Mathematica Moravica
Vol. **16-2** (2012), 59–61

On the Location of Zeros of Some Polynomials

Dragomir M. Simeunović

Abstract. In this paper we determine the regions in the complex plane containing zeros of some polynomials.

In this paper we consider the polynomial

$$P(z) = z^n + a_p z^{n-p} + a_{p+1} z^{n-p-1} + \cdots + a_n,\ a_p \neq 0,\ p < n. \tag{1}$$

The location of zeros of the polynomials in the complex plane, depending on its coefficients was studied by many authors. Here we cite a result obtained by P. Montel [1] and a result by H. Guggenheimer [2] which are, respectively, as follows:
(R_1): All the zeros of the polynomial (1) are in the region

$$|z| < 2 \max |a_k|^{\frac{1}{k}},\ p \leq k \leq n. \tag{2}$$

(R_2): All the zeros of the polynomial (1) are in the region

$$|z| < r, \tag{3}$$

where $r > 1$ is the root of the equation

$$r^p - r^{p-1} - |a_q| = 0 \tag{4}$$

and where

$$|a_q| = \max |a_k|,\ p \leq k \leq n. \tag{5}$$

In this paper we prove the following theorem.

Theorem 1. *Let c_k, $p \leq k \leq n$ be the positive parameters, where*

$$A_c = \max \left(\frac{|a_k|}{c_k} \right),\ p \leq k \leq n, \tag{6}$$

and

$$M_c = \max (c_k)^{\frac{1}{k}},\ p \leq k \leq n. \tag{7}$$

Then all zeros of the polynomial (1) *are in the region*

$$|z| < r_c M_c, \tag{8}$$

2010 *Mathematics Subject Classification.* Primary: 12D10.
Key words and phrases. Zeros of the polynomials, region of zeros.

where $r_c > 1$ *is the root of the equation*

(9) $$r^p - r^{p-1} - A_c = 0.$$

Proof of Theorem 1. From (7) we have

$$c_k^{\frac{1}{k}} \le M_c$$

that is

(10) $$c_k \le M_c^{\,k},\ p \le k \le n.$$

From (1), for $|z| > M_c$, having (6) and (10) in mind, we obtain

$$\begin{aligned}
|P(z)| &= |z|^n\Big(1 - \Big(\frac{|a_p|}{|z|^p} + \frac{|a_{p+1}|}{|z|^{p+1}} + \cdots + \frac{|a_n|}{|z|^n}\Big)\Big)\\
&= |z|^n\Big(1 - \Big(\frac{|a_p|}{c_p}\cdot\frac{c_p}{|z|^p} + \frac{|a_{p+1}|}{c_{p+1}}\cdot\frac{c_{p+1}}{|z|^{p+1}} + \cdots + \frac{|a_n|}{c_n}\cdot\frac{c_n}{|z|^n}\Big)\Big)\\
&\ge |z|^n\Big(1 - A_c\Big(\frac{c_p}{|z|^p} + \frac{c_{p+1}}{|z|^{p+1}} + \cdots + \frac{c_n}{|z|^n}\Big)\Big)\\
&> |z|^n\Big(1 - A_c\Big(\frac{M_c^{\,p}}{|z|^p} + \frac{M_c^{p+1}}{|z|^{p+1}} + \cdots + \frac{M_c^{\,n}}{|z|^n} + \cdots\Big)\Big)\\
&= |z|^n\Big(1 - \frac{A_c M_c^{\,p}}{|z|^p}\Big(1 + \frac{M_c}{|z|} + \Big(\frac{M_c}{|z|}\Big)^2 + \cdots + \Big(\frac{M_c}{|z|}\Big)^{n-p} + \cdots\Big)\Big)\\
&= |z|^n\Big(1 - \frac{A_c M_c^{\,p}}{|z|^p - |z|^{p-1}M_c}\Big),
\end{aligned}$$

that is

(11) $$\big|P(z)\big| > |z|^n\Big(1 - \frac{A_c M_c^{\,p}}{|z|^p - |z|^{p-1}M_c}\Big).$$

For

(12) $$|z| \ge r_c M_c$$

from (11) we have $\big|P(z)\big| > 0$, that is $\big|P(z)\big| \ne 0$.

This means that all zeros of the polynomial (1) are in the region (8). □

Taking different positive values for parameters c_k, we obtain several particular results from Theorem 1.

For

(13) $$c_k = \frac{|a_k|}{2^{p-1}},\ p \le k \le n,$$

the following result from Theorem 1 is obtained:
(R_3): All zeros of the polynomial (1) are in the region

(14) $$|z| < 2\max\Big(\frac{|a_k|}{2^{p-1}}\Big)^{\frac{1}{k}},\ p \le k \le n.$$

Proof of (R_3). Having (13) in mind, from (6) and (7) we obtain

$$A_c = 2^{p-1} \tag{15}$$

and

$$M_c = \max\left(\frac{|a_k|}{2^{p-1}}\right)^{\frac{1}{k}},\ p \le k \le n. \tag{16}$$

In this case the equation (9) reduces to equation

$$r^p - r^{p-1} - 2^{p-1} = 0, \tag{17}$$

whose positive root is

$$r_s = 2 \tag{18}$$

and region (8) reduces to region (14).

The region (14) for $p \ge 2$ is smaller than the region (2). For $p = 1$ the region (14) reduces to the region (2). □

We demonstrate the other case by giving an example.

Example 1. *The zeros of the polynomial*

$$P(z) = z^5 - 8z^2 + 11z + 20, \tag{19}$$

where $p = 3$*, according to result* (R_2) *are in the region*

$$|z| < 3.1, \tag{20}$$

where $r > 1$ *is the root of the equation*

$$r^3 - r^2 - 20 = 0, \quad (3 < r < 3.1), \tag{21}$$

and from (14) follows that all zeros of the polynomial (19) are in the region

$$|z| < 2.76. \tag{22}$$

References

[1] P. Montel, *Sur quelques limites pour les modules des zéros des polynomes*, Comment. Math. Helv., Vol. **7** (1934-35), 178-200.

[2] H. Guggenheimer, *On a note by Q.G. Mohammad*, Amer. Math. Monthly, **71** (1964), 54-55.

[3] M. Marden, *Geometry of Polynomials*, Amer. Math. Soc. Providence, RI. 2005.

Dragomir M. Simeunović
Mike Alasa 8
11000 Belgrade
Serbia