THE NUMERICAL FUNCTION OF A
∗–REGULARLY VARYING SEQUENCE

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Abstract. In this paper, we impose some conditions under which there is a close relation between the asymptotic behaviour of a ∗–regularly varying sequence and the asymptotic behaviour of its numerical function δc(x), x > 0.

1. Introduction and results

A sequence of positive numbers (cn) is called O–regularly varying [2], if we have
\[ \lim_{n \to +\infty} c\left[ \lambda n \right] c_n < +\infty, \quad \lambda > 0. \] (1)

The class of all O–regularly varying sequences is denoted ORV.

An O–regularly varying sequence (cn) is called ∗–regularly varying [6], if it is nondecreasing, and if
\[ \lim_{\lambda \to 1+} k_c(\lambda) = 1. \] (2)

The class of all ∗–regularly varying sequences is denoted ∗RV.

The above two classes of sequences represent the important objects in the sequential theory of regular variability in the Karamata sense [1], and in particular in the theory of statements of Tauberian type [4], as well as in some other parts of qualitative analysis of divergent processes [7].

The class \( K^*_c \) [5], consists of all ∗–regularly varying sequences which satisfy the condition
\[ k_c(\lambda) = \lim_{n \to +\infty} \frac{c\left[ \lambda n \right]}{c_n} > 1, \quad \lambda > 1. \] (3)

### AMS (MOS) Subject Classification 1991. Primary: 26A12.

**Key words and phrases:** Regularly varying sequence, numerical function.

*Research supported by Science Fund of Serbia under Grant 1457.
Notice that, in particular, the class $K^*_c$ contains all nondecreasing regularly varying sequences in the Karamata sense [1] whose index $\rho > 0$, and also all sequences whose general term is the $n$-th ($n \in \mathbb{N}$) partial sum of a $*$-regularly varying sequence, but does not contain slowly varying sequences in the Karamata sense [1].

If next, $(c_n)$ is an increasing sequence of positive numbers, then its numerical function $\delta_c(x)$, $x > 0$, is defined by $\delta_c(x) = \sum_{c_n \leq x} 1$, $x > 0$.

We shall prove several statements about the mentioned classes.

By $\sim$ we shall denote the weak, while by $\asymp$ the strong asymptotic equivalence.

**Theorem 1.** Let $(c_n)$ be an increasing sequence from the class $K^*_c$ and assume that $g: [1, +\infty) \mapsto (0, +\infty)$ is a continuous and increasing function. Then we have
\begin{equation}
(4) \quad c_n \sim g(n), \quad n \to \infty,
\end{equation}
if and only if
\begin{equation}
(5) \quad \delta_c(x) \sim g^{-1}(x), \quad x \to +\infty.
\end{equation}

Notice that if $(c_n)$ is an arbitrary increasing sequence of positive number which is not in the class $K^*_c$, it is easy to construct a continuous and increasing function $g: [1, +\infty) \mapsto (0, +\infty)$, so that (4) is true but not (5) or, (5) is true but not (4).

**Corollary 1.** Let $(c_n)$ be an increasing sequence from the class $K^*_c$, and $(d_n)$ be an increasing sequence of positive numbers. Then we have
\begin{equation}
(4') \quad c_n \sim d_n, \quad n \to \infty
\end{equation}
if and only if
\begin{equation}
(5') \quad \delta_c(x) \sim \delta_d(x), \quad x \to \infty.
\end{equation}

Corollary 1 follows easily from the theorem above.

**Corollary 2.** Let $(c_n)$ be an increasing sequence from the class $K^*_c$ and let $g: [1, +\infty) \mapsto (0, +\infty)$ be a continuous and increasing function. If (4) holds, then we have
\begin{equation}
(6) \quad \sum_{c_n \leq x} c_n \sim x g^{-1}(x), \quad x \to +\infty.
\end{equation}

**Corollary 3.** Let $(c_n)$ be an increasing sequence from the class $K^*_c$ and $(d_n)$ be an increasing sequence of positive numbers. If (4') holds, then we have
\begin{equation}
(6') \quad \sum_{c_n \leq x} c_n \sim \sum_{d_n \leq x} d_n, \quad x \to +\infty.
\end{equation}
2. Proofs of statements

**Proof of the theorem.** Consider the function \( f(x), x \geq 1 \), for which we have \( c_n = f(n) \). It is obviously linear on intervals \([n, n+1], n \in \mathbb{N}\).

For any \( \delta > 0 \), there is some \( n_0 = n_0(\delta) \in \mathbb{N} \), so that for all \( n \geq n_0 \) we have \( 1 \leq 1 + \frac{1}{n} \leq \delta + 1 \), so that we find \( 1 \leq \lim_{n \to +\infty} \frac{c_{n+1}}{c_n} \leq K_c(1+\delta) \). Since by assumption \( (c_n) \in K^*_c \), it is \( * \)-regularly varying, so that \( \lim_{n \to \infty} \frac{c_{n+1}}{c_n} = 1 \). If (4) holds true, then we have \( f(x) \sim g(x), x \to +\infty \), because for all \( n \leq x < n+1, n \in \mathbb{N} \), we have that

\[
\frac{c_n}{c_{n+1}} \cdot \frac{c_{n+1}}{g(n+1)} \leq \frac{f(x)}{g(x)} \leq \frac{c_n}{g(n)} \cdot \frac{c_{n+1}}{c_n},
\]

Next, let for any \( \lambda > 0 \), \( \overline{K}_f(\lambda) = \lim_{x \to +\infty} \frac{f(\lambda x)}{f(x)} \). Then for every \( \delta > 0 \) we have

\[
\overline{K}_c(\lambda) \leq \overline{K}_f(\lambda) \leq \lim_{x \to +\infty} \frac{f([\lambda x] + 1)}{f([x])} \leq \leq \lim_{x \to +\infty} \frac{c_{[\lambda x]} + 1}{c_{[x]}} \cdot \lim_{x \to +\infty} \frac{c_{[\lambda x]} + 1}{c_{[\lambda x]}} \leq \overline{K}_c(\lambda) \cdot \overline{K}_c(1+\delta),
\]

because

\[
\lim_{x \to +\infty} \frac{[\lambda x] + 1}{[\lambda x]} = 1 + .
\]

This means that for every \( \lambda > 0 \) we have \( \overline{K}_c(\lambda) = \overline{K}_f(\lambda) \).

If we next redefine \( f(x) \) by \( f(0) = 0 \), and on the interval \([0, 1]\) as a linear function, then we have that \( f \in K^*_c \) (see [5]). If we in a similar way redefine \( g(x) \) for \( 0 \leq x < 1 \), and we suppose (4), then by [3] we have

\[
(7) \quad f^{-1}(x) \sim g^{-1}(x), \quad x \to +\infty.
\]

Since \( \delta_c(x) = [f^{-1}(x)], x > 0 \), we obtain (5).

Conversely, supposing that (5) holds true, then with the so redefined functions \( f \) and \( g \) we have (7). Since \( f \in K^*_c \), we get \( f(x) \sim g(x), x \to +\infty \), so that we obtain (4).

**Remark.** If \( (c_n) \) is an increasing and unbounded \( * \)-regularly varying sequence, out the class \( K^*_c \), then (5) implies (4) for every function \( g \) described in the Theorem. But it is not difficult to see that there is a function \( g \) which has properties from the Theorem, such that (4) does not implies (5).

If a sequence \( (c_n) \) is increasing and unbounded, and it is not \( * \)-regularly varying, it is not clear if, in the general case, (4) and (5) are equivalent to each other for an arbitrary function \( g \) described in the Theorem.
Proof of Corollary 2. By assumptions, we have that
\[
\sum_{c_n \leq x} c_n = \int_0^x t \, d\delta_c(t) \leq x \delta_c(x), \quad x > 0.
\]
On the other side, we have
\[
\sum_{c_n \leq x} c_n \geq \int_{x/2}^x t \, d\delta_c(t) \geq \frac{x}{2} \left( \delta_c(x) - \delta_c\left(\frac{x}{2}\right) \right), \quad x > 0.
\]
Since \((c_n) \in K^*_c\) we have that \(k_c(\lambda) > 1, \lambda > 1\), so that \(k_c(2) > 1\). In other words, \(k_c\left(\frac{1}{2}\right) < 1\). Next, define \(p = 1 - k_c\left(\frac{1}{2}\right)\). Then for all \(x \geq x_0\) we have that
\[
\frac{p}{4} \leq \sum_{c_n \leq x} c_n x \delta_c(x) \leq 1,
\]
so that \(\sum_{c_n \leq x} c_n \asymp x \delta_c(x), x \to +\infty\). By assumptions of the colollary, and the Theorem, we have that then \(\delta_c(x) \sim g^{-1}(x), x \to +\infty\), so that (6) holds true.

Finally, Corollary 3 is a direct consequence of the Theorem and the Corollary 2.

3. References


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Received: January 15, 2002.