SOME REMARKS NEAR-\(P\)-POLYAGROUPS
AND POLYAGROUPS

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Abstract. In this paper the Hosszú–Gluskin Theorem for near-\(P\)-polyagroups (polyagroups) is proved.

1. Introduction

1.1. Definition [1]: Let \(n \geq 2\) and let \((Q, A)\) be an \(n\)-groupoid. We say that \((Q, A)\) is a Dörnte \(n\)-group [briefly: \(n\)-group] iff is an \(n\)-semigroup and an \(n\)-quasigroup as well. (See, also [12].)

1.2. Definition (cf. [9],[10]): Let \(k > 1, s \geq 1, n = k \cdot s + 1\) and let \((Q; A)\) be an \(n\)-groupoid. Then: we say that \((Q; A)\) is a polyagroup of the type \((s, n - 1)\) iff the following statements hold:

1° For all \(i, j \in \{1, \ldots, n\} (i < j)\) if \(i \equiv j (\text{mod } s)\), then \(< i, j >\) -associative law holds in \((Q; A)\); and

2° \((Q; A)\) is an \(n\)-quasigroup.

1.3. Definition[11]: Let \(k > 1, s \geq 1, n = k \cdot s + 1\) and let \((Q; A)\) be an \(n\)-groupoid. Then: we say that \((Q; A)\) is a near-\(P\)-polyagroup [briefly: \(NP\)-polyagroup] of the type \((s, n - 1)\) iff the following statements hold:

\(1°\) For all \(i, j \in \{1, \ldots, n\} (i < j)\) if \(i, j \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}\), then the \(< i, j >\) -associative law holds in \((Q; A)\); and

\(2°\) For all \(i \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}\), and for every \(a^n_i \in Q\) there is exactly one \(x_i \in Q\) such that the equality

\[ A(a^{i-1}_1, x_i, a^{n-1}_i) = a_n \]

holds.\(^1\)

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\(1°\) For \(s = 1\) \((Q; A)\) is a \((k + 1)\)-group, where \(k + 1 \geq 3; k > 1\).
1.4. Proposition: Every polyagroup of the type \((s, n - 1)\) is an NP-polyagroup of the type \((s, n - 1)\). [By Def.1.2 and by Def. 1.3.]

2. Auxiliary propositions

2.1. Proposition [8]: Let \(n \geq 2\) and let \((Q; A)\) be an \(n\)-groupoid. Then, the following statements are equivalent: (i) \((Q; A)\) is an \(n\)-group; (ii) there are mappings \(-1\) and \(e\), respectively, of the sets \(Q^{-1}\) and \(Q^n\) into the set \(Q\) such that the following laws hold in the algebra \((Q; A, e)\) [of the type \(<n, n-1, n-2>\)]

\[
\begin{align*}
(a) \quad & A(x_{n-2}^n, A(x_{n-1}^{2n-2}), x_{n-1}) = A(x_{n-1}^n, A(x_{n-1}^{2n-2})), \\
(b) \quad & A(e(a_{n-1}^{-2}), a_{n-1}^{-2}, x) = x \quad \text{and} \\
(c) \quad & A((a_{n-1}^{-2}, a_{n-2}^{-1}), a_{n-2}^{-2}) = e(a_{n-2}^{-2}); \quad \text{and}
\end{align*}
\]

(iii) there are mappings \(-1\) and \(e\), respectively, of the sets \(Q^{-1}\) and \(Q^n\) into the set \(Q\) such that the following laws hold in the algebra \((Q; A, e)\) [of the type \(<n, n-1, n-2>\)]

\[
\begin{align*}
(\pi) \quad & A(A(x_1^n, x_{n+1}^{2n-1}), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}, x_{n+2}^{2n-1})), \\
(\beta) \quad & A(x, a_{n-2}^{n-2}, e(a_{n-1}^{n-2})) = x \quad \text{and} \\
(\gamma) \quad & A(a, a_{n-2}^{n-2}, (a_{n-2}^{n-2}, a_{n-2}^{n-2})^{-1}) = e(a_{n-2}^{n-2}).
\end{align*}
\]

2.2. Remark: \(e\) is an \(\{1, n\}\)-neutral operation of \(n\)-groupoid \((Q; A)\) iff algebra \((Q; A, e)\) [of the type \(<n, n-1, n-2>\)] satisfies the laws \((\beta)\) and \((\gamma)\) from 2.1 [5]. Operation \(-1\) from 2.1. \((c), (\gamma)\) is a generalization of the inverse operation in a group [6]. Cf. [12].

2.3. Definition[7]: We say that an algebra \((Q; B, \varphi, b)\) [of the type \(<2, 1, 0>\)] is a Hosszú-Gluskin algebra of order \(n\) (\(n \geq 3\)) [briefly: nHG-algebra] iff the following statements hold:

\((Q; B)\) is a group;

\(\varphi \in \text{Aut} (Q; B)\);

\(\varphi(b) = b\); and

For every \(x \in Q\), \(B(\varphi^{n-1}(x), b) = B(b, x)\).

2.4. Proposition (Hosszú-Gluskin Theorem [2,3][7]): Let \((Q; A)\) be an \(n\)-group, \(e\) its \(\{1, n\}\)-neutral operation (cf. 2.2) and \(n \geq 3\). Let also, \(c_1^{n-2}\) be an arbitrary (fixed) sequence over a set \(Q\), and let

\[
\begin{align*}
B(x, y) \overset{\text{def}}{=} & A(x, c_1^{n-2}, y), \\
\varphi(x) \overset{\text{def}}{=} & A(e(c_1^{n-2}), x, c_1^{n-2}) \quad \text{and} \\
b \overset{\text{def}}{=} & A(e(c_1^{n-2}))
\end{align*}
\]

for all \(x, y \in Q\). Then, the following statements hold:

1. \((Q; B, \varphi, b)\) is an nHG-algebra; and
(2) For every \( x^n \in Q \) the equality
\[
A(x^n) = B(x_1, \varphi(x_2), \ldots, \varphi^{n-1}(x_n), b)^2
\]
holds.\(^3\)

2.5. Proposition [11]: Every \( NP \)-polygroup has a \( \{1, n\} \)-neutral operation.

3. Results

3.1. Theorem: Let \( k > 1 \), \( s \geq 1 \), \( n = k \cdot s + 1 \), \((Q; A)\) be an \( NP \)-polygroup of the type \((s, n-1)\), \( e \) its \( \{1, n\} \)-neutral operation and \( Y \) stands for sequence
\[
y_1^{s-1}, \ldots, y_s^{s-1} \in Q \quad \Rightarrow \quad y_1^{s-1} \mid_{k=1} \quad \text{over } Q.\]
Also let \( c_1^{s-1} \) be an arbitrary \( (fixed) \) sequence over a set \( Q \). Further on, let
\[
\begin{align*}
(1) \quad & B(Y, x, y) \overset{\text{def}}{=} A(x, y_1^{s-1}, c_j)_{j=1}^{k-1}y_1^{s-1}, y), \\
(2) \quad & \varphi(Y, x) \overset{\text{def}}{=} A(e( y_1^{s-1}, c_j)_{j=1}^{k-1}y_1^{s-1}, x, y_1^{s-1}, c_j)_{j=1}^{k-1}y_1^{s-1})\quad \text{and} \\
(3) \quad & b(Y) \overset{\text{def}}{=} A(e(y_1^{s-1}, c_j)_{j=1}^{k-1}y_1^{s-1})
\end{align*}
\]
for all \( x, y, y_1^{s-1}, \ldots, y_s^{s-1} \in Q \). Finally, let for all \( x, y \in Q \)
\[
\begin{align*}
(1) \quad & B_Y(x, y) \overset{\text{def}}{=} B(Y, x, y), \\
(2) \quad & \varphi_Y(x) \overset{\text{def}}{=} \varphi(Y, x) \quad \text{and} \\
(3) \quad & b_Y \overset{\text{def}}{=} b(Y),
\end{align*}
\]
where \( Y \) is an arbitrary \( (fixed) \) sequence over \( Q \). Then, the following statements hold:

(i) For all sequence \( Y \) over \( Q \) \((Q; B_Y, \varphi_Y, b_Y)\) is an \( (k+1)HG \)-algebra; and

(ii) For all \( x_1^{k+1}, \ldots, y_1^{s-1} \in Q \) the following equality holds
\[
A(x_j, y_1^{s-1}, \ldots, y_1^{s-1})_{j=1}^{k+1} B(Y, x_1, Y, \varphi(Y, x_2), \ldots, Y, \varphi^{k}(Y, x_{k+1}), Y, b(Y)),
\]
where \( \varphi^i \overset{\text{def}}{=} \varphi \) and for all \( x \in Q \), for all sequence \( Y \) over \( Q \) and for every \( i \in N \) \( \varphi^{i+1}(Y, x) \overset{\text{def}}{=} \varphi(Y, \varphi^i(Y, x)) \).

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\(^2\)B \overset{\text{def}}{=} B_{(s+1)}(x_n^{(s+1)+1}, x_n^{(s+1)+1}, t \in N); \text{ cf. [12], VI-6.}

\(^3\)The formulation and the proof of the theorem follow the idea of E.I. Sokolov from [4].
See also [12]; Chapter IV and Appendix 2.
Proof. Firstly, we observe that under the assumptions the following statements hold

1° Let \( Y \) be an arbitrary (fixed) sequence over a set \( Q \). Also let

\[
A(a_{k+1}) = A( x_j, y_{s-1}^{k} 
\]

for all \( x_{k+1} \in Q \). Further on, let \( c_{n-2}^{1} \) be an arbitrary (fixed) sequence over a set \( Q \). Then \((Q;\ A)\) is an \((k+1)-\)group;

2° Let \((Q; A)\) \((k + 1)-\)group from 1°. Also let

\[
E(a_{n-2}^{1}) = E( y_{s-1}^{k}, a_{j}^{k-1}, y_{s-1}^{1})
\]

for all \( a_{n-2}^{1} \in Q \). Then \( E \) is an \( \{1, k+1\}-\)neutral operation of the \((k+1)-\)group \((Q;\ A)\); and

3° Let \((Q; A)\) \((k+1)-\)group from 1°. Then:

3° a \((Q; B_{Y}, \varphi_{Y}, b_{Y})\) is an \((k+1)H G-\)algebra; and

3° b \(A(x_{k+1}) = B_{Y}(x_{1}, \varphi_{Y}(x_{2}), \ldots, \varphi_{Y}^{k}(x_{k+1}), b_{Y})\) for all \( x_{k+1} \in Q \).

The proof of 1° : Def. 1.1 and by Def. 1.3.

Sketch of the proof of 2°:

\[
A(E(a_{n-2}^{1}), a_{n-2}^{1}, x) = A(E( y_{s-1}^{k}, a_{j}^{k-1}, y_{s-1}^{1}), a_{n-2}^{1}, x)
\]

\[
= A( y_{s-1}^{k}, a_{j}^{k-1}, y_{s-1}^{1}), a_{n-2}^{1}, x) = x.
\]

Whence, by Def. 1.1, Prop. 2.1 and by Rem. 2.2, we obtain 2°.

Sketch of the proof of 3°:

\[
B_{Y}(x, y) = A(x, y_{s-1}^{1}, c_{j}^{k-1}, y_{s-1}^{1}, y)
\]

\[
= A(x, y_{s-1}^{1}, c_{j}^{k-1}, y)
\]

\[
\varphi_{Y}(x) = A( y_{s-1}^{k}, a_{j}^{k-1}, y_{s-1}^{1}, y_{s-1}^{1}, x)
\]

\[
= A(E( c_{n-2}^{1}, x, c_{j-1}^{k-1})
\]

\[
= A(E( c_{n-2}^{1}, x, c_{j-1}^{k-1}) and
\]

\[
by\ A( y_{s-1}^{k}, a_{j}^{k-1}, y_{s-1}^{1}, y_{s-1}^{1}, c_{j-1}^{k-1})
\]

\[
= A(E( c_{n-2}^{1}, x, c_{j-1}^{k-1}))
\]
Whence, by Prop. 2.4, we obtain $3^\circ$.

In addition, by $3^\circ$, since $Y$ is an arbitrary sequence over $Q$, we conclude that the statement $(i)$ holds.

Finally, by $3^\circ \beta_k^\circ (1) - (3)$ and $(1) - (3)$, since $Y$ is an arbitrary sequence over $Q$, we obtain also $(ii)$. □

By Th.3.1 and by Prop.1.4, we have:

3.2. Theorem: Let $k > 1, s > 1, n = k \cdot s + 1, (Q; A)$ be an polyagroup of the type $(s,n-1)$, e its $\{1, n\}$—neutral operation and $Y$ stands for sequence $(1)^{s-1}, \ldots, y_1^{s-1} = \left[ y_1^{s-1} \right]_{j=1}^k$ over $Q$. Also let $c_1^{k-1}$ be an arbitrary (fixed) sequence over $Q$. Further on, let $(1) - (3)$ from Th.3.1 for all $x, y, y_1^{s-1}, \ldots, y_1^{s-1} \in Q$. Finally, let for all $x, y \in Q (1) - (3)$ form Th.3.1, where $Y$ is an arbitrary (fixed) sequence over $Q$. Then, the statements $(i)$ and $(ii)$ from Th. 3.1 hold.

4. References


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