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Fixed Points of Occasionally Weakly Compatible Maps Satisfying General Contractive Conditions of Integral Type

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Abstract. In this paper, two common fixed point theorems for four occasionally weakly compatible maps satisfying a contractive condition of integral type are obtained. Our results improve some results especially Theorem 2.1 of [3] and Theorem 1 of [1].

1. Introduction and Preliminaries

In 1982 Sessa [6] generalized the concept of commuting maps by calling self-maps f and g of a metric space $(\mathcal{X}, d)$ a weakly commuting pair if and only if for all $x \in \mathcal{X}$:

$$d(fgx, gfx) \leq d(gx, fx).$$

In 1986 Jungck [4] made a generalization of concept of weakly commutativity called compatibility. f and g are compatible if:

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

Further, the same author [5] defined the concept of weak compatibility which generalized the notion of compatibility. f and g above are said to be weakly compatible if they commute at their coincidence points.

Recently in 2008, Al-Thagafi with Shahzad [2] gave a proper generalization of weakly compatible maps by introducing the concept of occasionally weakly compatible maps (shortly (owc)). Two self-maps f and g of a set $\mathcal{X}$ are owc if and only if there is a point $t \in \mathcal{X}$ which is a coincidence point of f and g at which f and g commute.

Before giving our main results, recall that a symmetric on a set $\mathcal{X}$ is a function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ satisfying the following conditions:

(1) $d(x, y) = 0$, if and only if $x = y$ for $x, y \in \mathcal{X}$,
(2) $d(x, y) = d(y, x)$, for all $x, y \in \mathcal{X}$.

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2. Main Results

Now we give our main results. We begin by citing and proving our first theorem.

Theorem 2.1. *Let d be a symmetric for $\mathcal{X}$. Let h, k, f and g be self-maps of $\mathcal{X}$ such that for all x, y in $\mathcal{X}$, there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(0) = 0$, $\psi(t) < t$ for $t > 0$, and*

$$\int_0^{d(fx,gy)} \varphi(t)\,\mathrm{d}\,t \leq \psi\left(\int_0^{M(x,y)} \varphi(t)\,\mathrm{d}\,t\right), \tag{1}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable map which is summable nonegative such that $\int_0^{\varepsilon} \varphi(t)\,\mathrm{d}\,t > 0$ for each $\varepsilon > 0$,

$$M(x,y) = \max\left\{d(hx,ky), d(fx,hx), d(gy,ky)\frac{1}{2}\Big(d(fx,ky) + d(gy,hx)\Big)\right\}.$$

If the pairs $\{f,h\}$ and $\{g,k\}$ are owc, then h, k, f and g have a unique common fixed point in $\mathcal{X}$.

Proof. By hypothesis, there are two points u and v in $\mathcal{X}$ such that $fu = hu$ and $fhu = hfu$, $gv = kv$ and $gkv = kgv$.

We claim that $fu = gv$. If not, from (1):

$$\int_0^{d(fu,gv)} \varphi(t)\,\mathrm{d}\,t \leq \psi\left(\int_0^{M(u,v)} \varphi(t)\,\mathrm{d}\,t\right) =$$

$$= \psi\left(\int_0^{\max\left\{d(hu,kv),d(fu,hu),d(gv,kv),\frac{1}{2}\left(d(fu,kv)+d(gv,hu)\right)\right\}}\right) =$$

$$= \psi\left(\int_0^{d(fu,gv)} \varphi(t)\,\mathrm{d}\,t\right) < \int_0^{d(fu,gv)} \varphi(t)\,\mathrm{d}\,t,$$

a contradiction. Therefore $fu = hu = gv = kv$.

Suppose that $f^2u \neq fu$, then inequality (1) gives:

$$\int_0^{d(f^2u,fu)} \varphi(t)\,\mathrm{d}\,t = \int_0^{d(ffu,gv)} \varphi(t)\,\mathrm{d}\,t \leq$$

$$\leq \psi\left(\int_0^{M(fu,v)} \varphi(t)\,\mathrm{d}\,t\right) =$$

$$= \psi\left(\int_0^{\max\left\{d(hfu,kv),d(f^2u,hfu),d(gv,kv),\frac{1}{2}\left(d(f^2u,kv)+d(gv,hfu)\right)\right\}} \varphi(t)\,\mathrm{d}\,t\right) =$$

$$= \psi\left(\int_0^{d(f^2u,fu)} \varphi(t)\,\mathrm{d}\,t\right) < \int_0^{d(f^2u,fu)} \varphi(t)\,\mathrm{d}\,t,$$

which is a contradiction. Hence, $f^2u = fu = hfu$.

Similarly, $g^2v = gv = kgv$, and $fu = hu = gv = kv$ is a common fixed point of f, h, g and k.

The uniqueness of the common fixed point follows easily from condition (1). ☐

If we let $f = g$ and $k = h$ in Theorem 2.1, we get the following corollary.

Corollary 2.1. *Let d be a symmetric for $\mathcal{X}$ and let f and h be self-maps of $\mathcal{X}$ such that for all x, y in $\mathcal{X}$, there is a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(0) = 0$, $\psi(t) < t$ for $t > 0$, and*

$$\int_0^{d(fx,fy)} \varphi(t)\,\mathrm{d}\,t \leq$$
$$\leq \psi\left(\int_0^{\max\left\{d(hx,hy),d(fx,hx),d(fy,hy),\frac{1}{2}\left(d(fx,hy)+d(fy,hx)\right)\right\}} \varphi(t)\,\mathrm{d}\,t\right),$$

where φ is as in Theorem 2.1. If f and h are owc, then f and h have a unique common fixed point in $\mathcal{X}$.

Now, if we put $k = h$ in Theorem 2.1, we get the following result.

Corollary 2.2. *Let d be a symmetric for $\mathcal{X}$. Let h, f and g be three self-maps of $\mathcal{X}$ such that for all x, y in $\mathcal{X}$, there is a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(0) = 0$, $\psi(t) < t$ for $t > 0$, and*

$$\int_0^{d(fx,gy)} \varphi(t)\,\mathrm{d}\,t \leq$$
$$\leq \psi\left(\int_0^{\max\left\{d(hx,hy),d(fx,hx),d(gy,hy),\frac{1}{2}\left(d(fx,hy)+d(gy,hx)\right)\right\}} \varphi(t)\,\mathrm{d}\,t\right),$$

where φ is as in Theorem 2.1. If f and h, as well as g and h, are owc, then f, g and h have a unique common fixed point in $\mathcal{X}$.

Theorem 2.2. *Let d be a symmetric for the set $\mathcal{X}$ and let h, k, f and g be self-maps of $\mathcal{X}$ such that*

$$\begin{aligned} &\int_0^{d(hx,ky)} \varphi(t)\,\mathrm{d}\,t \leq \\ &\leq \psi\left(\int_0^{\max\left\{d(fx,gy),d(fx,ky),d(ky,gy)\right\}} \varphi(t)\,\mathrm{d}\,t\right), \end{aligned} \tag{2}$$

for all x, y in $\mathcal{X}$, where φ and ψ are as in Theorem 2.1. If the pairs $\{h, f\}$ and $\{k, g\}$ are owc, then h, k, f, and g have a unique common fixed point in $\mathcal{X}$.

Proof. **Existence.** Since the pairs $\{h,f\}$ and $\{k,g\}$ are owc, then there are two elements u,v in $\mathcal{X}$ such that $hu = fu$ and $hfu = fhu$, $kv = gv$ and $kgv = gkv$. First, we prove that $hu = kv$. Suppose not, then, by using inequality (2) we get:

$$\int_0^{d(hu,kv)} \varphi(t)\,\mathrm{d}\,t \leq$$
$$\leq \psi\left(\int_0^{\max\left\{d(fu,gv),d(fu,kv),d(kv,gv),\right\}} \varphi(t)\,\mathrm{d}\,t\right) =$$
$$= \psi\left(\int_0^{d(hu,kv)} \varphi(t)\,\mathrm{d}\,t\right) < \int_0^{d(hu,kv)} \varphi(t)\,\mathrm{d}\,t,$$

which is a contradiction. Therefore $fu = hu = kv = gv$

Now, we claim that $hhu = fhu = hu$. If not, then the use of condition (2) gives:

$$\int_0^{d(h^2u,hu)} \varphi(t)\,\mathrm{d}\,t = \int_0^{d(hhu,kv)} \varphi(t)\,\mathrm{d}\,t \leq$$
$$\leq \psi\left(\int_0^{\max\left\{d(fhu,gv),d(fhu,kv),d(kv,gv),\right\}} \varphi(t)\,\mathrm{d}\,t\right) =$$
$$= \psi\left(\int_0^{d(h^2u,hu)} \varphi(t)\,\mathrm{d}\,t\right) < \int_0^{d(h^2u,hu)} \varphi(t)\,\mathrm{d}\,t,$$

a contradiction. Hence $h^2u = fhu = hu$.

Similarly, $k^2v = gkv = kv$, and $hu = kv = z$ is a common fixed point of both h, k, f and g.

Uniqueness. Suppose that there are two distinct points z, z' in $\mathcal{X}$, then by (2) we have the contradiction:

$$\int_0^{d(z,z')} \varphi(t)\,\mathrm{d}\,t = \int_0^{d(hz,kz')} \varphi(t)\,\mathrm{d}\,t \leq$$
$$\leq \psi\left(\int_0^{\max\left\{d(fz,gz'),d(fz,kz'),d(kz',gz'),\right\}} \varphi(t)\,\mathrm{d}\,t\right) =$$
$$= \psi\left(\int_0^{d(z,z')} \varphi(t)\,\mathrm{d}\,t\right) < \int_0^{d(z,z')} \varphi(t)\,\mathrm{d}\,t.$$

The proof is completed. □

Corollary 2.3. *Let d be a symmetric for $\mathcal{X}$ and let h, k be two owc self-maps of $\mathcal{X}$ such that:*

$$\int_0^{d(hx,hy)} \varphi(t)\,\mathrm{d}\,t \leq$$
$$\leq \psi\left(\int_0^{\max\left\{d(kx,ky),d(kx,hy),d(hy,ky),\right\}} \varphi(t)\,\mathrm{d}\,t\right)$$

for all x, y in $\mathcal{X}$, where φ and ψ are as in Theorem 2.1, then h and k have a unique common fixed point in $\mathcal{X}$.

Corollary 2.4. *Let d be a symmetric for $\mathcal{X}$ and let h, k and f be self-maps of $\mathcal{X}$ satisfying the following inequality:*

$$\int_0^{d(hx,ky)} \varphi(t)\,\mathrm{d}\,t \leq$$
$$\leq \psi\left(\int_0^{\max\left\{d(fx,fy),d(fx,ky),d(ky,fy),\right\}} \varphi(t)\,\mathrm{d}\,t\right)$$

for all x, y in $\mathcal{X}$, where φ and ψ are as in Theorem 2.1. If pairs $\{h, f\}$ and $\{k, f\}$ are owc, then h, k and f have a unique common fixed point in $\mathcal{X}$.

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