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Fixed Points of Some Classes of Nonexpansive Mappings

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Abstract. In this paper we proves the convergence of a convex sequence $x_n = \lambda x_{n-1} + (1-\lambda)f(x_{n-1}), \lambda \in (0,1)$, to a fixed point of the nonexpansive completely continuous operator in the normed f_λ-orbitally complete spaces with λ-uniformly convex sphere. Further we shall prove some fixed point theorems of the star-shaped sets.

1. Introduction

Let X be a normed space. The mapping $f : X \to X$ where is called **nonexpansive** if it satisfies one of the following conditions:

1) $\|f(x) - f(y)\| \leq \|x - y\|$; (L)
2) $\|f(x) - f(y)\| \leq \frac{1}{2}(\|x - f(x)\| + \|y - f(y)\|)$; (K)
3) $\|f(x) - f(y)\| + \|y - f(y)\| \leq \|x - f(y)\|$. (B)

Let X be a vector space, $f : X \to X$ and $x \in X$. Let $\lambda \in (0,1)$ and $O_\lambda(x,f) \subseteq X$ be a set defined by

$$O_\lambda(x,f) = \{g_0(x,f(x)),\ g_1(x,f(x)),\ g_2(x,f(x)),\ldots\},$$

where

$$g_0(x,f(x)) = x,\ g_1(x,f(x)) = \lambda x + (1-\lambda)f(x),$$
$$g_n(x,f(x)) = g(g_{n-1}(x,f(x)), f(g_{n-1}(x,f(x)))).$$

Then $O_\lambda(x,f)$ is called convex orbit or **λ-orbit** of the point x defined by f.

Let (X,d) be a metric linear space, $f : X \to X$ and $\lambda \in (0,1)$. X is f_λ-orbitally complete if each Cauchy's sequence from $O_\lambda(x,f)$ is convergent.

Large number of papers presents fixed point results for nonexpansive mappings (for (L) type see: Browder [1], Karlowitz [4], Göhde [2], Kirk [5],…; for results on star-shaped sets see Reinermann's papers [8], [9]).

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2. The Convergence of the Convex Sequence $x_n = \lambda x_{n-1} + (1-\lambda)f(x_{n-1})$ to the Fixed Point of Nonexpansive (L) Type Mapping

Let $\lambda \in (0,1)$. The normed space X is the space with λ-uniformly convex sphere, if for each $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x, y \in X$ from $\|x-y\| > \varepsilon$ follows:

$$\|\lambda x + (1-\lambda)y\| \le (1-\delta)\max\{\|x\|, \|y\|\}.$$

For $f : E \to E$ we define $J(f,E) = \{x \mid f(x) = x\}$.

Lemma 2.1. *Let $f : E \to E$ be a completely continuous linear operator, E bounded subset of normed space X, and J set of all solutions of the equation $x = f(x)$. Let*

$$R(J(f,E),\alpha) = \{x | x \in E, d(x, J(J(f,E)) \ge \alpha\}.$$

Then for each $x \in R(J(J(f,E),\alpha)$ and each $\alpha > 0$, there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that

$$\|f(x) - x\| > \varepsilon$$

and the $J(f,E)$ is a convex set.

The proof of the above Lemma can be found in [7].

Theorem 2.1. *Let $\lambda \in (0,1)$ and $f : E \to E$ be a completely continuous operator, where E is closed, bounded, and convex subset of the normed vector space X, which has λ-uniformly convex sphere. If X is f_λ-orbitally complete space, and if f satisfies the condition (L), then the sequence $x_n = \lambda x_{n-1} + (1-\lambda)f(x_{n-1})$, $n \in N$ is convergent for arbitrary $x_0 \in E$, and its limit is the solution of the equation $x = f(x)$.*

Proof. From definition of the sequence x_n and condition (L), follows:

$$\begin{aligned} d(x_{n+1}, J(f,E)) &= \inf_{y \in J(f,E)} \|x_{n+1} - y\| = \\ &= \inf_{y \in J(f,E)} \|\lambda x_n + (1-\lambda)f(x_n) - \lambda y - (1-\lambda)y\| \le \\ &\le \inf_{y \in J(f,E)} (\lambda\|x_n - y\| + (1-\lambda)\|x_n - y\|) = \\ &= d(x_n, J(f,E)), \end{aligned}$$

and so the sequence of numbers $d(x_n, J(f,E))$ is non-increasing.

Let $x_1, \ldots, x_k \in R(J(f,E),\alpha)$. Since the space X has λ-uniformly convex sphere, then for any $y \in J(f,E)$, we have:

$$\begin{aligned} \|x_2 - y\| &= \|\lambda(x_1 - y) + (1-\lambda)(f(x_1) - y)\| \le \\ &\le (1-\delta)max\{\|x_1 - y\|, \|f(x_1) - f(y)\|\} \le \\ &\le 2M(1-\delta), \end{aligned}$$

where $M = \sup_{t \in E} \|t\|$.

Similarly, we can prove that

$$\|x_k - y\| \le 2M \cdot (1-\delta)^{k-1}. \tag{2.1}$$

So,

$$d(x_k, J(f,E)) \le 2M \cdot (1-\delta)^{k-1}.$$

From the triangle inequality of and the (L) condition, follows:

$$2\|x_i - y\| \ge \|f(x_i) - f(y)\| + \|y + x_i\| \ge \|f(x_i) - x_i\| \ge \varepsilon$$

for $i = 1, 2, \ldots, k$, and $y \in E$.

From inequality (2.1) follows

$$2M \cdot (1-\delta)^{k-1} \ge \frac{\varepsilon}{2}$$

and so

$$k \le 1 + \frac{\ln 4M - \ln \varepsilon}{-\ln(1-\delta)}.$$

The sequence $\{d(x_n, J(f,E))\}_{n \in N}$ is non-increasing for

$$n > 1 + \frac{\ln(4M) - \ln(\varepsilon)}{-\ln(1-\delta)} \qquad \text{and} \qquad d(x_n, J(f,E)) < \alpha.$$

So

$$\lim_{n\to\infty} d(x_n, J(f,E)) = 0. \tag{2.2}$$

From (2.2) follows that for each $\beta > 0$ there exists $n_0 \in N$ and $y_0 \in J(f,E)$, such that $d(x_n, J(f,E)) < \dfrac{\beta}{2}$ and $d(x_{n_0}, y_0) < \dfrac{\beta}{2}$, for $n_1, n_2 > n_0$. It follows

$$\|x_{n_1} - x_{n_2}\| \le \|x_{n_1} - y_0\| + \|y_0 - x_{n_2}\| \le \frac{\beta}{2} + \frac{\beta}{2} = \beta.$$

So $\{x_n\}_{n\in N}$ is a Cauchy sequence. Since the space X is f_λ-orbitally complete, this sequence is convergent in E. Let $\lim_{n\to\infty} x_n = \xi$. From complete continuity of the operator f and the definition of the sequence x_n, we obtain that $\xi = \lambda\xi + (1-\lambda)f(\xi)$ and $\xi = f(\xi)$. □

Theorem 2.2. *Let $\lambda \in (0,1)$ and $p \in \{2,3,\ldots\}$. Let $f : E \to E$ be a completely continuous operator, where E is closed, bounded, and convex subset of the normed vector space X which has λ-uniformly convex sphere. If X is f_λ^p-orbitally complete space and for every $x, y \in E$:*

$$\|f^p(x) - f^p(y)\| \le \|x - y\|,$$

then the sequence

$$x_n = \lambda x_{n-1} + (1-\lambda) f^p(x_{n-1}), \qquad n \in N \tag{2.3}$$

is convergent for an arbitrary $x_0 \in E$. Its limit is common solution of the equations $x = f^p(x)$ and $x = f(x)$.

Proof. The operator $f^p : E \to E$ is completely continuous and maps the closed, convex and bounded set E into E. The operator $f^p : E \to E$ is nonexpansive. So from Lemma 2.1 and Theorem 2.1, follows that the sequence defined by (2.3) is a Cauchy's sequence in f^p_λ-orbitally complete space and converges to a fixed point of operator f^p. The $J(f, E)$ and $J(f^p, E)$ are convex sets and $J(f, E) \subseteq J^p(f, E)$, which implies that the sequence (2.3) converges to the common solution of the equations $x = f(x)$ and $x = f^p(x)$. □

3. Fixed Points and Star-Shaped Sets

Fixed point result of nonexpansive mapping of type (L), defined on star-shaped subsets of Hilbert's spaces was given in [8].

Let X be a linear space. The $A \subseteq X$, is **star-shaped** if there exists $a \in A$, such that for each point $x \in A$ $\lambda a + (1-\lambda)x \in A$, $\lambda \in (0, 1)$. The point a is called the **star** of the set A. The point $x \in A$, X is is called **extremal point of A** if from $x = \lambda x_1 + (1-\lambda)x_2, x_1, x_2 \in A$ follows that $x_1 = x_2 = x$.

Let X be a vector space, $f : X \to X$ and $x, a \in X$. Let $\lambda \in (0, 1)$ and $O_\lambda(x, f) \subseteq X$ be a set defined by

$$O_\lambda(a, x, f) = \{g_0(a, f(x)),\ g_1(a, f(x)),\ g_2(a, f(x)), l \dots\},$$

where

$$g_0(a, f(x)) = x,\ g_1(x, f(x)) = \lambda a + (1-\lambda)f(x),$$
$$g_n(a, f(x)) = g(g_{n-1}(a, f(x)), f(g_{n-1}(a, f(x)))).$$

Then $O_\lambda(a, x, f)$ is called convex λ, a**-orbit** of the point x defined by f.

Let $f : X \to X$, where (X, d) is metric linear space and $\lambda \in (0, 1)$. X is f^a_λ-orbitally complete if each Cauchy's sequence from $O_\lambda(a, x, f)$ is convergent.

Theorem 3.1. *Let $\lambda \in (0, 1)$ and $f : E \to E$ be completely continuous operator, where E is closed, bounded, and star-shaped subset of the normed space X, which is f^a_λ-orbitally complete. Then a is the star of the set E. If 0 is an extermal point of the set E, and if f satisfies the condition (L) or the condition (B), then operator f has a fixed point which is the limit of sequence*

$$x_n = \lambda a + (1-\lambda)f(x_{n-1}), \tag{3.1}$$

for any $x_0 \in E$.

Proof. The Theorem will be proved only for operator which satisfies the condition (B). The proof of the Theorem is similar operator f satisfies the condition (L).

From condition (B) we obtain

$$\|f(x_{n-1}) - f(x_n)\| + \|x_n - f(x_n)\| \le \|x_{n-1} - f(x_n)\|.$$

From the definition of the sequence x_n, follows:

$$\|(x_n - \lambda a)(1-\lambda)^{-1} - (x_{n+1} - \lambda a)(1-\lambda)^{-1}\| + \\ + \|x_n - (x_{n+1} - \lambda a)(1-\lambda)^{-1}\| \le \|x_{n-1} - (x_{n+1} - \lambda a)(1-\lambda)^{-1}\|,$$

which implies

$$\|x_n - x_{n+1}\| \leq (1-\lambda)\|x_{n-1} - x_n\|.$$

It follows that the sequence defined by (2.3) is a Cauchy's sequence. Let $\lim_{n\to\infty} x_n = \xi$. From (3.1) follows $\xi = \lambda a + (1-\lambda)f(\xi)$, which implies

$$\lambda(\xi - a) + (1-\lambda)(\xi - f(\xi)) = 0. \tag{3.2}$$

From (3.2) follows that

$$\xi = a = f(\xi),$$

because 0 is an extremal point of set E. So the sequence (3.1) tends to the fixed point $x = a = \xi$. □

In [7] was proved that mapping $f : E \to E$, where E is a closed and convex subset of f_λ-orbitally complete space X, which satisfies the condition

$$\|f(x) - f(y)\| \leq q(\|x - f(x)\| + \|y - f(y)\|), \tag{3.3}$$

has a fixed point for $q \in \left[0, \dfrac{1-\lambda}{2-\lambda}\right), \qquad \lambda \in (0,1)$.

Theorem 3.2. *Let $\lambda \in (0,1)$ and E be bounded and closed subset of normed f_λ-orbitally complete space X. If 0 is a star of the set E and $f : E \to E$ is completely continuous operator satisfying the condition*

$$\|f(x) - f(y)\| \leq \frac{1-\lambda}{2-\lambda}(\|x - f(x)\| + \|y - f(y)\|), \lambda \in (0,1), \tag{3.4}$$

then operator f has at least one fixed point.

Proof. From the boundness of the set E follows that there exists the ball $B(0,r)$ of the radius $r > 0$ and center 0, which contains it.

The mappings $q \cdot \dfrac{2-\lambda}{1-\lambda} f$ satisfy the condition (3.3), because from (3.4) follows

$$\begin{aligned}\|q \cdot \frac{2-\lambda}{1-\lambda}f(x) - q \cdot \frac{2-\lambda}{1-\lambda}f(y)\| &\leq q \cdot \frac{2-\lambda}{1-\lambda}\frac{1-\lambda}{2-\lambda}(\|x - f(x)\| + \|y - f(y)\|) \leq \\ &\leq q \cdot (\|x - f(x)\| + \|y - f(y)\|)\end{aligned}$$

for each $q \in \left[0, \dfrac{1-\lambda}{2-\lambda}\right)$. Then there exists a fixed point $z(\lambda, q)$ of the mapping $q \cdot \dfrac{2-\lambda}{1-\lambda} \cdot f$, that is $q \cdot \dfrac{2-\lambda}{1-\lambda} \cdot f(z(\lambda,q)) = z(\lambda,q)$.

Now there is

$$\begin{aligned}\|f(z(\lambda,q)) - z(\lambda,q)\| &= \left\| f(z(\lambda,q)) - q \cdot \frac{2-\lambda}{1-\lambda} \cdot f(z(\lambda,q)) \right\| = \\ &= \left(1 - q \cdot \frac{2-\lambda}{1-\lambda}\right)\|f(z(\lambda,q))\| \leq \left(1 - q \cdot \frac{2-\lambda}{1-\lambda}\right) r.\end{aligned} \tag{3.5}$$

If $q \to \dfrac{1-\lambda}{2-\lambda}$, then $\left(1 - q \cdot \dfrac{2-\lambda}{1-\lambda}\right) \cdot r \to 0$. Hence, for each $\varepsilon > 0$ there exists $z(\lambda, q)$ such that

$$\|f(z(\lambda,q)) - z(\lambda,q)\| < \varepsilon \quad \text{if there is} \quad q > (1 - \frac{\varepsilon}{r}) \cdot \frac{1-\lambda}{2-\lambda}. \tag{3.6}$$

□

Let $\varepsilon \in \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. According to (3.6), for every $\varepsilon > 0$ from

$$q_1 > (1 - \frac{\varepsilon}{r}) \cdot \frac{1-\lambda}{2-\lambda}, q_2 > (1 - \frac{\varepsilon}{2r}) \cdot \frac{1-\lambda}{2-\lambda}, \dots \quad 3.7$$

that there exists a sequence of fixed points $y_1, y_2, \dots$, such that

$$\begin{aligned} \|f(y_1) - y_1\| &< 1 \\ \|f(y_2) - y_2\| &< \frac{1}{2} \\ &\vdots \\ \|f(y_n) - y_n\| &< \frac{1}{n} \\ &\vdots \end{aligned}$$

It follows

$$\|f(y_n) - y_n\| \to 0 \tag{3.8}$$

when $n \to \infty$.

Since the operator f is completely continuous, the sequence $\{f(y_n)\}_{n \in N}$ has at least one convergent subsequence $\{f(y_{n_p})\}_{n_p \in N}$. Let $\lim\limits_{n\to\infty} f(y_{n_p}) = \xi$. According to (3.8), we also have $\lim\limits_{n\to\infty} y_{n_p} = \xi$, and it follows that $\|f(\xi) - \xi\| = 0$. So, $\xi = f(\xi)$.

If the condition (3.4) of Theorem (3.2) is replaced by the condition (K) following from the result of paper [3] and from the condition of Theorem (3.2), it can similarly be shown that, if E is a subset of Banach's space X, the mapping $f : E \to E$ has at least one fixed point.

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