On a result of Jachymski, Matkowski, and Świątkowski

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Abstract. We prove on a new manner, via Monotone Principle (of fixed point), a result of Jachymski, Matkowski, and Świątkowski [Journal of Applied Analysis, 1 (1995), 125–134, Theorem 1, p. 130].

1. Introduction

In recent years a great number of papers have presented generalizations of the well-known Banach-Picard contraction principle.

Recently, Jachymski-Matkowski-Świątkowski have proved the following statement (see [1, Theorem 1, p. 130]).

Theorem 1 (Jachymski-Matkowski-Świątkowski [1]). Let $(X,d)$ be a Hausdorff semimetric and $d$-Cauchy complete space such that the diameters of open balls with radius $r$ are equibounded. Let $T$ be a selfmap on $X$ such that

\[(Jm) \quad d(Tx,Ty) \leq \psi\left(d(x,y)\right) \quad \text{for all} \quad x,y \in X\]

where $\psi : \mathbb{R}^+_0 \to \mathbb{R}^+_0 := [0, +\infty)$ is a nondereasing function such that satisfies $\lim_{n \to \infty} \psi^n(t) = 0$ for every $t > 0$. Then $T$ has a unique fixed point $\xi \in X$ and $T^n(z) \to \xi$ ($n \to \infty$) for every $z \in X$.

We notice that this result is an essential extension of a former result in 1975 of Matkowski [2].

In connection with the preceding, a topological space $(X, \tau)$ is semimetrizable iff there is a distance function $d$ such that for any $A \subset X$ is $\overline{A} = \{x \in X : d(x, A) = 0\}$. In this case $d$ is said to be a semimetric.

Further, a symmetric or semimetric space $(X,d)$ is $d$-Cauchy complete if every $d$-Cauchy sequence is $\tau$-convergent. Otherwise, a sequence $\{x_n\}_{n \in \mathbb{N}}$ is $d$-Cauchy if given $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq k$.

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Also, a **distance function** for a set $X$ is a function $d$ from $X \times X$ into $\mathbb{R}_+^0$ such that $d(x, y) = 0$ if and only if $x = y$, and $d(x, y) = d(y, x)$ for all $x, y \in X$. A distance function is also called a **symmetric**.

2. **Further facts**

Let $X$ be a topological space and $T : X \to X$ a mapping. For $x \in X$, $O(x) := \{x, Tx, T^2x, \ldots\}$ is called the **orbit** of $x$. A function $g$ mapping $X$ into the reals is $T$-**orbitally lower semicontinuous** at $p$ if $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in $O(x)$ and $x_n \to p$ ($n \to \infty$) implies that $g(p) \leq \liminf_{n \to \infty} g(x_n)$. A mapping $T : X \to X$ is said to be **orbitally continuous** if $x, y \in X$ are such that $x$ is a cluster point of $O(x)$, then $T\xi$ is a cluster point of $T(O(x))$.

In connection with the preceding facts we notice that in 1985 and 1990 Tasković proved so-called Monotone Principle (of fixed point) on arbitrary topological spaces.

In this sense, let $X$ be a topological space, $T : X \to X$, and let $A : X \times X \to \mathbb{R}_+^0$. In 1985 we introduced the concept of TCS-convergence in a space $X$, i.e., a topological space $X$ satisfies the condition of **TCS-convergence** iff $x \in X$ and if $A(T^n x, T^{n+1} x) \to 0$ ($n \to \infty$) implies that $\{T^n x\}_{n \in \mathbb{N}}$ has a convergent subsequence. In 1985 and 1990 we have the following result.

**Theorem 2** (Monotone Principle, Tasković [4]). Let $T$ be a mapping of a topological space $X$ into itself, where $X$ satisfies the condition of TCS-convergence. Suppose that there exists a mapping $\varphi : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ satisfying

1. **(I\varphi)** \hspace{1cm} $(\forall t \in \mathbb{R}_+ := (0, +\infty)) \left( \varphi(t) < t \text{ and } \limsup_{z \to t+0} \varphi(z) < t \right)$

such that

2. **(MP)** \hspace{1cm} $A(Tx, Ty) \leq \varphi(A(x, y))$ for all $x, y \in X,$

where $A : X \times X \to \mathbb{R}_+^0$, $x \mapsto A(x, Tx)$ is $T$-orbitally lower semicontinuous or $T$ is orbitally continuous, and $A(a, b) = 0$ implies $a = b$. Then $T$ has a unique fixed point $\xi \in X$ and $T^n(z) \to \xi$ ($n \to \infty$) for each $z \in X$.

**Proof of Theorem 1.** Let $A(x, y) = d(x, y)$, and $\varphi(t) = \psi(t)$. It is easy to see that $A$ and $\varphi$, by (MP) and (I\varphi), satisfy all the required hypotheses in Theorem 2.

Since $d$-Cauchy completeness implies TCS-convergence it follows, from Theorem 2, that $T$ has a unique fixed point $\xi \in X$ and that $\{T^n(z)\}_{n \in \mathbb{N}}$ converges to $\xi$ for each $z \in X$. The proof is complete.

An illustration elementary of Theorem 2 is an extension direct of Banach’s contraction principle on topological spaces which we give in the following form.

**Proposition 1** (Corollary 2 of [4]). Let $T$ be a mapping of a topological space $X$ into itself, where $X$ satisfies the condition of TCS-convergence. Suppose
that there exists an $\alpha \in [0, 1)$ such that
\[ A(Tx, Ty) \leq \alpha A(x, y) \quad \text{for all } x, y \in X, \]
where $A : X \times X \rightarrow \mathbb{R}^+_0$, $x \mapsto A(x, Tx)$ is $T$-orbitally lower semicontinuous or $T$ is orbitally continuous, and $A(a, b) = 0$ implies $a = b$. Then $T$ has a unique fixed point $\xi \in X$ and $T^n(z) \rightarrow \xi$ ($n \rightarrow \infty$) for each $z \in X$.

The proof of this statement based on Monotone Principle (of fixed point) may be found in [4]. Also, the proof of this statement we can give and elementary without of Theorem 2 in usual manner.

REFERENCES


