Mathematica Moravica
Vol. **18-1** (2014), 89–98

Curves of Restricted Type in Euclidean Spaces

Bengü (Kiliç) Bayram and Nergiz Önen

Abstract. Submanifolds of restricted type were introduced in [7]. In the present study we consider restricted type of curves in $\mathbb{E}^m$. We give some special examples. We also show that spherical curve in $S^2(r) \subset \mathbb{E}^3$ is of restricted type if and only if either $f(s)$ is constant or a linear function of s of the form $f(s) = \pm s + b$ and every closed $W - curve$ of rank k and of length $2\pi r$ in $\mathbb{E}^{2k}$ is of restricted type.

1. Introduction

Let M^n be an $n-$dimensional submanifold of a Euclidean space $\mathbb{E}^m$. Let x,H and Δ respectively be the position vector field, the mean curvature vector field and the Laplace operator of the induced metric on M^n. Then, as is well known (see e.g. [2])

$$\Delta x = -nH, \tag{1}$$

which shows, in particular, that M^n is a minimal submanifold in $\mathbb{E}^m$ if and only if its coordinate functions are harmonic (i.e. they are eigenfunctions of Δ with eigenvalue 0).

As a generalization of T. Takahashi's condition and following an idea of O. Garay [13], some of the authors together with J. Pas [10] initiated the study of submanifolds M^n in $\mathbb{E}^m$ such that

$$\Delta x = Ax + B \tag{2}$$

for some fixed vector $B \in \mathbb{E}^m$ and a given matrix $A \in \mathbb{R}^{m\times m}$. This study was continued by the first author together with M. Petrovic [5] and independently by T. Hasanis and T. Vlachos [14].

During the study of submanifolds of $\mathbb{R}^m$ satisfying (2), it was observed that all these matrices A_p are equal for all $p \in M$, or equivalently there exists a fixed matrix $A \in \mathbb{E}^{m\times m}$ (determining, of course, a linear endomorphism of $\mathbb{E}^m$) such that for all $p \in M$ and for all $X \in T_pM$,

$$A_H X = (AX)^T. \tag{3}$$

2010 *Mathematics Subject Classification.* Primary: 53A04.
Key words and phrases. Curves of restricted type.

As the relation (3) expresses a strong relationship between differential geometry and linear algebra, we do think it would be worthwhile to study submanifolds satisfying this condition; such submanifolds are said to be *of restricted type.*

Submanifolds of restricted type were introduced in [7] by the author B.Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken. The class of submanifolds of restricted type is large which includes $1-$type submanifolds, pseudo-umbilical submanifolds with constant mean curvature, submanifolds satisfying either Gray's condition or Dillen Pas Verstraelen's condition, all $k-$type curves lying fully in $\mathbb{E}^{2k}$, all null $k-$type curves lying fully in $\mathbb{E}^{2k-1}$, the products of submanifolds of restricted type, the diagonal immersions of restricted type submanifolds and equivariant isometric immersions of compact homogeneous spaces. In [7], it is shown that a hypersurface of restricted type is either minimal, or a part of the product of a sphere and a linear subspace, or a cylinder on a plane curve of restricted type, and all planar curves of restricted type are classified.

2. Basic Concepts

In the present section we recall definitions and results of [1]. Let $x : M \to \mathbb{E}^m$ be an immersion from an $n-$dimensional connected Riemannian manifold M into an $m-$dimensional Euclidean space $\mathbb{E}^m$. We denote by g the metric tensor of $\mathbb{E}^m$ as well as the induced metric on M. Let $\tilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^m$ and ∇ the induced connection on M. Then the Gaussian and Weingarten formulas are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{4}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{5}$$

where X, Y are vector fields tangent to M and ξ normal to M. Moreover, h is the second fundamental form, D is the linear connection induced in the normal bundle $T^\perp M$, called normal connection and A_ξ the shape operator in the direction of ξ that is related with h by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle . \tag{6}$$

For an $n-$dimensional submanifold M in $\mathbb{E}^m$. The mean curvature vector $\vec{H}$ is given by

$$\vec{H} = \frac{1}{n} trace h.$$

A submanifold M is said to be minimal (respectively, totally geodesic) if $\vec{H} \equiv 0$ (respectively, $h \equiv 0$).

Consider an $n-$dimensional Riemannian manifold M and denote by (g_{ij}) the local components of its metric. Put $G = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$.

Then the Laplacian Δ of the metric g can be locally defined by

$$\Delta u = -\frac{1}{\sqrt{G}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{G} g^{ij} \frac{\partial u}{\partial x_j}\right), \tag{7}$$

for any function u on M, where $x_1, x_2, ..., x_n$ are local coordinates [11].

M is said to be of finite type if each component of the position vector x has a finite spectral decomposition [2]

$$x = x_0 + x_1 + x_2 + \cdots + x_k, \tag{8}$$

where x_0 is a constant vector in $\mathbb{E}^m$ and $x_1, x_2, \ldots, x_k$ are non-constant maps which satisfy $\Delta x_i = \lambda_i x_i$, $1 \leq i \leq k$, $\lambda_1 < \lambda_2 < \cdots < \lambda_k$.

If all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are mutually distinct, then the immersion x (or the submanifold M) is said to be of $k-$type [2].

3. W-curves in $\mathbb{E}^m$

Let $\gamma = \gamma(t) : I \to \mathbb{E}^m$ be a regular curve in $\mathbb{E}^m$ (i.e. $\|\gamma'\|$ is nowhere zero), where I is interval in $\mathbb{R}$. γ is called a *Frenet curve of rank* r ($r \in \mathbb{N}_0$, $r \leq m$) if $\gamma'(t), \gamma''(t), \ldots, \gamma^{(r)}(t)$ are linearly independent and $\gamma'(t), \gamma''(t), \ldots, \gamma^{(r+1)}(t)$ are no longer linearly independent for all t in I. In this case, $Im(\gamma)$ lies in an r-dimensional Euclidean subspace of $\mathbb{E}^m$. To each Frenet curve of rank r there can be associated orthonormal $r-$frame $\{V_1, V_2, \ldots, V_r\}$ along γ, the Frenet $r-$frame and $r-1$ functions $\kappa_1, \kappa_2, \ldots, \kappa_{r-1} : I \to \mathbb{R}$, the Frenet curvatures, such that

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ \vdots \\ V_{r-1}' \\ V_r' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa_1 & 0 & \cdots & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & \cdots & 0 & 0 \\ 0 & -\kappa_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & \kappa_{r-1} \\ 0 & 0 & \cdots & \cdots & -\kappa_{r-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{r-1} \\ V_r \end{bmatrix}, \tag{9}$$

where v is the speed of the curve.

In fact, to obtain $V_1, V_2, \ldots, V_r$ it is sufficient to apply the Gram-Schmidt orthonormalization process to $\gamma'(t), \gamma'(t), \ldots, \gamma^{(r)}(t)$. Moreover, the functions $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are easily obtained as by-product during this calculation. More precisely, $V_1, V_2, \ldots, V_r$ and $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are determined by the following formulas:

$$
(10)\qquad
\begin{aligned}
E_1(t) &:= \gamma'(t); \qquad V_1 := \frac{E_1(t)}{\|E_1(t)\|}\\
E_k(t) &:= \gamma^{(k)}(t) - \sum_{i=1}^{k-1} \left\langle \gamma^{(k)}(t), E_i(t) \right\rangle \frac{E_i(t)}{\|E_i(t)\|}\\
\kappa_{k-1}(t) &:= \frac{E_k(t)}{\|E_{k-1}(t) E_1(t)\|}\\
V_k &:= \frac{E_k(t)}{\|E_k(t)\|}
\end{aligned}
$$

where $k \in \{2, 3, \dots, r\}$. It is natural and convenient to define Frenet curvatures $\kappa_r = \kappa_{r+1} = \cdots = \kappa_{m-1} = 0$. It is clear that $V_1, V_2, \dots, V_r$ and $\kappa_1, \kappa_2, \dots, \kappa_{r-1}$ can be defined for any regular curve (not necessary a Frenet curve) in the neighborhood of a point t_0 for which $\gamma'(t_0), \gamma''(t_0), \dots, \gamma^{(r)}(t_0)$ are linearly independent.

Definition 1. Frenet curve of rank r for which $\kappa_1, \kappa_2, \dots, \kappa_{r-1}$ are constant is called (generalized) screw line or helix [6]. Since these curves are trajectories of the 1-parameter group of the Euclidean transformations, so, F. Klein and S. Lie [9] called them $W-$ *curves*.

A unit speed $W-$*curve of rank* $2k$ has the parametrization form

$$
(11)\qquad \gamma(s) = a_0 + \sum_{i=1}^{k} (a_i \cos \mu_i s + b_i \sin \mu_i s),
$$

and a unit speed $W-$*curve* of rank $(2k+1)$ has the parametrization form

$$
(12)\qquad \gamma(s) = a_0 + b_0 s + \sum_{i=1}^{k} (a_i \cos \mu_i s + b_i \sin \mu_i s),
$$

where $a_0, b_0, a_1, \dots, a_k, b_1, \dots, b_k$ are constant vectors in $\mathbb{E}^m$ and $\mu_1 < \mu_2 < \cdots < \mu_k$ are positive real numbers.

So, a $W-$*curve* of rank 1 is a straight line, a $W-$*curve* of rank 2 is a circle and a $W-$*curve* of rank 3 is a right circular helix [6].

A $W-$*curve* is closed if and only if its rank is even and all μ_i are rational multiples of a real number. Therefore, up to rigid motions of a Euclidean space, a closed $W-$*curve* of rank $2k$ and of length $2\pi r$ in $\mathbb{E}^{2k}$ has an arc length parameterization of the form:

(13)

$$
\gamma(s) = \frac{r}{\sqrt{k}} \left(\frac{1}{t_1} \cos\left(\frac{t_1 s}{r}\right), \frac{1}{t_1} \sin\left(\frac{t_1 s}{r}\right), \dots, \frac{1}{t_k} \cos\left(\frac{t_k s}{r}\right), \frac{1}{t_k} \sin\left(\frac{t_k s}{r}\right) \right)
$$

where $t_1 < \cdots < t_k$ are positive integers [8].

4. Curves of restricted type

Definition 2. A submanifold M^n in $\mathbb{E}^m$ is said to be restricted type if the shape operator A_H is the restriction of a fixed endomorphism A of $\mathbb{E}^m$ on the tangent space of M^n at every point of M^n, i.e.

$$A_H X = (AX)^T \tag{14}$$

for any vector X, tangent to M^n, where $(AX)^T$ denotes the tangential component of AX [7].

Remark 1. Equation (14) is equivalent to $\langle A_H X, Y\rangle = \langle AX, Y\rangle$ for all tangent vectors X, Y [7].

Proposition 1. *Every submanifold M^n in $\mathbb{E}^m$ whose position vector field satisfies $\Delta x = \tilde{A}x + B$, where Δ is the Laplacian of M^n, $\tilde{A} \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{E}^m$, is of restricted type. The endomorphism A is given by $\frac{1}{n}\tilde{A}$ in this case [7].*

Let γ be a regular curve in $\mathbb{E}^m$. The Laplacian of γ can be expressed as

$$\Delta\gamma(t) = -\frac{d^2\gamma(t)}{dt^2} = -\gamma''(t). \tag{15}$$

By the using of (1) and (15),

$$H = -\Delta\gamma(t) = \gamma''(t) \tag{16}$$

where H is the mean curvature of γ.

Proposition 2. *Let γ be a curve in $\mathbb{E}^m$. If γ has the equation*

$$-\gamma''(t) = \Delta\gamma(t) = A\gamma(t) + B \tag{17}$$

such that B is a fixed vector in $\mathbb{E}^m$ and A a symmetric matrix in $\mathbb{R}^{m\times m}$, then γ is of restricted type.

Proof. From Preposition 1 we have the equation

$$\Delta\gamma(t) = A\gamma(t) + B. \tag{18}$$

Thus using (16) and (18), we get (17). □

Corollary 1. *Let γ be a curve in $\mathbb{E}^m$. γ is of restricted type if and only if*

$$-\gamma'''(t) = A\gamma'(t), \tag{19}$$

where A is a symmetric matrix in $\mathbb{R}^{m\times m}$.

Example 1. $S^1(a) \subset \mathbb{E}^2$ is of restricted type.

$S^1(a)$ is given by the parametrization $\gamma(t) = (a\cos t, a\sin t)$. From higher order derivatives of γ we get

$$\gamma'''(t) = -I_2\gamma'(t). \tag{20}$$

Thus $S^1(a) \subset \mathbb{E}^2$ is of restricted type.

Example 2. A helix which is given by the parametrization

$$\gamma(t) = (r\cos(ct+d), r\sin(ct+d), at+b)$$

is of restricted type.

From higher order derivatives of γ we get $\gamma'''(t) = -A\gamma'(t)$ where

$$A = \begin{bmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus helix is of restricted type.

Example 3. Every $k-$type curve which lies fully in $\mathbb{E}^{2k}$ is of restricted type [7].

Example 4. Every $2-$type curve in $\mathbb{E}^m$ is of restricted type [7].

Example 5. Although every $2-$type curve in $\mathbb{E}^m$ and every $k-$type curve which lies fully in $\mathbb{E}^{2k}$ are curves of restricted type, not every curve of finite type (in the sense of [2,4]) is of restricted type. For instance the following $6-$type curve in $\mathbb{E}^3$ is not of restricted type [7]

$$\begin{aligned}\gamma(s) = \Big(&-\frac{2}{3}\cos\frac{12}{17}s + \frac{3}{4}\cos\frac{16}{17}s + \frac{3}{10}\cos\frac{20}{17}s + \frac{1}{8}\cos\frac{24}{17}s + \frac{1}{14}\cos\frac{28}{17}s, \\ &-\frac{2}{3}\sin\frac{12}{17}s + \frac{3}{4}\sin\frac{16}{17}s + \frac{3}{10}\sin\frac{20}{17}s + \frac{1}{8}\sin\frac{24}{17}s + \frac{1}{14}\sin\frac{28}{17}s, \sin\frac{8}{17}s\Big).\end{aligned}$$

Proposition 3. *Let γ be a spherical space curve given with*

$$\gamma(s) = (r\cos s\sin(f(s)), r\sin s\sin(f(s)), r\cos(f(s))\,, \tag{21}$$

where $f(s)$ is polynomial function. Then γ is of restricted type if and only if $f(s)$ is either constant or a linear function of s of the form $f(s) = \pm s + b$.

Proof. Suppose that γ is of restricted type, then by the use of (19) the equality

$$\begin{bmatrix} \gamma_1'''(s) \\ \gamma_2'''(s) \\ \gamma_3'''(s) \end{bmatrix} = \begin{bmatrix} -c_{11} & 0 & 0 \\ 0 & -c_{22} & 0 \\ 0 & 0 & -c_{33} \end{bmatrix} \cdot \begin{bmatrix} \gamma_1'(s) \\ \gamma_2'(s) \\ \gamma_3'(s) \end{bmatrix} \tag{22}$$

holds. Here $\gamma_i', \gamma_i'''(s)$ are the first and the third derivatives of i^{th} component of γ and c_{ii} is the entry of the matrix A.

From higher order derivatives of γ we get

$$\begin{aligned}\gamma'(s) = \big(&-r\sin s\sin(f(s)) + r\cos s\cos(f(s))f'(s), r\cos s\sin(f(s)) \\ &+ r\sin s\cos(f(s))f'(s), -r\sin(f(s))f'(s)\big)\end{aligned} \tag{23}$$

(24)
$$\gamma'''(s) = \big(r\cos s\cos(f(s))(f'''(s) - (f'(s))^3 - 3f'(s)) + r\sin s\sin(f(s))(1 + 3(f'(s))^2) + r\cos s\sin(f(s))(-3f'(s)f''(s)) + r\sin s\cos(f(s))(-3f''(s)), r\cos s\cos(f(s))(3f''(s)) + r\sin s\sin(f(s))(-3f'(s)f''(s)) + r\cos s\sin(f(s))(-1 - 3(f'(s))^2) + r\sin s\cos(f(s))(f'''(s) - (f'(s))^3 - 3f'(s)), r\sin(f(s))(-f'''(s) + (f'(s))^3) + r\cos(f(s))(-3f'(s)f''(s))\big).$$

Using (22), (23) and (24) we have

$$f'''(s) - (f'(s))^3 - 3f'(s) + c_{11}f'(s) = 0, \tag{25}$$
$$1 + 3(f'(s))^2 - c_{11} = 0, \tag{26}$$
$$-3f'(s)f''(s) = 0, \tag{27}$$
$$-3f''(s) = 0, \tag{28}$$
$$-1 - 3(f'(s))^2 + c_{22} = 0, \tag{29}$$
$$f'''(s) - (f'(s))^3 - 3f'(s) + c_{22}f'(s) = 0, \tag{30}$$
$$-f'''(s) + (f'(s))^3 - c_{33}f'(s) = 0. \tag{31}$$

From (27) and (28) it can be seen that either $f(s)$ is constant or a linear function of s of the form $f(s) = as+b$ where $a, b \in \mathbb{R}$. If $f(s)$ is constant, then $f(s)$ is a circle which is of restricted type. If $f(s)$ is a linear function of s of the form $f(s) = as+b$, then using (25) and (26) we get $c_{11} = 1+3a^2 = a^2+3$. Then $a = \pm 1$ and $c_{11} = 4$. Similarly, from (29), (30) and (31) we get $c_{22} = 4$ and $c_{33} = 1$. So we obtain

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{32}$$

Conversely, if $f(s) = const.$ or $f(s) = \pm s + b$ then it is easy to show that the curve given with the parametrization (21) is of restricted type. □

We also get the following result.

Proposition 4. *Let γ be closed $W-$curve of rank k and of length $2\pi r$ in $\mathbb{E}^{2k}$ given by the parametrization* (13). *Then γ is of restricted type.*

Proof. From higher order derivatives of γ we get

$$\gamma'(s) = \frac{1}{\sqrt{k}}\left(-\sin\left(\frac{t_1 s}{r}\right), \cos\left(\frac{t_1 s}{r}\right), \ldots, -\sin\left(\frac{t_k s}{r}\right), \cos\left(\frac{t_k s}{r}\right)\right)$$

$$\gamma''(s) = \frac{-1}{\sqrt{k}}\left(\frac{t_1}{r}\cos\left(\frac{t_1 s}{r}\right), \frac{t_1}{r}\sin\left(\frac{t_1 s}{r}\right), \ldots, \frac{t_k}{r}\cos\left(\frac{t_k s}{r}\right), \frac{t_k}{r}\sin\left(\frac{t_k s}{r}\right)\right)$$

$$\gamma'''(s) = \frac{1}{\sqrt{k}}\left(\frac{t_1^2}{r^2}\sin\left(\frac{t_1 s}{r}\right), -\frac{t_1^2}{r^2}\cos\left(\frac{t_1 s}{r}\right), \ldots, \frac{t_k^2}{r^2}\sin\left(\frac{t_k s}{r}\right), -\frac{t_k^2}{r^2}\cos\left(\frac{t_k s}{r}\right)\right).$$

So, we have $\gamma'''(t) = -A\gamma'(t)$ where

$$A = \begin{bmatrix} \frac{t_1^2}{r^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{t_1^2}{r^2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{t_k^2}{r^2} & 0 \\ 0 & 0 & \cdots & 0 & \frac{t_k^2}{r^2} \end{bmatrix}. \tag{33}$$

Thus $W-$*curve* is of restricted type. □

Example 6. A closed W$-$curve of rank 4 and of length 2π given by the parametrization

$$\gamma(s) = (\cos ms, \sin ms, \cos ns, \sin ns)$$

is of restricted type, where m, n are positive integers. From higher order derivatives of γ we get $\gamma'''(t) = -A\gamma'(t)$ where

$$A = \begin{bmatrix} m^2 & 0 & 0 & 0 \\ 0 & m^2 & 0 & 0 \\ 0 & 0 & n^2 & 0 \\ 0 & 0 & 0 & n^2 \end{bmatrix}.$$

Thus γ is of restricted type.

Theorem 1 ([7])**.** *Up to rigid motions of* $\mathbb{E}^2$*, a curve in* $\mathbb{E}^2$ *is of restricted type if and only if it is an open portion of one of the following plane curves:*

(1) *a circle,*

(2) *a line,*
(3) *a curve with equation :* $\cos(cx) = e^{-cy}$*, where* $c \neq 0$*,*
(4) *a curve with equation :* $a\sin^2(\sqrt{c}x) + b\sinh^2(\sqrt{c}x) = c$*, where* $a > b > 0$*,* $c = a - b$*,*
(5) *a curve with equation :* $a\sin^2(\sqrt{c}x) - b\cosh^2(\sqrt{c}x) = c$*, where* $a > 0 > b$*,* $c = a - b$*.*

Proposition 5 ([7]). *Let* γ *be a planar curve.* γ *is of restricted type if and only if the curvature* κ *of* γ *satisfies the following differential equation*

$$\kappa\kappa''' - \kappa'\kappa'' + 4\kappa^3\kappa' = 0 \tag{34}$$

where the derivatives are taken with respect to the arc length parameter.

References

[1] B.Y. CHEN, *Geometry of Submanifolds and Its Applications*, Science University of Tokyo, Tokyo, 1981.

[2] B.Y. CHEN, *Total mean curvature and submanifolds of finite type*, World Scientific, Singapore, 1984.

[3] B.Y. CHEN, *A report on submanifolds of finite type*, Soochow J. Math., **22** (1996), 117–337.

[4] B.Y. CHEN, J. DEPREZ, F. DİLLEN, L. VERSTRAELEN AND L. VRANCKEN, *Curves of finite type*, Geometry and Topology of Submanifolds, II, World Scientific, Singapore, 1990, 76-110.

[5] B.Y. CHEN, M. PETROVIC, *On spectral decomposition of immersions of finite type*, Bull. Austr. Math. Soc. **44** (1991), 117–129.

[6] B.Y. CHEN, J. DEPREZ, P. VERHEYEN, *Immersions with geodesics of 2-type*, In: Geometry and Topology of Submanifolds IV, Belgium, 1992.

[7] B.Y. CHEN, F. DİLLEN, L. VERSTRAELEN AND L. VRANCKEN, *Submanifolds of restricted type*, Journal of Geometry, **46** (1993), 20–32.

[8] B.Y. CHEN, *A report on submanifolds of finite type*, Soochow J. Math., **22** (1996), 117–337.

[9] F. KLEİN, S. LİE, *Uber diejenigen ebenenen kurven welche durch ein geschlossenes system von einfach unendlich vielen vartauschbaren linearen Transformationen in sich übergehen*, Math. Ann., **4** (1871), 50–84.

[10] F. DILLEN, J. PAS, L. VERSTRAELEN, *On surfaces of finite type in Euclidean 3-space*, Kodai Math. J., **13** (1990), 10–21.

[11] G. ZAFİNDRATAFA, *Hypersurfaces whose mean curvature function is of finite type*, J. Geom., **55** (1996), 182–191.

[12] H. GLUCK, *Higher curvatures of curves in Euclidean space*, Am. Math. Month., **73** (1966), 699–704.

[13] O.J. GARAY, *An extension of Takahashi's theorem*, Geom. Dedicate, **34** (1990), 105–112.

[14] T. HASANIS, T. VLACHOS, *Hypersurfaces of* $\mathbb{E}^{n+1}$ *satisfying* $\Delta x = Ax + B$, Jour. Austr. Math. Soc. **53** (1992), 377–384.

BENGÜ (KILIÇ) BAYRAM
DEPARTMENT OF MATHEMATICS
BALIKESIR UNIVERSITY
BALIKESIR
TURKEY
E-mail address: benguk@balikesir.edu.tr

NERGIZ ÖNEN
DEPARTMENT OF MATHEMATICS
ÇUKUROVA UNIVERSITY
ADANA
TURKEY
E-mail address: nonen@cu.edu.tr