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On Kirk's Fixed Point Main Theorem for Asymptotic Contractions

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Abstract. We prove that main result of asymptotic contractions by Kirk [J. Math. Anal. Appl. **277** (2003), 645–650, Theorem 2.1, p. 647] has been for the first time proved 17 years ago in Tasković [Fundamental elements of the fixed point theory, ZUNS-1986, Theorem 4, p. 170]. But, the author (and next other authors) this historical fact is to neglect and to ignore.

1. Introduction

In recent years a great number of papers have appeared presenting a various generalizations of the well known Banach-Picard contraction principle (via linear and nonlinear conditions). The following result is a statement with nonlinear conditions given in 2003 by W.A. Kirk.

Theorem 1 (Kirk [2])**.** *Let (X, ρ) be a complete metric space, $T : X \to X$ continuous function, and $\{\varphi_n\}_{n\in\mathbb{N}}$ sequence of continous functions such that $\varphi_n : \mathbb{R}^0_+ \to \mathbb{R}^0_+ := [0, +\infty)$ and*

$$\rho[T^n(x), T^n(y)] \leq \varphi_n\big(\rho[x, y]\big) \text{ for all } x, y \in X,$$

and $n \in \mathbb{N}$. Assume also that there exists function $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ such that for any $r > 0$, $\varphi(r) < r$, $\varphi(0) = 0$ and $\varphi_n \to \varphi$ $(n \to \infty)$ uniformly of the range of ρ. If there exists $x \in X$ such that orbit of T at x is bounded, then T has a unique fixed point $\xi \in X$ and all sequences of Picard iterates defined via T converges to ξ.

2. Main results and facts

Let X be a topological space, $T : X \to X$, and let $A : X \times X \to \mathbb{R}^0_+$. In 1986 Tasković [3] investigated the concept of TCS-convergence in a space

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X, i.e., a topological space $X := (X, A)$ satisfies the **condition of TCS-convergence** iff $x \in X$ and if $A(T^n x, T^{n+1} x) \to 0$ $(n \to \infty)$ implies that $\{T^n(x)\}_{n\in\mathbb{N}}$ has a convergent subsequence.

For $x \in X$ the set $\mathcal{O}(x, \infty) := \{x, Tx, T^2x, \dots\}$ is called the **orbit** of x. A function f mapping X into reals is a T**-orbitally lower semicontinuous** at the point p iff for all sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to p$ $(n \to \infty)$ it follows that $f(p) \le \liminf_{n\to\infty} f(x_n)$. A mapping $T : X \to X$ is said to be orbitally continuous if ξ, $x \in X$ are such that ξ is a cluster point of $\mathcal{O}(x, \infty)$, then $T(\xi)$ is a cluster point of $T(\mathcal{O}(x, \infty))$.

The following results, given in the next two theorems are given in 1986 by M. R. Tasković [3] as a natural extension of characterization statements of asymptotically conditions of fixed point theorem given in 1985 by Tasković [4]. These results are according to topological spaces.

Theorem 2 (Tasković [3]). *Let T be a mapping of topological space $X := (X, A)$ into itself, where X satisfies the condition of TCS-convergence. Suppose that there exist a sequence of nonnegative real functions $\{\alpha_n(x, y)\}_{n\in\mathbb{N}}$ such that $\alpha_n(x, y) \to 0$ $(n \to \infty)$ and positive integer $m(x, y)$ such that*

$$\text{(B)} \qquad A\Big(T^n(x), T^n(y)\Big) \le \alpha_n(x, y) \text{ for all } n \ge m(x, y),$$

and for all $x, y \in X$, where $A : X \times X \to \mathbb{R}^0_+$. If $x \mapsto A(x, T(x))$ is a T-orbitally lower semicontinuous function or T is orbitally continuous and $A(a, b) = 0$ implies $a = b$, then T has a unique fixed point $\xi \in X$ and $T^n(x) \to \xi$ $(n \to \infty)$ for each $x \in X$.

Proof. For $y = T(x)$ from (B) we have that $A(T^n x, T^{n+1} x) \le \alpha_n(x, Tx)$ for all $n \ge m(x, y)$, and thus we obtain that $A(T^n x, T^{n+1} x) \to 0$ $(n \to \infty)$. This implies (from TCS-convergence) that the sequence of iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{T^{n(i)}(x)\}_{i\in\mathbb{N}}$ with the limit point $\xi \in X$. Since $x \mapsto A(x, T(x))$ is T-orbitally lower semicontinuous we get

$$A(\xi, T(\xi)) \le \liminf_{i\to\infty} A\Big(T^{n(i)}x, T^{n(i)+1}x\Big) = \liminf_{n\to\infty} A\Big(T^n x, T^{n+1} x\Big) = 0$$

which implies that $A(\xi, T(\xi)) = 0$, i.e. $\xi = T(\xi)$. On the other hand, if T is orbitally continuous the proof of previous fact is trivially. We complete the proof by showing that T can have at most one fixed point. Indeed, if we suppose that $\xi \ne \eta$ were two fixed points, then from (B) we have

$$0 < A(\xi, \eta) = A\Big(T^n(\xi), T^n(\eta)\Big) \le \alpha_n(\xi, \eta) \text{ for } n \ge m(\xi, \eta);$$

taking limits as $n \to \infty$ we obtain a contradiction. The proof is complete. □

Note that, from the preceding proof of Theorem 2, we can give the following local form of this statement.

Theorem 3 (Localization of (B), Tasković [3]). *Let T be a mapping of topological space $X := (X, A)$ into itself, where X satisfies the condition of TCS-convergence. Suppose that there exist a sequence of nonnegative real functions $\{\alpha_n(x,y)\}_{n\in\mathbb{N}}$ such that $\alpha_n(x,Tx) \to 0$ $(n \to \infty)$ and positive integer $m(x)$ such that*

$$A\Big(T^n(x), T^{n+1}(x)\Big) \le \alpha_n(x,Tx) \text{ for all } n \ge m(x),$$

and for every $x \in X$, where $A : X \times X \to \mathbb{R}^0_+$. If $x \mapsto A(x,Tx)$ is a T-orbitally lower semicontinuous function or T is orbitally continuous and $A(a,b) = 0$ implies $a = b$, then T has at least one fixed point in X.

The proof of this statement is an analogous with the preceding proof of Theorem 2. A brief broof of this statement may be found in Tasković [3].

Annotation. The Theorem 1 is a consequence of Theorem 2 (In this sense in next we give the following proof of this essential fact).

Proof. (Application of Theorem 2).

Suppose that all the conditions of Theorem 1 are satisfied. We prove that all conditions of Theorem 2 are satisfied, too. Since $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ is a continuous function such that $\varphi(t) < t$ for every $t > 0$ and $\varphi(0) = 0$, from Wong's lemma ([5], Lemma 4, p. 201) it follows that there exists nondecreasing continuous function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ such that $\varphi(t) < \psi(t) < t$ for every $t > 0$ and $\psi(0) = 0$. Let us define $A : X \times X \to \mathbb{R}^0_+$ by $A(a,b) = \psi(\rho[a,b])$, and define a sequence of functions $\{\alpha_n(a,b)\}_{n\in\mathbb{N}}$ by $\alpha_n(a,b) = \rho[T^n(a), T^n(b)]$ for any $a, b \in X$. Since $\psi(t) < t$ we get that

$$A\Big(T^n(x), T^n(y)\Big) = \psi\Big(\rho[T^n(x), T^n(y)]\Big) < \rho[T^n(x), T^n(y)] = \alpha_n(x,y)$$

this is that the condition (B) is satisfied. Since $\psi(t) = 0$ implies $t = 0$, from $A(a,b) = \psi(\rho[a,b]) = 0$ it follows that $\rho[a,b] = 0$, i.e., $a = b$. From the proof given by W.A. Kirk [2] it follows that $\rho[T^n(x), T^n(y)] \to 0$ $(n \to \infty)$ for all $x, y \in X$. Consequently, $\alpha_n(x,y) \to 0$ $(n \to \infty)$. Since T and ψ are continuous mappings the function $x \mapsto A(x,Tx) := \psi(\rho[x,Tx])$ is a T-orbitally lower semicontinouos. Since X is a complete metric space it satisfies the condition of TCS-convergence. Applying Theorem 2 we obtain that T has a unique fixed point $\xi \in X$ and all sequences of Picard iterates converge to ξ. The proof is complete. □

Further, applying the Theorem 2 we get an asymptotic version of a statement due to Ivanov [1]. This is the following result which is an extension of Kirk's theorem on asymptotic contractions.

Theorem 4. *Let (X, ρ) be a complete metric space, $T : X \to X$ a continuous function, and $\varphi_n : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ for $n \in \mathbb{N}$ a sequence such that for all*

$n \in \mathbb{N}$ *satisfy*

$$(1) \quad \rho[T^n(x), T^n(y)] \leq$$
$$\leq \max\Big\{\varphi_n(\rho[x,y]), \varphi_n(\rho[x,Tx]), \varphi_n(\rho[y,Ty]), \varphi_n(\rho[x,Ty]), \varphi_n(\rho[y,Tx])\Big\}$$

for all $x, y \in X$; and assume also that there exists a function $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ such that for any $t > 0$, $\varphi(t) < t$, $\varphi(0) = 0$ and $\varphi_n \to \varphi$ $(n \to \infty)$ uniformly of the range of ρ. If there exists $x \in X$ such that orbit of T at x is bounded, then T has a unique fixed point $\xi \in X$ and all sequences of Picard iterates defined by T converges to ξ.

Proof. (Application of Theorem 2).

Again, since $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ is a continuous function such that $\varphi(t) < t$ for every $t > 0$ and $\varphi(0) = 0$, from Wong's lemma ([5] Lemma 4, p. 201) it follows that there exists nondecreasing continuous function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ such that $\varphi(t) < \psi(t) < t$ for every $t > 0$ and $\psi(0) = 0$. We define a function $A : X \times X \to \mathbb{R}^0_+$ by

$$A(a,b) := \max\Big\{\psi(\rho[a,b]), \psi(\rho[a,Ta]), \psi(\rho[b,Tb]), \psi(\rho[a,Tb]), \psi(\rho[b,Ta])\Big\}$$

and define a sequence of functions $\{\alpha_n(x,y)\}_{n\in N}$ by

$$\alpha_n(x,y) := \max\Big\{\rho[T^n x, T^n y], \rho[T^n x, T^{n+1} x], \rho[T^n y, T^{n+1} y], \rho[T^n x, T^{n+1} y], \rho[T^n y, T^{n+1} x]\Big\}.$$

It is easy to show that A and $\{\alpha_n(x,y)\}_{n\in\mathbb{N}}$ satisfy all the required hypothesis (similarly as in the proof of Kirk's theorem) in Theorem 2. Applying Theorem 2 we get conclusion of Theorem 4. This completes the proof. $\square$

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