta). \]

Also, the following statements hold.

4° $\theta \in \text{Con}(\Gamma, \ast)$.

5° Let $C(\alpha) \cdot C(\beta) \overset{\text{def}}{=} C(\alpha \ast \beta)$ for all $\alpha, \beta \in \Gamma$. Then: a) $(\Gamma/\theta, \cdot)$ is a semigroup; and b) $C(\emptyset)$ is a neutral element of the semigroup $(\Gamma/\theta, \cdot)$. [See: 1° and 4°.]

6° Let $\alpha \neq \emptyset$ and let for all $y \in Q$ the following equality holds

\[ A(\alpha, y) = y \quad [A(y, \alpha) = y]. \]

Then $\alpha \in C(\emptyset)$.

7° For every $\alpha \in \Gamma$ there is at least one $\beta \in \Gamma \{\gamma \in \Gamma\}$ such that for all $y \in Q$ the following equality holds

\[ A(\beta, \alpha, y) = y \]

\[ [A(y, \alpha, \gamma) = y]. \]

8° Let $x \in Q$, $y \in Q$ and $y \in C(x)$. Then $y = x$.

9° Let $a_i^n \in Q$. Then the following equality holds

\[ C(a_1) \cdots C(a_n) = C(A(a_1^n)). \]

The sketch of the proof of 4°:

a) For all $\alpha \in \Gamma$ there is $\delta, \varphi \in \Gamma$ and $k \in N$ such that the following equalities hold

\[ |\delta, \alpha, \varphi| = k(n - 1) + 1 \quad \text{and} \quad A(\delta, \alpha, \varphi) = A(\delta, \alpha, \varphi) \]

[2°].

b) $A(\delta, \alpha, \varphi) = A(\delta, \beta, \varphi) \Rightarrow A(\delta, \beta, \varphi) = A(\delta, \alpha, \varphi)$

[2°].

c) $A(\delta, \alpha, \varphi) = A(\delta, \beta, \varphi) \land A(\delta, \gamma, \varphi) \Rightarrow A(\delta, \alpha, \varphi) = A(\delta, \beta, \varphi) \land A(\delta, \gamma, \varphi) \Rightarrow A(\delta, \alpha, \varphi) = A(\delta, \gamma, \varphi)$

[2°, 3°].
d) $\alpha \theta \bar{\alpha}, \beta \theta \bar{\beta}, |\gamma, \alpha, \beta, \delta| = k(n - 1) + 1$;

$$A(\gamma, \alpha, \beta, \delta) = A(\gamma, \bar{\alpha}, \beta, \delta) = A(\gamma, \bar{\alpha}, \bar{\beta}, \delta)$$

[\ref{2.2}, \ref{3.3}].

The sketch of the proof of $6^\circ$:

$$A(\alpha, y) = y \Rightarrow A(\alpha, A(x_1^{k(n-1)+1})) = A(x_1^{k(n-1)+1}) \Rightarrow$$

$$A(\alpha, x_1^{k(n-1)+1}) = A(\emptyset, x_1^{k(n-1)+1}) \Rightarrow \alpha \theta \emptyset$$

[\ref{1.1 - (Q, A)} is an $n$-quasigroup, \ref{1.5}, \ref{2.0}].

The sketch of the proof of $7^\circ$:

Let $n \geq 3$. Then the following statements hold:

a) $\left\{ (1)^{n-2}, (k)^{n-2}, a_1, \ldots, a_1, b_1^t, k \geq 0 \wedge 0 \leq t < n - 2 \wedge (1)^{n-2}, a_1, b_1^t \in \Gamma \right\} = \Gamma$; and

b) For each $(1)^{n-2}, a_1, \ldots, (k)^{n-2}, a_1, \bar{b}_1^t, c_{t+1}^{n-2} \in Q$ and for all $x \in Q$ the following equality holds

$$A \left( (c_{t+1}^{n-2}, b_1^t), (c_{t+1}^{n-2}, a_1^t), \ldots, (c_{t+1}^{n-2}, a_1^t), (1)^{n-2}, a_1^t, b_1^t, x \right) = x$$

[\ref{1.2-1.5}]. (Remark: For $k = t = 0$:

$$A(1)^{n-2}, a_1, \ldots, a_1, b_1^t = \emptyset.$$

The sketch of the proof of $8^\circ$:

a) $y \in C(x) \Leftrightarrow y \theta x \Leftrightarrow (\exists \alpha \in \Gamma)(\exists \beta \in \Gamma)A(\alpha, y, \beta) = A(\alpha, x, \beta); k \in N$ [\ref{2.0}].

b) $A(\alpha, y, \beta) = A(\alpha, x, \beta) \Rightarrow y = x$ [\ref{1.1,1.5}].

The sketch of the proof of $9^\circ$:

a) $b = A(a_1^n) \Leftrightarrow A(b, x_1^{n-1}) = A(a_1^n, x_1^{n-1}) \Leftrightarrow b \theta a_1^n$ [\ref{1.1,1.5,2.0}].

b) $C(b) = C(a_1^n) = C(a_1) \cdots C(a_n) [\ref{4.0}, \ref{5.0}].$

c) $C(A(a_1^n)) = C(a_1) \cdots C(a_n) [\ref{a,b}].$

By $4^\circ - 7^\circ$, we conclude that $(\Gamma / \theta, \cdot)$ is an group.

Finally, let

$$A(C(a_1), \ldots, C(a_n)) \overset{def}{=} C(a_1) \cdots C(a_n)$$

for each $C(a_1), \ldots, C(a_n) \in \{C(x)|x \in Q\}$. Also, let

$$F(a) \overset{def}{=} C(a)$$

for all $a \in Q$. Then, by $8^\circ$ and $9^\circ$, we conclude that the following statements hold.
1) \((\forall a_i \in Q)^n F A(a_1^n) = A(F(a_1), \ldots, F(a_n))\); and
2) \(F\) is a bijection which maps the set \(Q\) onto the set \(\{C(x)|x \in Q\}\). □

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