CR-Warped Product Submanifolds of Lorentzian Manifolds*

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Abstract. In this paper, we study warped product CR-submanifolds of a Lorentzian Sasakian manifold. We show that the warped product of the type $M = N_\bot \times_f N_T$ in a Lorentzian Sasakian manifold is simply CR-product and obtain a characterization of CR-warped product submanifolds.

1. Introduction

Warped product manifolds were introduced by Bishop and O’Neill in [3] to construct new examples of negatively curved manifolds. These manifolds are obtained by warping the product metric of a product manifold onto the fibers and thus provide a natural generalization to the product manifolds. Let $(N_1, g_1)$ and $(N_2, g_2)$ be semi-Riemannian manifolds of dimensions $m$ and $n$, respectively and $f$, a positive differentiable function on $N_1$. Then the warped product [3] of $(N_1, g_1)$ and $(N_2, g_2)$ with warping function $f$ is defined to be the product manifold $M = N_1 \times N_2$ with metric tensor $g = g_1 + f^2 g_2$. The warped product manifold $(N_1 \times N_2, g)$ is denoted by $N_1 \times_f N_2$. If $U$ is tangent to $M = N_1 \times_f N_2$ at $(p, q)$ then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2,$$

where $\pi_1$ and $\pi_2$ are the canonical projections of $M$ onto $N_1$ and $N_2$, respectively. The function $f$ is called the warping function of the warped product manifold. In particular, if the warping function is constant, then the warped product manifold $M$ is said to be trivial. Let $X$ be vector field on $N_1$ and $Z$ be vector field on $N_2$, then from Lemma 7.3 of [3], we have

$$\nabla_X Z = \nabla_Z X = \left(\frac{Xf}{f}\right) Z,$$

2000 Mathematics Subject Classification. Primary: 53C25, 53C42, 53C50.

Key words and phrases. Warped product, CR-submanifold, contact CR-warped product, Lorentzian Sasakian manifold.

*This work is supported by the research grant RG117/10AFR (University of Malaya, Kuala Lumpur, Malaysia).
where $\nabla$ is the Levi-Civita connection on $M$. Let $M = N_1 \times f N_2$ be a warped product manifold, this means that $N_1$ is totally geodesic and $N_2$ is totally umbilical submanifold of $M$, respectively.

The notion of CR-submanifolds of Kaehler manifolds was introduced by A. Bejancu [2] as a generalization of totally real and holomorphic submanifolds of a Kaehler manifold. Later, the concept of CR-submanifold has been also considered in various manifolds. In [6] and [1], as analogous of submanifolds of Lorentzian paracontact and Lorentzian manifolds, respectively. Furthermore H. Gill and K.K. Dube have recently introduced generalized CR-submanifolds of a trans Lorentzian Sasakian manifold [7].

Recently, B.Y. Chen has introduced the notion of CR-warped product in Kaehler manifolds and showed that there exist no proper warped product CR-submanifolds in the form $M = N_{\perp} \times f N_T$ in a Kaehler manifold. He considered only the warped product of the type $M = N_T \times f N_{\perp}$ and called it a CR-warped product submanifold [4, 5]. Later on, Hasegawa and Mihai proved that warped product CR-submanifolds $N_{\perp} \times f N_T$ in Sasakian manifolds are trivial i.e. simply contact CR-product submanifolds, where $N_T$ and $N_{\perp}$ are $\phi$–invariant and anti-invariant submanifolds of a Sasakian manifold respectively [8].

In this paper, we study warped product CR-submanifolds of a Lorentzian Sasakian manifold. We, show that the warped product in the form $M = N_{\perp} \times f N_T$ does not exist except for the trivial case, where $N_T$ and $N_{\perp}$ are invariant and anti-invariant submanifolds of a Lorentzian Sasakian manifold $\bar{M}$, respectively. Also, we obtain a characterization result of the warped product CR-submanifold of the type $M = N_T \times f N_{\perp}$.

2. Preliminaries

A $(2m+1)$–dimensional manifold $\bar{M}$ is said to be a Lorentzian almost contact manifold with an almost contact structure and compatible Lorentzian metric, $(\bar{M}, \phi, \xi, \eta, g)$, that is, $\phi$ is a $(1, 1)$ tensor field, $\xi$ is a structure vector field, $\eta$ is a $1$–form and $g$ is Lorentzian metric on $\bar{M}$, satisfying [1]:

\begin{align}
\phi^2 &= -X + \eta(X)\xi, \eta(\xi) = 1, \phi \xi = 0, \eta \circ \phi = 0, \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \eta(X) = -g(X, \xi)
\end{align}

for all $X, Y \in T\bar{M}$. It is Lorentzian Sasakian if

\begin{align}
(\bar{\nabla}_X \phi)Y &= -g(X, Y)\xi - \eta(Y)X, \\
\bar{\nabla}_X \xi &= -\phi X,
\end{align}

for any vector fields $X, Y$ on $\bar{M}$, where $\bar{\nabla}$ denotes the Levi-Civita connection with respect to $g$.

Let $M$ be a $n$–dimensional submanifold of a Lorentzian almost contact manifold $\bar{M}$ with Lorentzian almost contact structure $(\phi, \xi, \eta, g)$. Let the
induced connection on $M$ be denoted by $\nabla$. Then the Gauss and Weingarten Formulae are respectively given by

$$\nabla_X Y = \nabla_X Y + h(X, Y)$$

(2.4)

$$\nabla_X N = -A_N X + \nabla^\perp_X N,$$

(2.5)

for any $X, Y \in TM$ and $N \in T^\perp M$, where $TM$ is the Lie algebra of vector fields in $M$ and $T^\perp M$ is the set of all vector fields normal to $M$. $\nabla^\perp$ is the connection in the normal bundle, $h$ the second fundamental form and $A_N$ is the Weingarten endomorphism associated with $N$. It is easy to see that

$$g(A_N X, Y) = g(h(X, Y), N).$$

(2.6)

For any $X \in TM$, we write

$$\phi X = PX + FX,$$

(2.7)

where $PX$ is the tangential component and $FX$ is the normal component of $\phi X$. Similarly for $N \in T^\perp M$, we write

$$\phi N = tN + fN,$$

(2.8)

where $tN$ is the tangential component and $fN$ is the normal component of $\phi N$.

The covariant derivatives of the tensor fields $\phi$, $P$ and $F$ are defined as

$$(\nabla_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y, \forall X, Y \in T\bar{M}$$

(2.9)

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y, \forall X, Y \in TM$$

(2.10)

$$(\nabla_X F)Y = \nabla^\perp_X FY - F\nabla_X Y, \forall X, Y \in TM.$$ 

(2.11)

Moreover, for a Lorentzian Sasakian manifold we have

$$(\nabla_X P)Y = A_{FY} X + th(X, Y) - g(X, Y)\xi - \eta(Y)X,$$

(2.12)

$$(\nabla_X F)Y = fh(X, Y) - h(X, PY).$$

(2.13)

A submanifold $M$ of a Lorentzian almost contact manifold, $(\bar{M}^{2m+1}, \phi, \eta, \xi, g)$ is called \textit{CR-submanifold} if it admits an invariant distribution $\mathcal{D}$ whose orthogonal complementary distribution $\mathcal{D}^\perp$ is anti-invariant i.e., $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \langle \xi \rangle$ with $\phi(\mathcal{D}_x) \subseteq \mathcal{D}_x$ and $\phi(\mathcal{D}^\perp_x) \subseteq T^\perp_x M$, for every $x \in M$.

Note that $\xi$ is a timelike vector field and all vector field in $\mathcal{D} \oplus \mathcal{D}^\perp$ are space like. Denoting orthogonal complementary subbundle to $\phi\mathcal{D}^\perp$ in $T^\perp M$ by $\mu$, then we have

$$T^\perp M = \phi\mathcal{D}^\perp \oplus \mu.$$ 

Invariant and anti-invariant submanifolds are the special cases of CR-submanifolds. A submanifold $M$ called an \textit{invariant} submanifold if $\mathcal{D}^\perp = \{0\}$ and $M$ is said to be an \textit{anti-invariant} submanifold if $\mathcal{D} = \{0\}$. A CR-submanifold is \textit{proper} if neither $\mathcal{D} = \{0\}$ nor $\mathcal{D}^\perp = \{0\}$.

In the following section we shall investigate the warped products of the type $M = N_T \times f N_\perp$ and $M = N_\perp \times f N_T$, where $N_T$ and $N_\perp$ are invariant and anti-invariant submanifolds of a Lorentzian Sasakian manifold $\bar{M}$. A
warped product CR-submanifold is simply *CR-product* with the integrable distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ if the warping function $f$ is constant.

### 3. Warped Product CR-submanifolds

Throughout the section structure vector field $\xi$ is either tangent to the invariant submanifold $N_T$ or tangent to the anti-invariant submanifold $N_\perp$. There are two types of warped product CR-submanifolds of a Lorentzian Sasakian manifold $\bar{M}$, namely $N_\perp \times fN_T$ and $N_T \times fN_\perp$. In the following theorem we deal the warped product CR-submanifold of the type $N_\perp \times fN_T$.

**Theorem 3.1.** Let $M = N_\perp \times fN_T$ be a warped product CR-submanifold of a Lorentzian Sasakian manifold $\bar{M}$, where $N_T$ and $N_\perp$ are invariant and anti-invariant submanifolds of $\bar{M}$, respectively. Then $M$ is CR-product.

**Proof.** For any $X \in TN_T$ and $Z \in TN_\perp$, by (1.1) we deduced that
\[ \nabla_X Z = \nabla_Z X = (Z \ln f)X. \]

There are two cases arise:

1. When $\xi \in TN_T$, then $\bar{\nabla}_Z \xi = -\phi Z$, i.e., $h(Z, \xi) = -\phi Z$ and $\nabla_Z \xi = 0$. On using (3.1) we get
   \[ (Z \ln f)\xi = 0, \ \forall \ Z \in TN_\perp. \]
2. When $\xi \in TN_\perp$, then for any $X \in TN_T$ we have $\bar{\nabla}_X \xi = -\phi X = -PX$. This means that $h(X, \xi) = 0$ and $\nabla_X \xi = -\phi X$. Using (3.1) we get
   \[ (\xi \ln f)X = -\phi X, \ \forall \ X \in TN_T. \]

Taking product in (3.3) with $X \in TN_T$ thus, we obtain
\[ (\xi \ln f)\|X\|^2 = 0, \ \forall \ X \in TN_T. \]

Now for any $X \in TN_T$ and $Z \in TN_\perp$, we have
\[ g(h(X, \phi X), \phi Z) = g(\bar{\nabla}_X \phi X, \phi Z) = g(\phi \bar{\nabla}_X X + (\bar{\nabla}_X \phi) X, \phi Z). \]

Then from (2.2), (2.3) and the fact that $\xi \in TN_\perp$, we obtain
\[ g(h(X, \phi X), \phi Z) = g(\bar{\nabla}_X X, Z) = -g(\bar{\nabla}_X Z, X). \]

Thus by (2.4) and (3.1), we get
\[ g(h(X, \phi X), \phi Z) = -(Z \ln f)\|X\|^2. \]

Interchanging $X$ by $\phi X$ in (3.5) and using the fact that $\xi$ is tangent to $N_\perp$, we get
\[ g(h(X, \phi X), \phi Z) = (Z \ln f)\|X\|^2. \]

Thus (3.5) and (3.6) imply
\[ (Z \ln f)\|X\|^2 = 0, \ \forall Z \in TN_\perp \ \& \ X \in TN_T. \]
Thus, from (3.2), (3.4) and (3.7) we conclude that \( f \) is constant i.e., \( M \) is CR-product. This completes the proof. \( \square \)

Now, the other case i.e., \( N_T \times fN_{\perp} \) with \( \xi \) tangential to \( N_T \) is dealt with the following. To prove the main theorem first we obtain some useful formulae for later use.

**Lemma 3.1.** Let \( M = N_T \times fN_{\perp} \) be a warped product CR-submanifold of a Lorentzian Sasakian manifold \( \bar{M} \), such that \( \xi \) is tangent to \( N_T \), where \( N_T \) and \( N_{\perp} \) are invariant and anti-invariant submanifolds of \( \bar{M} \), respectively. Then

1. \( \xi \ln f = 0 \),
2. \( g(h(X, Y), FZ) = 0 \),
3. \( g(h(X, Z), FW) = g(h(X, W), FZ) \),
4. \( g(h(\phi X, Z), FW) = (X \ln f)g(Z, W) = g(h(\phi X, W), FZ) \)

for any \( X, Y \in TN_T \) and \( Z, W \in TN_{\perp} \).

**Proof.** The first part is obtained from (1.1), (2.3) and (2.4). Now for any \( X \in TN_T \) and \( Z \in TN_{\perp} \), we have

\[ \nabla_X Z = \nabla_Z X = (X \ln f)Z. \]

On the other hand for any \( X, Y \in TN_T \) and \( Z \in TN_{\perp} \), by formula (2.4) we have

\[ g(h(X, Y), \phi Z) = g(\bar{\nabla}_X Y, \phi Z). \]

On using (2.3) and (2.9), we get

\[ g(h(X, Y), \phi Z) = -g(\bar{\nabla}_X \phi Y, Z) = g(\phi Y, \bar{\nabla}_X Z) = g(\phi Y, \nabla_X Z). \]

Taking account of the formula (3.8), the above equation yields

\[ g(h(X, Y), \phi Z) = (X \ln f)g(\phi Y, Z) = 0. \]

That proves \( g(h(X, Y), FZ) = 0 \). For (iii), for any \( X \in TN_T \) and \( Z, W \in TN_{\perp} \) we have

\[ g(h(X, Z), \phi W) = g(\bar{\nabla}_X Z, \phi W) \]
\[ = -g(\bar{\nabla}_X \phi Z, W) \]
\[ = g(A_{\phi Z} X, W) \]
\[ = g(h(X, W), \phi Z), \]

or equivalently, \( g(h(X, Z), FW) = g(h(X, W), FZ) \). This proves (iii). Now, for any \( X \in TN_T \) and \( Z, W \in TN_{\perp} \) and using (2.2), (2.3), (2.4), (2.9) and the fact that \( \xi \) is tangent to \( N_T \), formula (3.8) gives

\[ g(\nabla_X Z, W) = g(\nabla_Z X, W) = g(\bar{\nabla}_Z X, W) \]
\[ = g(\phi \bar{\nabla}_Z X, \phi W) - \eta(\nabla_Z X)\eta(W). \]
That is
\[(X \ln f)g(Z, W) = g(\nabla Z \phi X, \phi W) - g((\nabla Z \phi)X, \phi W)\]
\[= g(\nabla Z \phi X + h(Z, \phi X), \phi W).\]

The above equation becomes
\[(X \ln f)g(Z, W) = g(h(Z, \phi X), \phi W) + (\phi X \ln f)g(Z, FW)\]
\[= g(h(Z, \phi X), \phi W).\]

This means that \((X \ln f)g(Z, W) = g(h(Z, \phi X), FW)\). This proves the first equality of (iv). For the second equality, by Gauss formula we may write
\[g(h(\phi X, Z), \phi W) = g(\nabla_{\phi X} Z, \phi W)\]
\[= -g(\phi \nabla_{\phi X} Z, W)\]
\[= g((\nabla_{\phi X} \phi)Z, W) - g(\nabla_{\phi X} \phi Z, W)\]
\[= g(A_{\phi X} X, W)\]
\[= g(h(\phi X, W), \phi Z),\]
i.e., \(g(h(\phi X, Z), FW) = g(h(\phi X, W), FZ)\). This proves the lemma completely. \(\square\)

**Theorem 3.2.** Let \(M\) be a proper CR-submanifold of a Lorentzian Sasakian manifold \(\bar{M}\) with integrable distribution \(\mathcal{D}^\perp\). Then \(M\) is locally a CR-warped product if and only if

\[(3.9) \quad A_{\phi Z} X = -(\phi X \mu)Z\]

for each \(X \in \mathcal{D} \oplus \langle \xi \rangle, \ Z \in \mathcal{D}^\perp\) and \(\mu\), a \(C^\infty\)-function on \(M\) such that \(V \mu = 0\), for each \(W \in \mathcal{D}^\perp\).

**Proof.** If \(M\) is CR-warped product submanifold \(N_T \times f N_\perp\), then on applying Lemma 3.1, we obtain (3.9). In this case \(\mu = \ln f\).

Conversely, suppose \(M\) is a proper CR-submanifold of a Lorentzian Sasakian manifold \(\bar{M}\) satisfying (3.9), then for any \(X, Y \in \mathcal{D} \oplus \langle \xi \rangle\)
\[g(h(X, Y), \phi Z) = g(A_{\phi Z} X, Y) = g(-(\phi X \mu)Z, Y) = 0\]
\[\Rightarrow g(\nabla_X \phi Y, Z) = 0,\]
which implies
\[g(\nabla_X Y, Z) = 0.\]

This means \(\mathcal{D} \oplus \langle \xi \rangle\) is integrable and its leaves are totally geodesic in \(M\). So far as anti-invariant distribution \(\mathcal{D}^\perp\) is concerned, it is involutive on \(M\)
Moreover, for any \( X \in D \oplus \langle \xi \rangle \) and \( Z, W \in D^\perp \), we have
\[
g(\nabla_Z W, X) = g(\bar{\nabla}_Z W, X)  
= g(\phi \bar{\nabla}_Z W, \phi X) - \eta(\bar{\nabla}_Z W) \eta(X)  
= g(\bar{\nabla}_Z \phi W, \phi X) - g((\bar{\nabla}_Z \phi) W, \phi X)  
= -g(A_{\phi W} Z, \phi X) - g((\bar{\nabla}_Z \phi) W, \phi X).  
\]
The second term in the right hand side of the above equation vanishes in view (2.3) and the fact that \( \xi \) tangential to \( N_T \) and the first term will be
\[
- g(A_{\phi W} Z, \phi X) = - g(h(Z, \phi X), \phi W) = - g(A_{\phi W} \phi X, Z).  
\]
Making use of (2.1), (3.9) and Lemma 3.1 (i), the above equation takes the form
\[
(3.10) \quad g(\nabla_Z W, X) = - g(A_{\phi W} Z, \phi X) = X \mu \ g(Z, W).  
\]
Now, by Gauss formula
\[
g(h'(Z, W), X) = g(\nabla_Z W, X)  
\]
where \( h' \) denotes the second fundamental form of the immersion of \( N_\perp \) into \( M \). On using (3.10), the last equation gives
\[
g(h'(Z, W), X) = X \mu \ g(Z, W).  
\]
The above relation shows that the leaves of \( D^\perp \) are totally umbilical in \( M \). Moreover, the fact that \( V \mu = 0 \), for each \( V \in D^\perp \), implies that the mean curvature vector on \( N_\perp \) is parallel along \( N_\perp \) i.e., each leaf of \( D^\perp \) is an extrinsic sphere in \( M \). Hence by virtue of a result in [9] we obtain that \( M \) is locally a CR-warped product submanifold \( N_T \times \mu N_\perp \) of \( \bar{M} \). This proves the theorem completely. \( \square \)

**Acknowledgement.** The authors are thankful to the anonymous referees and Professor Viqar Azam Khan (AMU, Aligarh) for their valuable suggestion and comments.

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