Mathematica Moravica
Vol. **16-2** (2012), 69–77

Some Fixed Point Theorems for Certain Contractive Mappings on Metric and Generalized Metric Spaces

Amit Singh*, M.S. Khan and Brian Fisher

Abstract. In the present paper we obtain sufficient conditions for the existence of a unique fixed point of Reich and Rhoades type contractive conditions on generalized, complete, metric spaces dependent on another function. Our results generalize and extend some well-known previous results.

1. Introduction and preliminaries

The fixed point theorem most frequently cited in the literature is the Banach contraction mapping principle (see [4] or [6]), which asserts that if (X,d) is a complete metric space and $S : X \to X$ is a contractive mapping, i.e., there exists $k \in [0,1)$ such that for all $x, y \in X$,

$$d(Sx,Sy) \leq kd(x,y). \tag{1}$$

Then S has a unique fixed point.

The above contractive definition implies that S is uniformly continuous. It is natural to ask if there is a contractive definition which does not force S to be continuous. To answer the above question, in 1968 Kannan [5] established a fixed point theorem for mappings satisfying the inequality:

$$d(Sx,Sy) \leq \lambda\big[d(x,Sx) + d(y,Sy)\big] \tag{2}$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{2})$.

Kannan's result [5] was followed by a spate of papers containing a variety of contractive definitions in metric spaces. Rhoades [10] in 1977 considered 250 types of contractive definitions and analyzed the relationship between them.

In 2000 Branciari [2] introduced a class of generalized metric spaces by replacing the triangular inequality by similar ones which involve four or more

2010 *Mathematics Subject Classification.* Primary: 54H25; Secondary: 47H10.
Key words and phrases. Fixed point, contractive mapping, sequently convergent, subsequently convergent.

*Corresponding author.

points instead of three and improved the Banach contraction mapping principle. Recently, Azam and Arshad [1] in 2008 extended Kannan's theorem for this kind of generalized metric space. In the present paper, we first of all extend Kannan's theorem [5] and then extend the theorem due to Azam and Arshad [1] and [8] for these new classes of functions.

The following definitions will be frequently used in the sequel.

Definition 1.1. [8] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be *sequentially convergent* if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ also is convergent. T is said *subsequentially convergent* if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ has a convergent subsequence.

Definition 1.2. [2] Let X be a nonempty set. Suppose that the mapping $d : X \to X$ satisfies:

(i) $d(x, y) \geq 0$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, w)+d(w, z)+d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x, y\}$ (the rectangular property).

Then d is said to be a *generalized metric* and (X, d) is said to be a *generalized metric space.*

Definition 1.3. Let $\{x_n\}$ be a sequence in X and let x be a point in X.

(i) If for every $\epsilon > 0$ there is an $n_0 \in N$ such that $d(x_n, x) < \epsilon$ for all $n > n_0$ then $\{x_n\}$ is said to be *convergent,* $\{x_n\}$ *converges* to x and x is the *limit* of $\{x_n\}$. We denote this by $\lim_n x_n = x$ or by $x_n \to x$ as $n \to \infty$.
(ii) If for every $\epsilon > 0$ there is an $n_0 \in N$ such that $d(x_n, x_m) < \epsilon$ for all $m, n > n_0$, then $\{x_n\}$ is said to be a *Cauchy sequence* in X.
(iii) If every Cauchy sequence is convergent in X, then X is said to be a *complete generalized metric space.*

Remark 1.1. [2]

(i) $d(a_n, y) \to d(a, y)$ and $d(x, a_n) \to d(x, a)$ whenever $\{a_n\}$ is a sequence in X and $\{a_n\}$ converges to $a \in X$.
(ii) X becomes a Hausdorff topological space with neighbourhood basis given by

$$B = \{B(x, r) : x \in X, r \in (0, \infty)\},$$

where

$$b(x, r) = \{y \in X : d(x, y) < r\}.$$

2. Fixed Point Theorems on Metric Spaces

Theorem 2.1. *Let (X, d) be a complete metric space and let $T, S : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially* *convergent and satisfies the inequality*

$$d(TSx, TSy) \leq ad(Tx, TSx) + bd(Ty, TSy) + cd(Tx, Ty) \tag{3}$$

for all $x, y \in X$ and $a, b, c \geq 0$ with $a + b + c < 1$, then S has a unique fixed point. Also, if T is sequentially convergent, then for every $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to x_0.

Proof. Let x_0 be an arbitrary point in X. We define the iterative sequence $\{x_n\}$ by $x_{n+1} = Sx_n = S^{n+1}x_0$ for $n = 0, 1, 2, \ldots$. Using inequality (3), we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TSx_{n-1}, TSx_n) \\ &\leq ad(Tx_{n-1}, TSx_{n-1}) + bd(Tx_n, TSx_n) + cd(Tx_{n-1}, Tx_n) \\ &\leq ad(Tx_{n-1}, Tx_n) + bd(Tx_n, Tx_{n+1}) + cd(Tx_{n-1}, Tx_n), \end{aligned}$$

which implies that

$$(1 - b)d(Tx_n, Tx_{n+1}) \leq (a + c)d(Tx_{n-1}, Tx_n).$$

Putting $h = \frac{a+c}{1-b}$, it follows that

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq hd(Tx_{n-1}, Tx_n) \leq h^2 d(Tx_{n-2}, Tx_{n-1}) \\ &\leq \ldots \leq h^n d(Tx_0, Tx_1). \end{aligned} \tag{4}$$

Hence, for every $m, n \in N$ with $m > n$, we have

$$\begin{aligned} d(Tx_m, Tx_n) &\leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + \ldots + d(Tx_{n+1}, Tx_n) \\ &\leq (h^{m-1} + h^{m-2} + \ldots + h^n)d(Tx_0, Tx_1) \\ &\leq \frac{h^n}{1-h} d(Tx_0, Tx_1). \end{aligned} \tag{5}$$

Letting $m, n \to \infty$ in (5), we see that $\{Tx_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists a point $v \in X$ such that

$$\lim_{n\to\infty} Tx_n = v. \tag{6}$$

Since T is subsequentially convergent, $\{x_n\}$ has a convergent subsequence $\{x_{n(k)}\}_{k=1}^{\infty}$ a point $u \in X$ such that $\lim_{k\to\infty} x_{n(k)} = u$.

Since T is continuous and $\lim_{k\to\infty} x_{n(k)} = u$ it follows that $\lim_{k\to\infty} Tx_{n(k)} = Tu$. By (6), we conclude that $Tu = v$.

We also have

$$\begin{aligned} d(TSu, Tu) &\leq d(TSu, TS^{n(k)}x_0) + d(TS^{n(k)}x_0, TS^{n(k)+1}x_0) + d(TS^{n(k)+1}x_0, Tu) \\ &\leq ad(Tu, TSu) + bd(TS^{n(k)-1}x_0, TS^{n(k)}x_0) + cd(Tu, TS^{n(k)-1}x_0) \\ &\qquad + h^{n(k)}d(Tx_0, TSx_0) + d(Tx_{n(k)+1}, Tu). \end{aligned}$$

Therefore,

$$(1-a)d(TSu,Tu) \leq bd(TS^{n(k)-1}x_0, TS^{n(k)}x_0) + cd(Tu, TS^{n(k)-1}x_0) + h^{n(k)}d(Tx_0, TSx_0) + d(Tx_{n(k)+1}, Tu)$$

and so

$$d(TSu,Tu) \leq \frac{b}{1-a}d(Tx_{n(k)-1}, Tx_{n(k)}) + \frac{c}{1-a}d(Tu, Tx_{n(k)-1}) + \frac{h^{n(k)}}{1-a}d(Tx_0, Tx_1) + \frac{1}{1-a}d(Tx_{n(k)+1}, Tu) \to 0 \quad \text{as} \quad k \to \infty.$$

Hence $d(TSu, Tu) = 0$, which implies that $TSu = Tu$. Since T is one to one, we have $Su = u$ and so S has a fixed point u.

To prove the uniqueness of u, let v be a second fixed point of S. Then from injectivity of T, we get $Su = Sv$, proving the uniqueness of the fixed point.

Finally, suppose that T is sequentially convergent. Then replacing $(n(k))$ by n, we conclude that $\lim_{n\to\infty} S^n x_0 = u$. This shows that $\{S^n x_0\}$ converges to the fixed point of S. □

Corollary 2.1. *Let (X,d) be a complete metric space and let $T, S : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $\lambda \in [0, \frac{1}{2})$ and*

$$d(TSx, TSy) \leq \lambda\big[d(Tx, TSx) + d(Ty, TSy)\big] \tag{7}$$

for all $x, y \in X$, then S has a unique fixed point. Further, if T is sequentially convergent, then for every $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Remark 2.1. By taking $Tx \equiv x$ in Theorem 2.1, we can conclude the Reich's theorem [9].

Remark 2.2. By taking $Tx \equiv x$ in Corollary 2.1, we can conclude the Kannan's theorem [5].

The following example shows that Theorem 2.1 and Corollary 2.1 are indeed proper extensions of Kannan's theorem.

Example 2.1. [8] Let $X = \{0\} \cup \{4^{-1}, 5^{-1}, 6^{-1}, \ldots\}$ endowed with the Euclidean metric. Define $S : X \to X$ by $S(0) = 0$ and $S(n^{-1}) = (n+1)^{-1}$ for all $n \geq 4$. Obviously the condition (2) is not true for every $\lambda > 0$ and so we cannot use the Kannan's theorem [5]. By defining $T : X \to X$ by

$T(0) = 0$ and $S(n^{-1}) = n^{-n}$ for all $n \geq 4$ we have, for $m, n \in N$ $(m > n)$,

$$\begin{aligned}
\left|TS(m^{-1}), TS(n^{-1})\right| &= (n+1)^{-n-1} - (m+1)^{-m-1} \\
&< (n+1)^{-n-1} \leq 3{-}1[n^{-n} - (n+1)^{-n-1}] \\
&\leq 3^{-1}[n^{-n} - (n+1)^{-n-1} + m^{-m} - (m+1)^{-m-1}] \\
&= 3^{-1}\left[\left|T(n^{-1}) - TSn^{-1}\right| + \left|T(m^{-1}) - TSm^{-1}\right|\right]. \qquad (8)
\end{aligned}$$

The inequality (8) shows that (7) is true for $\lambda = 3^{-1}$. Therefore by Corollary 2.1, S has a unique fixed point.

Similarly, we can prove the following theorem.

Theorem 2.2. *Let (X, d) be a complete metric space and let $T, S : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent and satisfies the inequality*

$$d(TSx, TSy) \leq ad(Tx, TSy) + bd(Ty, TSx) + cd(Tx, Ty) \qquad (9)$$

for all $x, y \in X$, $a, b, c \geq 0$ with $a+b+c < 1$, then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

3. Fixed Point Theorems on Generalized Metric Spaces

Theorem 3.1. *Let (X, d) be a complete generalized metric space and let $T, S : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent and satisfies the inequality*

$$d(TSx, TSy) \leq ad(Tx, TSx) + bd(Ty, TSy) + cd(Tx, Ty) \qquad (10)$$

for all $x, y \in X$, $a, b, c \geq 0$ with $a+b+c < 1$, then S has a unique fixed point. Also if T is sequentially convergent, then for every $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Proof. Let x_0 be any arbitrary point in X and put $x_1 = Tx_0$. If $x_0 = Tx_0$, this means that x_0 is a fixed point of T and there is nothing to prove.

Assume that $x_1 \neq x_0$ and put $x_2 = Tx_1$. Proceeding in this way, we can define the iterative sequence of points in X as follows:

$$x_{n+1} = Sx_n = S^{n+1}x_0, x_n \neq x_{n+1} \; n = 0, 1, 2, \ldots.$$

Using inequality(10), we have

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) &= d(TSx_{n-1}, TSx_n) \\
&\leq ad(Tx_{n-1}, TSx_{n-1}) + bd(Tx_n, TSx_n) + cd(Tx_{n-1}, Tx_n) \\
&\leq ad(Tx_{n-1}, Tx_n) + bd(Tx_n, Tx_{n+1}) + cd(Tx_{n-1}, Tx_n),
\end{aligned}$$

implying that

$$(1-b)d(Tx_n, Tx_{n+1}) \leq (a+c)d(Tx_{n-1}, Tx_n),$$

and so

$$d(Tx_n, Tx_{n+1}) \le hd(Tx_{n-1}, Tx_n),$$

where

$$h = \frac{a+c}{1-b} < 1.$$

We can also suppose that x_0 is not a periodic point. In fact if $x_n = x_0$, then

$$\begin{aligned} d(Tx_0, Tx_1) &= d(Tx_0, TSx_0) = d(Tx_n, TSx_n) = d(TS^n x_0, TS^{n+1} x_0) \\ &\le hd(TS^{n-1}x_0, TS^n x_0) \le h^2 d(TS^{n-2}x_0, TS^{n-1}x_0) \\ &\le \ldots \le h^n d(Tx_0, TSx_0). \end{aligned}$$

Since $h < 1$, it follows that x_0 is a fixed point of S. Thus in the sequel of the proof, we can suppose that $S^n x_0 \neq x_0$ for $n = 1, 2, \ldots$.

Now inequality (10) implies that

$$\begin{aligned} d(Tx_n, Tx_{n+m}) &= d(TS^n x_0, TS^{n+m} x_0) \\ &\le ad(TS^{n-1}x_0, TS^n x_0) + bd(TS^{n+m-1}x_0, TS^{n+m}x_0) \\ &\qquad + cd(TS^{n-1}x_0, TS^{n+m-1}x_0) \\ &\le ad(TS^{n-1}x_0, TS^n x_0) + bd(TS^{n+m-1}x_0, TS^{n+m}x_0) \\ &\qquad + c\big[d(TS^{n-1}x_0, TS^n x_0) + d(TS^n x_0, TS^{n+m}x_0) \\ &\qquad + d(TS^{n+m}x_0, TS^{n+m-1}x_0)\big]. \end{aligned}$$

Hence

$$\begin{aligned} (1-c)d(Tx_n, Tx_{n+m}) &\le (a+c)d(TS^{n-1}x_0, TS^n x_0) \\ &\qquad + (b+c)d(TS^{n+m-1}x_0, TS^{n+m}x_0) \end{aligned}$$

and so

$$\begin{aligned} d(Tx_n, Tx_{n+m}) &\le hd(TS^{n-1}x_0, TS^n x_0) \\ &\qquad + \frac{b+c}{1-c} d(TS^{n+m-1}x_0, TS^{n+m}x_0) \\ &\le h^n d(Tx_0, Tx_1) + \frac{b+c}{1-c} h^{n+m-1} d(Tx_0, Tx_1). \end{aligned}$$

Therefore, $d(Tx_n, Tx_{n+m}) \to 0$ as $n \to \infty$. This implies that $\{Tx_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $u \in X$ such that $\lim_{n\to\infty} Tx_n = u$.

By the rectangular property, we have

$$\begin{aligned} d(TSu, Tu) &\le d(TSu, TS^n x_0) + d(TS^n x_0, TS^{n+1}x_0) + d(TS^{n+1}x_0, Tu) \\ &\le ad(Tu, TSu) + bd(TS^{n-1}x_0, TS^n x_0) + cd(Tu, TS^{n-1}x_0) \\ &\qquad + h^n d(Tx_0, TSx_0) + d(Tx_{n+1}, Tu). \end{aligned}$$

Therefore,

$$
\begin{aligned}
(1-a)d(TSu,Tu) \le\ & bd(TS^{n-1}x_0, TS^n x_0) + cd(Tu, TS^{n-1}x_0) \\
& + h^n d(Tx_0, TSx_0) + d(Tx_{n+1}, Tu)
\end{aligned}
$$

and so

$$
\begin{aligned}
d(TSu,Tu) &\le \frac{b}{1-a} d(TS^{n-1}x_0, TS^n x_0) + \frac{c}{1-a} d(Tu, TS^{n-1}x_0) \\
&\quad + \frac{h^n}{1-a} d(Tx_0, TSx_0) + \frac{1}{1-a} d(Tx_{n+1}, Tu) \\
&= \frac{b}{1-a} d(Tx_{n-1}, Tx_n) + \frac{c}{1-a} d(Tu, Tx_{n-1}) \\
&\quad + \frac{h^n}{1-a} d(Tx_0, Tx_1) \\
&\quad + \frac{1}{1-a} d(Tx_{n+1}, Tu).
\end{aligned}
$$

Letting $n \to \infty$ and using Remark 1.1, we have $TSu = Tu$. Since T is one to one, we have $Su = u$. and so S has a fixed point.

To prove uniqueness, let v be another fixed point of S. Then by (10), we have

$$
\begin{aligned}
d(Tv,Tu) &= d(TSv, Tsu) \\
&\le ad(TSv,Tv) + bd(TSu,Tu) + cd(Tv,Tu) \\
&\le \frac{a}{1-c} d(Tv,Tv) + \frac{b}{1-c}(Tu,Tu) = 0.
\end{aligned}
$$

Hence $Tv = Tu$ and so $u = v$. The fixed point is therefore unique.

Finally, if T is sequentially convergent, we conclude that $lim_{n\to\infty} S^n x_0 = u$. This shows that $\{S^n x_0\}$ converges to the fixed point of S. □

Corollary 3.1. *Let (X,d) be a complete generalized metric space and let $T, S : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $\lambda \in [0, \frac{1}{2})$ and*

$$
d(TSx, TSy) \le \lambda\big[d(Tx,TSx) + d(Ty,TSy)\big] \tag{11}
$$

for all $x, y \in X$, then S has a unique fixed point. Further, if T is sequentially convergent then for every $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Remark 3.1. By taking $Tx \equiv x$ in Theorem 3.1, we can conclude the Reich's theorem [9].

Remark 3.2. By taking $Tx \equiv x$ in Corollary 3.1, we can conclude the Kannan's theorem [5].

Example 3.1. [1] Let $X = \{1, 2, 3, 4\}$. Define $d : X \times X \to R$ as follows:

$$d(1,2) = d(2,1) = 3,$$
$$d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1,$$
$$d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = 4.$$

Obviously (X, d) is a generalized metric space but not a metric space.

The following example shows that Theorem 3.1 and Corollary 3.1 are indeed a proper extensions of Azam and Arshad theorem [1].

Example 3.2. Define a mapping $S : X \to X$ as follows:

$$Sx = \begin{cases} 2, & \text{if } x \neq 1, \\ 4, & \text{if } x = 1. \end{cases}$$

Obviously the inequality (2) does not holds for S and every $\lambda \in [0, \frac{1}{2})$, and so we cannot use the Azam and Arshad theorem for S.

Now define $T : X \to X$ by

$$Tx = \begin{cases} 2, & \text{if } x = 4, \\ 3, & \text{if } x = 2, \\ 4, & \text{if } x = 1, \\ 1, & \text{if} x = 3. \end{cases}$$

and so

$$TSx = \begin{cases} 3, & \text{if } x \neq 1, \\ 2, & \text{if } x = 1. \end{cases}$$

It follows that

$$d(TSx, TSy) \leq \frac{1}{3}\big[d(Tx, TSx) + d(Ty, TSy)\big].$$

Therefore by Corollary 3.1, S has a unique fixed point.

Similarly, we can prove the following theorem:

Theorem 3.2. *Let (X, d) be a complete generalized metric space and let $T, S : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent and satisfies the inequality*

$$(12) \qquad d(TSx, TSy) \leq ad(Tx, TSy) + bd(Ty, TSx) + cd(Tx, Ty)$$

for all $x, y \in X$, where $a, b, c \geq 0$ and $a + b + c < 1$, then S has a unique fixed point. Further, if T is sequentially convergent, then for every $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Acknowledgement. The authors would like to thank the editor of the paper and the referees for their careful reading of the paper.

References

[1] A. Azam and M. Arshad, *Kannan fixed point theorem on generalized metric spaces*, J. Nonlinear Sci. Appl., **1**(1)(2008), 45-48.

[2] A. Branciari, *A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces*, Publ. Math. Debrecen, **57**(1-2) (2000), 31-37.

[3] S.K. Chatterjea, *Fixed point theorems*, C.R. Acad. Bulgare Sci., **25**(1972), 727-730.

[4] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, New York, 1990.

[5] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **60**(1968), 71-76.

[6] R. Kannan, *Some results on fixed points II*, Amer. Math. Monthly, **76**(1969), 405-408.

[7] M.A. Khamsi and W. A. Kirk, *An introduction to metric spaces and fixed point theory*, John Wiley and Sons, Inc., 2001.

[8] S. Moradi, *Kannan fixed-point theorem on complete metric spaces and on generalized metric spaces depended an another function*, arXiv:0903.1577v1 [math.FA] 9 Mar 2009.

[9] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull., **14**(1971), 121-124.

[10] B.E. Rhoades, *A Comparison of Various Definitions of Contractive Mappings*, Trans. Amer. Math. Soc. **226**(1977), 257-290.

Amit Singh
Department of Mathematics
Government Degree College Billawar
Jammu and Kashmir
INDIA-184204
E-mail address: singhamit841@gmail.com

M.S. Khan
College of Science
Department of Mathematics and Statistics
Sultan Qaboos University
Post Box 36, Postal Code 123
Al-Khod, Muscat
Sultanate of Oman
E-mail address: mohammad@squ.edu.om

Brian Fisher
Department Of Mathematics
University of Leicester
Leicester, LE1 7RH
England
E-mail address: fbr@le.ac.uk