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On a Result of T. Suzuki for Generalized Distance and Fixed Points

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Abstract. We prove that a main result of T. Suzuki [J. Math. Anal. Appl. **253** (2001), 440–458, Theorem 1, p. 451] has been for the first time proved 20 years ago by Tasković [Proc. Amer. Math. Soc. **94** (1985), 427–432, Theorem 2, p. 430], and second time proved by Tasković [Math. Japonica, **35** (1990), 645–666, Theorem 2, p. 654] as a very special case of the so-called Localization Monotone Principle.

1. Introduction

In recent years a great number of papers have presented generalizations of the well-known Banach-Picard contraction principle.

Recently, Suzuki have proved the following statement (see [2, Theorem 1, p. 451]).

Theorem 1 (Suzuki [2]). *Let X be a complete metric space and let T be a mapping on X. Suppose that there exist a τ-distance p on X and $r \in [0,1)$ such that $p(Tx, T^2x) \leq rp(x, Tx)$ for every $x \in X$. Assume that either of the following holds:*

(a) *T is a continuous mapping;*
(b) *if $\{x_n\}_{n\in\mathbb{N}}$ and $\{T(x_n)\}_{n\in\mathbb{N}}$ converge to y, then $Ty = y$;*
(c) *if $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_{n\to\infty} p(x_n, Tx_n) = 0$, and $\lim_{n\to\infty} p(x_n, y) = 0$, then $Ty = y$;*

then there exists $x_0 \in X$ such that $Tx_0 = x_0$. Moreover, if $Tz = z$, then $p(z, z) = 0$.

In connection with this statement, by the proof of Suzuki [2], we first notice that (b) implies (c), and that (b) implies (a). Clearly (a) implies (b). Thus, de facto, Theorem 1, is essentially only with the condition (c).

In this paper we discuss some properties of τ-distance. Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers. Let (X, d) be a metric

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space. Then, by Suzuki [2], a function $p : X \times X \to \mathbb{R}^0_+ := [0, +\infty)$ is called a τ-*distance* on X if there exists a function η from $X \times \mathbb{R}^0_+$ into $\mathbb{R}^0_+$ and the following are satisfied:

(τ1) $p(x, z) \le p(x, y) + p(y, z)$ for all $x, y, z \in X$;

(τ2) $\eta(x, 0) = 0$ and $\eta(x, t) \ge t$ for all $x \in X$ and $t \in \mathbb{R}^0_+$, and η is concave and continuous in its second variable;

(τ3) $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \sup\{\eta(z_n, p(z_n, z_m)) : m \ge n\} = 0$ imply $p(w, x) \le \liminf_{n\to\infty} p(w, x_n)$ for every $w \in X$;

(τ4) $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m \ge n\} = 0$ and $\lim_{n\to\infty} \eta(x_n, t_n) = 0$ imply $\lim_{n\to\infty} \eta(y_n, t_n) = 0$;

(τ5) $\lim_{n\to\infty} \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_{n\to\infty} \eta(z_n, p(z_n, y_n)) = 0$ imply that we have $\lim_{n\to\infty} d(x_n, y_n) = 0$.

The metric d is a τ-distance on X. Many useful examples are stated in more other papers. It is very meaningful that one τ-distance generates other τ-distances. We notice that Suzuki [2] proved, using the notion of τ-distance, several fixed point theorems.

2. Further facts

Let X be a topological space and $T : X \to X$ a mapping. For $x \in X$, $O(x) := \{x, Tx, T^2x, \ldots\}$ is called the *orbit* of x. A function g mapping X into the reals is T-*orbitally lower semicontinuous* at p if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $O(x)$ and $x_n \to p$ $(n \to \infty)$ implies that $g(p) \le \liminf_{n\to\infty} g(x_n)$. A mapping $T : X \to X$ is said to be *orbitally continuous* if $\xi, x \in X$ are such that ξ is a cluster point of $O(x)$, then $T\xi$ is a cluster point of $T(O(x))$.

In connection with the preceding facts we notice that in 1985 and 1990 Tasković proved so-called Localization Monotone Principle (of fixed point) on arbitrary topological spaces.

In this sense, let X be a topological space, $T : X \to X$, and let $B : X \to \mathbb{R}^0_+$ be a given functional. In 1985 (and later in 1990) we introduced the concept of LTCS-convergence in a space X, i.e., a topological space $X := (X, B)$ satisfies the **condition of LTCS-convergence** iff $x \in X$ and $B(T^n(x)) \to 0$ $(n \to \infty)$ implies that $\{T^n(x)\}_{n\in\mathbb{N}}$ has a convergent subsequence in X. In 1985 and 1990 we have the following result.

Theorem 2 (Localization Monotone Principle, Tasković [3]). *Let T be a mapping of a topological space $X := (X, B)$ into itself, where X satisfies the condition of LTCS-convergence. Suppose that there exists a mapping $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying*

$$(\mathrm{I}\varphi) \qquad \Big(\forall t \in \mathbb{R}_+ := (0, +\infty)\Big)\Big(\varphi(t) < t \ \textit{ and } \ \limsup_{z\to t+0} \varphi(z) < t\Big)$$

such that

$$(\mathrm{L}) \qquad B(T(x)) \le \varphi(B(x)) \quad \textit{for every} \quad x \in X,$$

where $B : X \to \mathbb{R}^0_+$ is a T-orbitally lower semicontinuous function and $B(x) = 0$ implies $T(x) = x$. Then T has at least one fixed point in X.

In the following stroke, we considered and proved that Theorem 1 is a very special case of the Localization Monotone Principle of Fixed Point.

Proof of Theorem 1. (*Application of Theorem 2*). Suppose that all the conditions of Theorem 1 are satisfied. We prove that all conditions of Theorem 2 are satisfied, too. Let $B(x) = \max\{p(x, Tx), p(Tx, x)\}$ and $\varphi(t) = rt$ for $r \in [0, 1)$. It is easy to see that B and φ, by (L) and (Iφ), satisfy all the required hypothesis in Theorem 2. Since X is a complete metric space it satisfies the condition of LTCS-convergence. In this sense further we prove that $B(x) = 0$ implies $T(x) = x$. If $B(x) = 0$, then $p(x, Tx) = p(Tx, x) = 0$. Also, by the properties of τ-distance, $p(x, x) \leq p(x, Tx) + p(Tx, x) = 0$, i.e., $p(x, x) = 0$. Thus, $B(x) = 0 = \max\{p(x, x), p(x, Tx)\}$, i.e., $T(x) = x$, because general holds that $p(z, x) = 0$ and $p(z, y) = 0$ implies that $x = y$.

On the other hand, $B(x)$ is a T-orbitally lower semicontinuous function. Indeed, fix $u \in X$ and put $u_n = T^n(u)$ for all $n \in \mathbb{N}$. Then, by Suzuki [2], since X is complete, $\{u_n\}_{n\in\mathbb{N}}$ converges to some point $x_0 \in X$ and $\lim_{n\to\infty} \sup\{p(u_n, u_m) : m > n\} = 0$. From ($\tau$3) we have that $\lim_{n\to\infty} p(u_n, Tu_n) = \lim_{n\to\infty} p(u_n, x_0) = 0$. Hence $Tx_0 = x_0$, which means that

$$B(x_0) = \max\Big\{p(Tx_0, x_0), p(x_0, Tx_0)\Big\} =$$

$$= p(x_0, x_0) \leq \liminf_{n\to\infty} \max\Big\{p(u_n, Tu_n), p(Tu_n, u_n)\Big\},$$

i.e., B is a T-orbitally lower semicontinuous function.

Thus, applying Theorem 2 we obtain that T has at least one fixed point $x_0 \in X$. The fact $p(x_0, x_0) = 0$ is a consequence of the preceding facts for τ-distance. The proof is complete. □

An illustration elementary of Theorem 2 is an extension local form of Banach's contraction principle on topological spaces which give in the following form.

Corollary 1 (Tasković [3]). *Let T be a mapping of a topological space X into itself, where X satisfies the condition of LTCS-convergence. Suppose that there exists an $\alpha \in [0, 1)$ such that*

$$\text{(A)} \qquad A\Big(T(x), T^2(x)\Big) \leq \alpha A(x, T(x)) \quad \textit{for every } x \in X,$$

where $A : X \times X \to \mathbb{R}^0_+$, $x \mapsto A(x, T(x))$ is T-orbitally lower semicontinuous and $A(a, b) = 0$ implies $a = b$. Then T has at least one fixed point in X.

The proof of this statement based on Localization Monotone Principle (of fixed point) may be found in [3]. Also, the proof of this statement we can give and elementary without of Theorem 2. Also, Theorem 1 is a special case and of Corollary 1.

An illustration of Theorem 2 is so-called Monotone Principle of Fixed Point by Tasković [3] and [4]. It is the following statement.

We notice that in 1985 and 1990 Tasković proved so-called Monotone Principle (of fixed point) on arbitrary topological spaces.

In this sense, let X be a topological space, $T : X \to X$, and let $A : X \times X \to \mathbb{R}^0_+$. In 1985 we introduced the concept of TCS-convergence in a space X, i.e., a topological space $X := (X, A)$ satisfies the condition of *TCS-convergence* iff $x \in X$ and if $A(T^n x, T^{n+1} x) \to 0$ $(n \to \infty)$ implies that $\{T^n x\}_{n\in\mathbb{N}}$ has a convergent subsequence. In 1985 and 1990 we have the following result.

Corollary 2 (Monotone Principle, Tasković [4]). *Let T be a mapping of a topological space $X := (X, A)$ into itself, where X satisfies the condition of TCS-convergence. Suppose that there exists a mapping $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying*

$$(\mathrm{I}\varphi) \qquad \Big(\forall t \in \mathbb{R}_+\Big)\Big(\varphi(t) < t \;\; and \;\; \limsup_{z\to t+0} \varphi(z) < t\Big)$$

such that

$$(\mathrm{MP}) \qquad A(Tx, Ty) \le \varphi(A(x, y)) \quad for\ all \quad x, y \in X,$$

where $A : X \times X \to \mathbb{R}^0_+$, $x \mapsto A(x, Tx)$ is T-orbitally lower semicontinuous and $A(a, b) = 0$ implies $a = b$. Then T has a unique fixed point $\xi \in X$ and $T^n(z) \to \xi$ $(n \to \infty)$ for each $z \in X$.

Proof. Let x be an arbitrary point in X. Then, for $y = T(x)$, from (MP), we have that $A(Tx, T^2x) \le \varphi(A(x, Tx))$. Hence, for $B(x) = A(x, T(x))$, and since X satisfies the condition of LTCS-convergence (in the context of the space $X := (X, B)$), applying Theorem 2 we obtain that T has at least one fixed point in X. Uniqueness follows immediately from the condition (MP). The proof is complete. □

In further, an illustration of Theorem 2 is a local form of well-known statement in 1978 by Dugundji-Granas.

Corollary 3 (Corollary 5 of [3]). *Let (X, ρ) be a complete metric space and $T : X \to X$ an arbitrary mapping. Suppose for all $x \in X$ that T satisfies*

$$\rho[T(x), T^2(x)] \le \rho[x, T(x)] - \theta(x, T(x)),$$

where θ is compactly positive on X, i.e., $\inf\{\theta(x, y) : \alpha \le \rho[x, y] \le \beta\} > 0$ for each finite closed interval $[\alpha, \beta] \subset \mathbb{R}_+$. If $x \mapsto \rho[x, T(x)]$ is T-orbitally lower semicontinuous, then T has at least one fixed point in X.

A brief write first proof of this statement may be found in Tasković [3] via Localization Monotone Principle of Fixed Point. See also Tasković [4].

Corollary 4 (Suzuki [2, p. 455]). *Let X be a complete metric space and let p be a τ-distance on X. Let T be a mapping from X into itself and let f*

be a function from X *into* $\mathbb{R} \cup \{+\infty\}$ *which is proper lower semicontinuous and bounded from below. Assume*

$$\text{(S)} \qquad p(x, Tx) \leq f(x) - f(Tx) \ \text{ for all } \ x \in X,$$

then there exists $x_0 \in X$ *such that* $T(x_0) = x_0$ *and that is* $p(x_0, x_0) = 0$.

Proof. (*Application of Theorem 2*). Letting $B(x) = f(x)$ and $\varphi(t) = t - p(x, Tx)$ for $t \geq p(x, Tx)$ and $\varphi(t) = 0$ for $0 \leq t < p(x, Tx)$ in (L) gives

$$f(Tx) \leq f(x) - p(x, Tx), \quad \text{for} \quad x \in X,$$

i.e., (S). Since X is a complete metric space and from (S) for the iterates sequence $\{x_n\}_{n\in\mathbb{N}}$ via T we obtain

$$\sum_{n=0}^{\infty} p(x_n, x_{n+1}) \leq f(x_0),$$

i.e., X satisfies the condition of LTCS-convergence. Hence, it follows from Theorem 2 that T has at least one fixed point. The fact $p(x_0, x_0) = 0$ it follows as in the proof of Theorem 1. The proof is complete. □

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